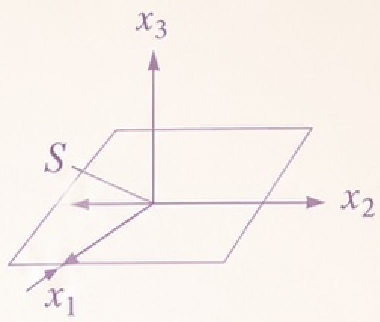


Lecture Notes
on
LINEAR ALGEBRA
P2S01NCMTH03

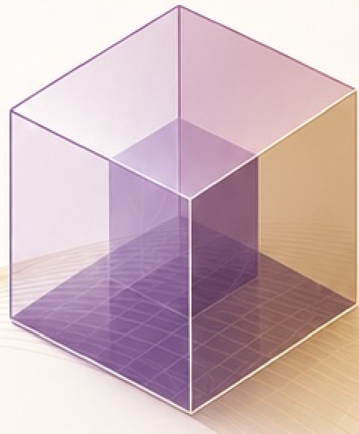
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$T : V \to W$$

$$T(v+u) = T(v) + T(u)$$

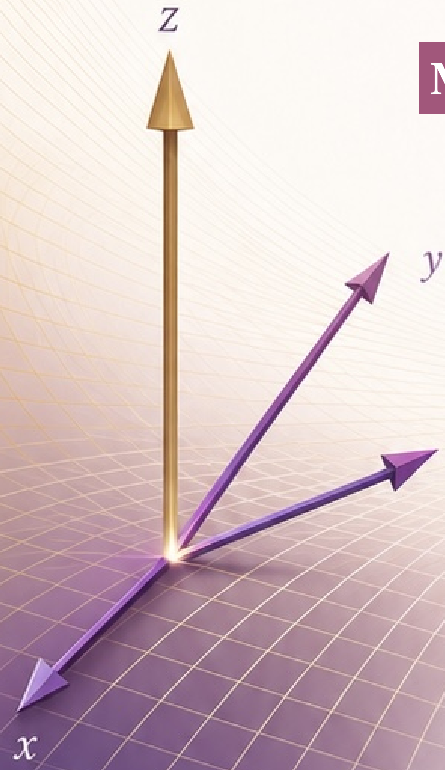
$$T(\alpha v) = \alpha T(v)$$

$$\langle u, v \rangle = u^T v$$



M.Sc. Mathematics – Semester I

As per NEP curriculum



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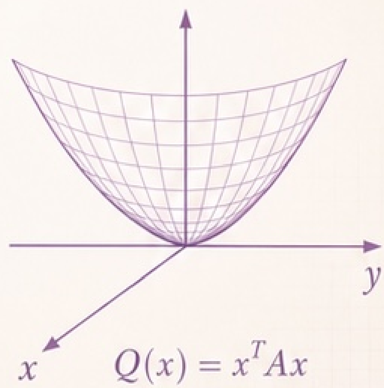
$$P_V(u) = \frac{\langle u, v \rangle}{\langle v, v \rangle} v$$

$$det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$



$$Q(x) = x^T A x$$

2026 – 27

Preface and Acknowledgments

Contents

SYLLABUS

P2S01NCMTH03: Linear Algebra

Unit I: Vector Spaces: Quick review of vector spaces, examples of vector spaces including sequence spaces and function spaces, Subspaces, Bases, Dimensions, Row and column spaces, Rank and nullity, Bases for subspaces, Invertibility, Applications: The Wronskian.

Unit II: Linear Transformations: Basic properties of linear transformations, Invertible linear transformations, Matrices of linear transformations, Vector spaces of linear transformations, Change of bases, Similarity, Applications: Dual spaces, Adjoint (only definition and examples).

Unit III: Inner Product Spaces: Dot products and inner products, the lengths and angles of vectors, Matrix representations of inner products, Gram-Schmidt orthogonalization, Projections, Orthogonal projections (only definitions and examples). Diagonalization: Eigenvalues and eigenvectors, Diagonalization of matrices, Exponential matrices, Diagonalization of linear transformations.

Unit IV: Jordan Canonical Forms: Basic properties of Jordan canonical forms, Generalized eigenvectors, the power A^k and the exponential e^A, Cayley-Hamilton theorem, the minimal polynomial of a matrix. Quadratic Forms: Basic properties of quadratic forms, Diagonalization of quadratic forms, A classification of level surfaces.

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CHAPTER 1

VECTOR SPACES

1.1 Vector Spaces

Definition 1.1.1. Let V be a non empty set and F be a field. Suppose there exists two opeartions: (1) **vector addition**, $+ : V \times V \to V$, which combines every pair elements u and v in V to a unique element $u + v$ in V and (2) **scalar multiplication**, $\cdot : F \times V \to V$, which combines an each element of α in F and an element of u in V to an element αu in V. Then V, along with the two operations, is called a vector space over the field F if the following properties hold:

(A1) Addition is associative, $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$.
(A2) Addition is commutative, $u + v = v + u$ for all $u, v \in V$.
(A3) There is an element $0 \in V$, called the zero vector, such that $u + 0 = u$ for all $u \in V$.
(A4) For each element $u \in V$ there exists an element $-u \in V$, called the inverse of u, such that $u + (-u) = 0$.
(M1) $\alpha(\beta u) = (\alpha\beta)u$ for all $\alpha, \beta \in F$ and $u \in V$.
(M2) Vector addition is distributive, $\alpha(u + v) = \alpha u + \alpha v$ for all $\alpha \in F$ and $u, v \in V$.
(M3) Distributivity over scalar addition, $(\alpha + \beta)u = \alpha u + \beta u$ for all $\alpha, \beta \in F$ and $u \in V$.
(M4) Unit scalar, $1u = u$ for all $u \in V$.

The elements, no matter what they might really be, of V are called *vectors* and the elements of the field F are called *scalars*. Obeserve that properties (A1)-(A4) shows that $(V, +)$ is an abelian group. Another important observation to make here is that the vector space V is a composite object consisting of a non empty set V along with the two operations defined above and a field F of scalars. The same underlying set V may be a part of a number of distinct vector spaces by considering a different field F (see Example 1.1.3 and Example 1.1.6) or altering the operations (see Example 1.1.3 and Example 1.1.8). When there is no ambiguity or scope of any confusion, we may just state that V is a vector space or else we shall say that V is a vector space over the field F thereby specifying the field.

We shall now see a variety of examples to be comfortable with the notion of vector space.

Example 1.1.2. The trivial vector space $V = \{0\}$ for any field F. One may define the singleton vector space $V = \{z\}$ for some element z with the following operations:

$$z + z = z \text{ and } \alpha z = z.$$

Here, the zero element is also z since V being singleton. So, z behaves like a zero vector is a singleton vector space.

Question: Is a field F also a vector space over itself? The answer is Yes. In general, we have the following example.

Example 1.1.3. The set of n-tuples, $F^{(n)} = \{(v_1, \dots, v_n) : v_i \in F,\ 1 \leq i \leq n\}$ is a vector space over the field F with vector addition and scalar multiplication defined as follows:

$$(u_1, \dots, u_n) + (v_1, \dots, v_n) := (u_1 + v_1, \dots, u_n + v_n);$$
$$\alpha(u_1, \dots, u_n) := (\alpha u_1, \dots, \alpha u_n).$$

In particular, for $n \in \mathbb{N}$, $\mathbb{R}^n$ is a vector over $\mathbb{R}$ of reals and $\mathbb{C}^n$ is a vector space over the field $\mathbb{C}$ of complex numbers.

Example 1.1.4. The set of infinite sequences of elements of a field F is a vector space over the field F. More precisely, $\mathcal{S}(F) = \{(x(n))_{n\in\mathbb{N}} : x(n) \in F\}$ is a vector space over the field F with vector addition and scalar multiplication defined as follows:

$$(x(1), x(2), \dots) + (y(1), y(2), \dots) := (x(1) + y(1), x(2) + y(2), \dots);$$
$$\alpha(x(1), x(2), \dots) := (\alpha x(1), \alpha x(2), \dots).$$

In particular, $\mathbb{C}^\infty$ and $\mathbb{R}^\infty$ are vector spaces over $\mathbb{C}$ and $\mathbb{R}$ respectively.

Example 1.1.5. Let X be a non-empty set and F be a field. Let $\mathcal{F}(X, F)$ denote the set of all functions $f : X \to F$. Then $\mathcal{F}(X, F)$ is a vector space over F with pointwise addition and scalar multiplication defined as follows:

$$(f + g)(x) := f(x) + g(x)$$
$$(\alpha f)(x) := \alpha f(x).$$

Example 1.1.6. Let F be a field and K be a subfield of F. Then F^n is a vector space over the field K with vector addition and scalar multiplication defined as in Example 1.1.3. In particular, $\mathbb{C}^n$ is a vector space over $\mathbb{R}$. Note that this is different from the one defined in Example 1.1.3.

Example 1.1.7. The vector space of matrices: Let F be a field and m, n be positive integers. Then the set $M_{m,n}(F)$ or $M_{m\times n}(F)$ of all $m \times n$ matrices with entries from F is a vector space over F with usual matrix (entrywise) addition and scalar multiplication, i.e.

$$(A + B)_{ij} := A_{ij} + B_{ij} \text{ and } (\alpha A)_{ij} := \alpha A_{ij}.$$

Example 1.1.8. Let $V = \{(x, y) : x, y \in \mathbb{C}\}$ is a vector space over $\mathbb{C}$ with the following opeations:

$$\begin{aligned} \text{Vector addition:} \quad & (x_1, y_1) + (x_2, y_2) && := (x_1 + x_2 + 1, y_1 + y_2 + 1); \\ \text{Scalar multiplication:} \quad & \alpha(x, y) && := (\alpha x + \alpha - 1, \alpha y + \alpha - 1). \end{aligned}$$

The zero vector here is $\mathbf{0} = (-1, -1)$ and the additive inverse of the element (x, y) is $(-x - 2, -y - 2)$. Note that the underlying set V here is $\mathbb{C}^2$. However, it is different from the vector space $\mathbb{C}^2$ over $\mathbb{C}$ considered in Example 1.1.3 due to the operations defined differently.

Example 1.1.9. Let F be a field. Then the set $F[x]$ of all polynomials in x over F is a vector space with usual operations (addition of two polynomials and multiplication of a polynomial by an element of F).

Example 1.1.10. The vector space of polynomials, $F_n[x]$: Let $F_n[x]$ be the set of all polynomials of degree less than or equal to n in the variable x with coefficients from the field F (say for example $\mathbb{R}$ or $\mathbb{C}$). Then $F_n[x]$ is a vector space over F with vector addition and scalar multiplication defined as follows:

$$\begin{aligned}(a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n) &:= (a_0 + b_0) + (a_1 + b_1)x + \cdots + \\ &\quad\ (a_n + b_n)x^n; \\ \alpha(a_0 + a_1x + \cdots + a_nx^n) &:= (\alpha a_0) + (\alpha a_1)x + \cdots + (\alpha a_n)x^n,\end{aligned}$$

where $\alpha, a_i \in F$, $1 \le i \le n$.

There is a relation between Example 1.1.3 and Example 1.1.10. Every element of $F_{n-1}[x]$ is of the form $a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$, where $a_i \in F,\ 0 \le i \le n-1$. If we map this element onto an element $(a_0, a_1, \ldots, a_{n-1})$ of $F^{(n)}$, then we have one-one correspondence between $F_{n-1}[x]$ and $F^{(n)}$. Once homomorphism and isomorphism are defined, we shall show that they are isomorphic.

Now, we see some properties of vector spaces. We have the following lemma. The proof is easy and left as an exercise.

Lemma 1.1.11. *If V is a vector space over F then*

(1) The zero vector $0 \in V$ is unique.
(2) Additive inverse is unique.
(3) $u + v = v$ implies $v = 0$.
(4) $\alpha 0 = 0$ for $\alpha \in F$.
(5) $0v = 0$ for all $v \in V$.
(6) $(-\alpha)v = -(\alpha v)$ for $\alpha \in F,\ v \in V$. In particular, for $\alpha = 1$ we have $-v = (-1)v$. Thus we get additive inverse from scalar multiplication.
(7) If $\alpha v = 0$ then either $\alpha = 0$ or $v = 0$.

Observe that $F_n[x]$ (in Example 1.1.10) is a subset of $F[x]$ (see Example 1.1.3), and the former is also a vector space over F under the same operations as the latter. Then we say that $F_n[x]$ is a subspace of $F[x]$. Before we give more examples we define subspace of a vector space.

1.2 Subspaces

Definition 1.2.1. Let V be a vector space over F. A subset W of V is called a subspace of V if W itself forms a vector space over F under the operations of V.

A vector space V itself and the zero vector $\{0\}$ are trivially subspaces of V.

To show that W is a subspace of a vector space V, it is not necessary to check all the properties in the definition of a vector space. Certain properties satisfied in a larger space inherently hold for its subset. The following result says that for a nonempty subset to be a subspace, one needs to check only closed under addition and scalar multiplications as in the larger space. The proof is easy and is left as an exercise.

Theorem 1.2.2. *A nonempty subset W of a vector space V over a field F is a subspace if and only if $x+y$ and kx are contained in W (or equivalently $x+ky \in W$) for any vectors x and y in W and any scalar $k \in F$.*

Exercise 1.2.3. Show that a nonempty subset W of V is a subspace if and only if for every $w_1, w_2 \in W$ and $\alpha, \beta \in F$, $\alpha w_1 + \beta w_2 \in W$, i.e., $w_1 + w_2 \in W$ and $\alpha w_1 \in W$ if and only if $\alpha w_1 + \beta w_2 \in W$.

Now, we consider some examples which are subspaces of the vector space $\mathcal{S}(K)$ where the field K is either $\mathbb{R}$ or $\mathbb{C}$.

Example 1.2.4. Vector space of summable sequences. Let ℓ^1 be the set of all sequences $(x(n))_{n\in\mathbb{N}}$ over $\mathbb{R}$ (or $\mathbb{C}$) such that $\sum_{n=1}^{\infty} |x(n)| < \infty$. Then ℓ^1 is a vector space over $\mathbb{R}$ (or $\mathbb{C}$). Hint: Using triangular inequality.

Example 1.2.5. Vector space of pth power summable sequences. Let ℓ^p is the set of all sequences $(x(n))_{n\in\mathbb{N}}$ over $\mathbb{R}$ (or $\mathbb{C}$) such that $\sum_{n=1}^{\infty} |x(n)|^p < \infty$. Then ℓ^p is a vector space over $\mathbb{R}$ (or $\mathbb{C}$).
Hint: $|x(n)+y(n)| \le |x(n)| + |y(n)| \le 2\max\{|x(n)|, |y(n)|\}$. Hence,

$$\begin{aligned} |x(n)+y(n)|^p \le 2^p(\max\{|x(n)|, |y(n)|\})^p &= 2^p \max\{|x(n)|^p, |y(n)|^p\} \\ &\le 2^p(|x(n)|^p + |y(n)|^p). \end{aligned}$$

One can also use Minkowski's inequality given below:

$$\left(\sum_{n=1}^{\infty} |x(n)+y(n)|^p\right)^{\frac{1}{p}} \le \left(\sum_{n=1}^{\infty} |x(n)|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y(n)|^p\right)^{\frac{1}{p}}.$$

Example 1.2.6. The vector space ℓ^∞ of all bounded sequences.

Example 1.2.7. The vector space **c** of all convergent sequences.

Note that if $x(n) \to a$ and $y(n) \to a$ then $x(n)+y(n) \to 2a$. The set of all sequences converging to a does not form a vector space unless $a = 0$. Thus, we have the following example:

Example 1.2.8. The vector space $\mathbf{c_0}$ of all convergent sequences whose limit is 0. $\mathbf{c_0}$ is a subspace of **c**.

Example 1.2.9. Let c_{00} be the set of all sequences with finitely many non-zero terms. Thus, if $(x(n))_{n\in\mathbb{N}} \in c_{00}$ then there exists $k_0 > 0$ such that $x(n) = 0$ for all $n > k_0$, i.e.

$$(x(n))_{n\in\mathbb{N}} = (x(1), \ldots, x(k_0), 0, 0, \ldots).$$

Then c_{00} is a vector space over the field $K = \mathbb{R}$ or $\mathbb{C}$ under the same operations of $\mathcal{S}(K)$.

Observe that, among all the spaces of sequences we saw so far, we have the following relation:

$$c_{00} \subset \ell^p \subset c_0 \subset c \subset \ell^\infty \subset \mathcal{S}(K), \qquad 1 \leq p < \infty$$

where $K = \mathbb{R}$ or $\mathbb{C}$. There is a relation between the spaces c_{00} and $K[x]$, where $K = \mathbb{R}$ or $\mathbb{C}$ (Example 1.2.9 and Example 1.1.9). For any element $(a(n)) = (a(1), \dots, a(k), 0, 0, \dots) \in c_{00}$ we get an element $p(x) = a(0) + a(1)x + \cdots + a(k)x^k \in K[x]$ and vice versa. Thus we have one to one correspondence between c_{00} and $K[x]$ and once we define isomorphism, we can see that they are isomorphic.

Example 1.2.10. Let X be a non-empty set and the field $K = \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{B}(X, K)$ denote the set of all bounded functions from X to K, i.e.

$$\mathcal{B}(X, K) = \{f : X \to K : \text{there exists } M > 0 \text{ such that } |f(x)| \leq M, \text{ for all } x \in X\}.$$

Then $\mathcal{B}(X, K)$ is a vector space over K under the same operations as in $\mathcal{F}(X, K)$. Thus, $B(X, K)$ is a subspace of $\mathcal{F}(X, K)$ (see Example 1.1.5).

Example 1.2.11. Let X be a metric (or topological) space and $K = \mathbb{R}$ or $\mathbb{C}$. Then $C(X, K)$, the set of all continuous functions from X to K, is a vector space over K under the same operations as in $\mathcal{F}(X, K)$.

Example 1.2.12. Is it true that intersection of two subspaces of a vector space is also a subspace? The answer is Yes. Let $C_b(X, K)$ denote the set of all bounded continuous functions from X to K. Then $C_b(X, K)$ is a vector space as

$$C_b(X, K) = C(X, K) \cap \mathcal{B}(X, K).$$

If we assume X to be a compact metric (or topological) space and $f \in C(X, K)$ then $f(X)(\subset \mathbb{R} \text{ or } \mathbb{C})$ is compact as we know that continuous image of compact set is compact. By Heine-Borel theorem, $f(X)$ is bounded and hence in case of compact space, $C_b(X, K) = C(X, K)$.

Definition 1.2.13. Let $f \in C(X, K)$ (continuity of function is not necessary) for a metric (or topological) space X. Then f is said to be vanishing at infinity if for every $\epsilon > 0$, there exists a compact subset Y of X such that

$$|f(x)| < \epsilon \text{ for } x \in Y^c,$$

where $x \in Y^c$ denoted the complement of Y. In other words, for each positive ϵ, the set $\{x \in X : |f(x)| \geq \epsilon\}$ is compact. The set of all continuous functions on X vanishing at infinity is denoted by $C_0(X)$.
For example, $X = K = \mathbb{R}$, $f(x) = e^{-|x|}$ vanishes at infinity. Also the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{1+x^2}$.

Definition 1.2.14. Let X be a metric (or a toplogical) space and $f \in C(X, K)$ for $K = \mathbb{R}$ or $\mathbb{C}$. The support of f denoted by $\text{supp}(f)$ is the set of points in X where f is non-zero, i.e.,

$$\text{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}.$$

A function $f : X \to K$ is said to have compact support if there exists a compact subset Y

of X such that
$$f(x) = 0 \text{ for } x \in Y^{c},$$
where $x \in Y^{c}$ denoted the complement of Y. In other words, the set $\{x \in X : f(x) \neq 0\}$ is compact. The set of all continuous functions on X with compact support is denoted by $C_c(X)$.
For example, $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} 1 - x^2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Example 1.2.15. $C_0(X)$ and $C_c(X)$ are vector spaces over the field K with the operations of $\mathcal{F}(X, K)$.
Hint: Let $f, g \in C_0(X)$. Given $\epsilon > 0$ there exists compact subset Y_1 and Y_2 of X such that
$$|f(x)| < \frac{\epsilon}{2} \text{ for all } x \in Y_1^{c}$$
$$|g(x)| < \frac{\epsilon}{2} \text{ for all } x \in Y_2^{c}$$
Take $Y = Y_1 \cup Y_2$. Then
$$|(f + g)(x)| = |f(x) + g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for all } x \in Y^{c}.$$
Thus, $f + g \in C_0(X)$. Similarly, one can show that $\alpha f \in C_0(X)$ which concludes that $C_0(X)$ is a vector space. The proof of $C_c(X)$ is similar.

Example 1.2.16. Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = 0\}$, where a, b, c are constants. If $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are points in W, then clearly $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ is also a point in W as it satisfies the equation in W. Similarly, αx also lies in W for any scalar α. Hence, W is a subspace of $\mathbb{R}^3$ which is a place passing through origin.

Example 1.2.17. Let A be a $m \times n$ matrix with entries from a field F. For instance, one may take a matrix with real entries, i.e., $F = \mathbb{R}$. Then the set,
$$W = \{x \in \mathbb{F}^{(n)} \mid Ax = 0\},$$
of all solutions of a homogeneous system $Ax = 0$ is a vector space over F. Moreover, since the operations in W coincides with the operations in $F^{(n)}$, W is a subspace of $F^{(n)}$.

Example 1.2.18. Let W be the set of all $n \times n$ real symmetric matrices. Then W is a subspace of the vector space $M_{n\times n}(\mathbb{R})$ of all $n \times n$ matrices, because the sum of two symmetric matrices is symmetric and a scalar multiplication of a symmetric matrix is also symmetric.

Similarly, the set of all $n \times n$ skew-symmetric matrices is also a subspace of $M_{n\times n}(\mathbb{R})$.

Definition 1.2.19. Let U and W be two subspaces of a vector space V.
(1) The *sum* of U and W is defined by
$$U + W = \{u + w \mid u \in U,\ w \in W\}.$$
(2) A vector space V is called the *direct sum* of two subspaces U and W, denoted by $V = U \oplus W$, if $V = U + W$ and $U \cap W = \{0\}$.

It is easy to see that $U+W$ and $U\cap W$ are also subspaces of V. If $V=\mathbb{R}^2$, $U=\{(x,0)\mid x\in\mathbb{R}\}$ (x-axis), and $W=\{(0,y)\mid y\in\mathbb{R}\}$ (y-axis), then $\mathbb{R}^2=U\oplus V=\mathbb{R}\oplus\mathbb{R}$ considering x-axis and y-axis as $\mathbb{R}$. Similarly, $\mathbb{R}^3=\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$.

Exercise 1.2.20. Let U and W be subspaces of V.

(1) Suppose that Z is a subspace of V contained in both U and W. Then show that Z is contained in $U\cap W$.

(2) Suppose that Z is a subspace of V containing both U and W. Then show that Z contains $U+W$.

Theorem 1.2.21. *A vector space V is the direct sum of subspaces U and W, i.e., $V=U\oplus W$, if and only if for any $v\in V$ there exist unique $u\in U$ and $w\in W$ such that $v=u+w$.*

Proof. ($\Rightarrow$) Suppose that $V=U+W$. Then for any $v\in V$, there exist vectors $u\in U$ and $w\in U$ such that $v=u+w$, since $V=U+W$. Further, suppose that $v=u'+w'$ for some $u'\in U$ and $w'\in W$. Then $u+w=u'+w'$ implies

$$u-u'=w'-w\in U\cap W=\{0\}.$$

Hence, $u=u'$ and $w=w'$.

($\Leftarrow$) Conversely, suppose that given any $v\in V$ there exist unique $u\in U$ and $w\in W$ such that $v=u+w$. Clearly, $V=U+W$. It remains to show that $U\cap W=\{0\}$. Clearly $0\in U\cap W$ as U and W are subspaces. Suppose if possible $v\in U\cap W$, $v\neq 0$. Then v can be written as a sum of vectors in U and W as

$$v=v+0=0+v.$$

This is a contradiction to the hypothesis. This completes the proof. ☐

Example 1.2.22 (Sum, but not direct sum)**.** In $\mathbb{R}^3$, consider the three vectors $e_1=(1,0,0)$, $e_2=(0,1,0)$, and $e_3=(0,0,1)$. Let $U=\{ae_1+ce_3\mid a,c\in\mathbb{R}\}$ be the xz-plane and $W=\{be_2+ce_3\mid b,c\in\mathbb{R}\}$ be the yz-plane. Then both U and W are subspaces of $\mathbb{R}^3$. Then a vector in $U+W$ is of the form

$$\begin{aligned}(ae_1+c_1e_3)+(be_2+c_2e_3)&=ae_1+be_2+(c_1+c_2)e_3\\&=ae_1+be_2+ce_3=(a,b,c),\end{aligned}$$

where $c=c_1+c_2$ and a,b,c are arbitrary numbers. Thus, $U+W=\mathbb{R}^3$. However, $\mathbb{R}^3\neq U\oplus W$ since clearly the vector $e_3\in U\cap W$ and so $U\cap W\neq\{0\}$. In fact, $e_3\in\mathbb{R}^3$ can be written and many linear combination of vectors in U and W:

$$e_3=\frac{1}{2}e_3+\frac{1}{2}e_3=\frac{1}{3}e_3+\frac{2}{3}e_3\in U+W.$$

Note that if we take $W=\{ye_2\mid y\in\mathbb{R}\}$ (y-axis), then it is easy to see that $\mathbb{R}^3=U\oplus W$.

Exercise 1.2.23. Let U and W be subspaces of $M_{n\times n}(\mathbb{R})$ consisting of all symmetric matrices and all skew-symmetric matrices, respectively. Show that $M_{n\times n}(\mathbb{R})=U\oplus W$. Therefore, such a decomposition of a square matrix A is unique (see Problem 1.11 (3) in our textbook).

1.3 Bases

Any vector $(x_1, x_2, x_3) \in \mathbb{R}^3$ can be written as a sum of scalar multiples of the three vectors $e_1 = (1, 0, 0)$, $e_=(0, 1, 0)$ and $e_3 = (0, 0, 1)$ as

$$(x_1, x_2, x_3) = x_1(1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0, 1) = x_1e_1 + x_2e_2 + x_3e_3.$$

Such an expression is called a linear combination of vectors e_1, e_2, e_3 and we have the following definition.

Definition 1.3.1. Let V be a vector space over a field F and let $\{x_1, x_2, \ldots, x_m\}$ be a set of vectors in V. Then any vector y of the form

$$y = \alpha_1x_1 + \alpha_2x_2 + \cdots \alpha_nx_n,$$

where $\alpha_i \in F, i = 1, 2, \ldots, n$ is called a *linear combination* (over F) of vectors $x_1, x_2, \ldots, x_n$.

The following result shows that the set of all linear combination of a finite set of vectors in a vector space forms a subspace.

Theorem 1.3.2. *Let $x_1, x_2, \ldots, x_n$ be vectors in a vector space V over F. Then the set*

$$W = \{\alpha_1x_1 + \alpha_2x_2 + \cdots \alpha_nx_n \mid \alpha_i \in F\}$$

of all linear combinations of $x_1, x_2, \ldots, x_n$ is a subspace of V. It is called the subspace of V spanned by $x_1, x_2, \ldots, x_n$. *Or,* $x_1, x_2, \ldots, x_n$ span the subspace W. *The subspace W is also denoted by $L(\{x_1, x_2, \ldots, x_n\})$.*

Proof. We have to show that W is closed under addition and scalar multiplication. Let $u, w \in W$. Then

$$\begin{aligned} u &= \alpha_1x_1 + \alpha_2x_2 + \cdots \alpha_nx_n, \\ w &= \beta_1x_1 + \beta_2x_2 + \cdots \beta_nx_n \end{aligned}$$

for some scalars $\alpha_i, \beta_i \in F$. Therefore,

$$u + w = (\alpha_1 + \beta_1)x_1 + (\alpha_2 + \beta_2)x_2 + \cdots (\alpha_n + \beta_n)x_n$$

and for any scalar $\lambda \in F$,

$$\lambda u = (\lambda\alpha_1)x_1 + (\lambda\alpha_2)x_2 + \cdots (\lambda\alpha_n)x_n.$$

Thus, $u + w$ and λu are linear combinations of $x_1, x_2, \ldots, x_n$ and so they are contained in W. Hence, W is a subspace of V. □

We shall denote this space W, the subspace spanned by the set $S = \{x_1, x_2, \ldots, x_n\}$, as $L(S)$.

Definition 1.3.3. Let V be a vector space over F and S be a non empty subset of V. Then the set of all linear combinations of finite sets of elements of S, denoted by $L(S)$, is called the *linear span* or the span of S, i.e.,

$$L(S) = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n : v_1, v_2, \ldots, v_n \in S,\ \alpha_1, \alpha_2, \ldots, \alpha_n \in F,\ n \in \mathbb{N}\}.$$

Example 1.3.4 (A space can be spanned in many ways)**.** (1) For a nonzero vector v in a vector space V over a field F, a linear combination of v is nothing but a scalar multiple of v. Thus the subspace W of V spanned by v is $W = \{\alpha v \mid \alpha \in F\}$.
Note that this subspace W can be spanned by any αv, $\alpha \in F$, $\alpha \neq 0$.

(2) Consider three vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $v = e_1 + e_2 = (1, , 1, 0)$ in $\mathbb{R}^3$. The subspace W_1 spanned by e_1 and e_2 is given by

$$\begin{aligned} W_1 &= \{\alpha_1 e_1 + \alpha_2 e_2 \mid \alpha_1, \alpha_2 \in \mathbb{R}\} \\ &= \{(\alpha_1, \alpha_2, 0) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}, \end{aligned}$$

where as the subspace W_2 spanned by e_1, e_2 and v is given by

$$\begin{aligned} W_2 &= \{\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 v \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\} \\ &= \{(\alpha_1 + \alpha_3, \alpha_2 + \alpha_3, 0) \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}. \end{aligned}$$

Clearly $W_1 \subseteq W_2$. On the other hand, since $v = e_1 + e_2 \in W_1$, we have $W_2 \subseteq W_1$. Hence, $W_1 = W_2$ which is the xy-plane in $\mathbb{R}^3$.
In general, a subspace in a vector space can have many different spanning sets.

Example 1.3.5 ($\mathbb{R}^m$ can be identified as a subspace of $\mathbb{R}^n$, $m \leq n$)**.** Let

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots, 0), \\ e_2 &= (0, 1, 0, \ldots, 0), \\ &\ \vdots \\ e_n &= (0, 0, 0, \ldots, 1) \end{aligned}$$

be n vectors in the n-space $\mathbb{R}^n$ ($n \geq 3$). Then a linear combination of e_1, e_2, e_3 is of the form

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = (\alpha_1, \alpha_2, \alpha_3, 0, \ldots, 0).$$

Hence, the set

$$W = \{(\alpha_1, \alpha_2, \alpha_3, 0, \ldots, 0) \in \mathbb{R}^n : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}$$

is the subspace of the n-space $\mathbb{R}^n$ spanned by the vectors e_1, e_2, e_3. Note that the subspace W can be identified with the 3-space $\mathbb{R}^3$ through the identification

$$(\alpha_1, \alpha_2, \alpha_3, 0, \ldots, 0) \equiv (\alpha_1, \alpha_2, \alpha_3)$$

with $\alpha_i \in \mathbb{R}$.

In general, for $m \leq n$, the m-space $\mathbb{R}^m$ can be identified as a subspace of the n-space $\mathbb{R}^n$.

Example 1.3.6 (Column space)**.** Let $A = [c_1\ c_2\ \cdots\ c_n]$ be a $m \times n$ matrix with columns c_i's. Then the column vectors c_i are in $\mathbb{R}^m$, and the matrix product Ax represents the linear combination of the column vector c_i whose coefficients are the components of $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, i.e.,

$$Ax = x_1 c_1 + x_2 c_2 + \cdots x_n c_n.$$

Therefore, the set

$$W = \{Ax \in \mathbb{R}^m \mid x \in \mathbb{R}^n\}$$

of all linear combinations of the column vectors of A is a subspace of $\mathbb{R}^m$ called the *column space* of A.

Thus, the system $Ax = b$ has a solution $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ if and only if the vector b belongs to the column space of A.

Now, we state the following lemma which gives the properties of linear span. The proof is straightforward and easy and left as an exercise.

Lemma 1.3.7. *Let S and T be two non-empty subsets of V. Then*

(1) $S \subseteq T$ implies $L(S) \subseteq L(T)$.
(2) $L(S \cup T) = L(S) + L(T)$.
(3) $L(L(S)) = L(S)$.
(4) S is a subspace of V if and only if $L(S) = S$.

Exercise 1.3.8. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a subset of the vector space V over F. Show that $L(S)$ is the smallest subspace of V containing the set S. That is, if U is a subspace of V containing $x_1, x_2, \ldots, x_n$, then $L(S) \subseteq U$.

Definition 1.3.9. Let V be a vector space over F and $v_1, \ldots, v_n \in V$. We say that they are *linearly dependent* over F if there exists $\alpha_1, \ldots, \alpha_n \in F$, not all of them 0, such that $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$.

$v_1, \ldots, v_n$ are said to be *linearly independent* if they are not linearly dependent, i.e., if $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ then $\alpha_i = 0, \forall i$. Note that if $v_1, \ldots, v_n$ are linearly independent then none of them can be zero, for if $v_1 = 0$ (say), then for any $\alpha \neq 0$ in F we have $\alpha v_1 + 0 v_2 + \cdots + 0 v_n = 0$.

Definition 1.3.10. A non-empty subset M of a vector space V is called linearly independent over F if every finite subset of M is linearly independent over F.

Note that a set of vectors $\{v_1, v_2, \ldots, v_n\}$ is linearly independent if the equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

has only the trivial solution $(\alpha_1, \alpha_2, \ldots, \alpha_n) = (0, 0, \ldots, 0)$. The set $\{v_1, v_2, \ldots, v_n\}$ is linearly dependent if the above equation has a non-trivial solution $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. If $\alpha_m \neq 0$, then the equation can be written as

$$v_m = -\frac{\alpha_1}{\alpha_m} v_1 - \frac{\alpha_2}{\alpha_m} v_2 - \cdots - \frac{\alpha_{m-1}}{\alpha_m} v_{m-1} - \frac{\alpha_{m+1}}{\alpha_m} v_{m+1} - \cdots - \frac{\alpha_n}{\alpha_m} v_m.$$

That is, a set of vectors is linearly dependent if and only if at least one of the vectors in the set can be written as a linear combination of the others. This means that at least one of these vectors can be written as a linear combination of the set in two different ways.

Exercise 1.3.11. Let V be a vector space over F, $v_1, v_2, \ldots, v_n \in V$ and $v_1 \neq 0$. Then either they are linearly independent or there exists a $k \leq n$ such that v_k is a linear combination of the preceding ones, $v_1, v_2, \ldots, v_{k-1}$.

Question: If $v_1, \ldots, v_n \in V$ are linearly independent then is it true that every element in their span has a unique representation (over F)?

The answer to this question is yes and it is proved in Theorem 1.3.23 later in the section. But first we see some examples.

Example 1.3.12. Let $x = (1, 2, 3)$ and $y = (3, 2, 1)$ be two vectors in $\mathbb{R}^3$. Then clearly $y \neq \lambda x$ for any $\lambda \in \mathbb{R}$ (or $\alpha x + \beta y = 0$ is possible only when $\alpha = \beta = 0$). Thus, $\{x, y\}$ is a linearly independent set in $\mathbb{R}^3$.

If $w = (3, 6, 9)$, then $\{x, w\}$ is linearly dependent as $w = 3x$ or $w - 3x = 0$.

By the abuse of language, for convenience, we shall say sometimes that "the vectors $x_1, x_2, \ldots, x_n$ are linearly independent" instead of say that "the set $\{x_1, x_2, \ldots, x_n\}$ is linearly independent".

Example 1.3.13. $p_1(x) = x + 1,\ p_2(x) = 3x^2 + x + 3$ and $p_3(x) = 3x^2 + 3x + 5$ are three linearly dependent elements of $\mathbb{R}_3[x]$ over $\mathbb{R}$ (Verify!).

Remark 1.3.14. Observe that the notion of linear dependence is considered over the field F. Hence it depends not only on the given vectors but also on the field. For example, the field of complex numbers $\mathbb{C}$ can be considered as a vector space over $\mathbb{R}$ and also over $\mathbb{C}$. The elements $v_1 = 1$ and $v_2 = i$ are linearly independent over $\mathbb{R}$ as there are no non-zero real numbers α_1 and α_2 such that $(\alpha_1 \times 1) + (\alpha_2 \times i) = 0$. However, when considered over $\mathbb{C}$, they are linearly dependent since $iv_1 + (-1)v_2 = 0$.

Example 1.3.15. The columns of the matrix

$$\begin{bmatrix} 1 & -2 & -1 & 0 \\ 4 & 2 & 6 & 8 \\ 2 & -1 & 1 & 3 \end{bmatrix}$$

are linearly dependent vectors in $\mathbb{R}^3$ since the third column is the sum of the first and the second column.

As in the above example, the concept of linear dependence can be applied to row or column vectors of a matrix.

Example 1.3.16. Consider the upper triangular matrix

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 4 \end{bmatrix}.$$

The linear dependence of the column vectors of A may be written as

$$\alpha_1 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which can be written in matrix notation as

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the third row, we get $c_3 = 0$. Then from the second row, we get $c_2 = 0$. Finally, using the fact that $c_2 = c_3 = 0$, from the first row, we get $c_1 = 0$. Thus, the homogeneous system has only the trivial solution and so the column vectors are linearly independent.

By the same argument one can prove the following result.

Theorem 1.3.17. *The nonzero rows of a matrix in row-echelon form are linearly independent, and so are the columns that contain leading* 1*'s.*

In particular, rows and columns of a triangular matrix with nonzero diagonal are linearly independent.

Lemma 1.3.18. *(1) If $n > m$, then any set of n vectors in the m-space $\mathbb{R}^m$ is linearly dependent.*
(2) If U is the reduced row-echelon form of A, then the columns of U are linearly independent if and only if the columns of A are linearly independent.

Proof. If $V = \mathbb{R}^m$ and $v_1, v_2, \ldots, v_n$ are n vectors in $\mathbb{R}^m$, then the form an $m \times n$ matrix $A = [v_1\ v_2\ \cdots\ v_n]$. On the other hand, as shown in Example 1.3.6, the linear dependence $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ is nothing but the homogeneous equation $Ax = 0$, where $x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Thus, we can say

(1) the columns v_i's of A are linearly independent in $\mathbb{R}^m$ if and only if the homogeneous system $Ax = 0$ has only the trivial solution, and
(2) they are linearly independent if and only if $Ax = 0$ has a non-trivial solution.

If U is the reduced row-echelon form of A, then we know that $Ax = 0$ and $Ux = 0$ have the same set of solutions. Moreover, we know that a homogeneous system $Ax = 0$ with more unknowns than the number of equations always has a non-trivial solution. This completes the proof. $\square$

Example 1.3.19. The matrix $A = [e_1\ e_2\ e_3]$ is the identity matrix, where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. Therefore, $Ax = 0$ has only the trivial solution. Hence, the set $\{e_1, e_2, e_3\}$ is linearly independent. Also, note that it spans $\mathbb{R}^3$.

One can check the linear independence of the vectors using the definition also.

Example 1.3.20 (The standard basis for $\mathbb{R}^n$)**.** The vectors $e_1, e_2, \ldots, e_n$ in $\mathbb{R}^n$ are clearly linear independent (Theorem 1.3.17). Moreover, they span $\mathbb{R}^n$ as any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ can be written as a linear combination of e_i's as

$$x = (x_1, x_2, \ldots, x_n) = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n.$$

Definition 1.3.21 (Basis)**.** Let V be a vector space over F. A subset S of V is called a *basis* of V if S is linearly independent (that is every finite subset of S is linearly independent) and spans V, i.e., $L(S) = V$.

For example, the set $\{e_1, e_2, \ldots, e_n\}$ forms a basis for $\mathbb{R}^n$, called the *standard basis for* $\mathbb{R}^n$.

Example 1.3.22 (A basis or not)**.** (1) The set of vectors $(1, 1, 0)$, $(0, -1, 1)$, and $(1, 0, 1)$ is not a basis for $\mathbb{R}^3$ since this set is linearly dependent. Note that, the third is the sum of the first two vectors. Also, it cannot span $\mathbb{R}^3$ because the vector $(1, 0, 0)$ cannot be obtained as a linear combination of the above three vectors (Prove!). Thus, this set does not have enough vectors spanning $\mathbb{R}^3$.

(2) The set of vectors $(1, 0, 0)$, $(0, 1, 1)$, $(1, 0, 1)$ and $(0, 1, 0)$ is not a basis either, since they are not linearly independent as the fourth vector is sum of the first two minus the third, even though they span $\mathbb{R}^3$. This set of vectors has some redundant vectors spanning $\mathbb{R}^3$.

(3) The set of vectors $(1, 1, 1)$, $(0, 1, 1)$, and $(0, 0, 1)$ is linearly independent and also spans $\mathbb{R}^3$ (Verify!). Hence, it is a basis for $\mathbb{R}^3$, which is different from the standard basis $\{e_1, e_2, e_3\}$. Note that the set cannot be reduced to a smaller set spanning $\mathbb{R}^3$. That is, no proper subset of the set of above three vector spans $\mathbb{R}^3$.

Let $x = (-1, 1, 0)$, $y = (1, -1, 1)$ and $z = (2, -2, 3)$ denote three elements of $\mathbb{R}^3$. Let $u = (-4, 4, 0) \in \mathbb{R}^3$. Then clearly $u \in L(\{x, y, z\})$ since u can be written in as $u = 5x + 3y - z$. Note that u can also be written as $u = 6x + 6y - 2z$. Thus in this case, the expression of an element in the span of x, y, z is not unique. Now consider another set $\{e_1, e_2, e_3\}$ where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ in $\mathbb{R}^3$. Clearly u also belongs to its span since u can be written as $u = -e_1 + 4e_2 + 0e_3$. As in the previous case, is it possible to find another representation of u in $L(\{e_1, e_2, e_3\})$? **No?** It seems that such expression is unique in this case. What is the difference¿ One obvious and immediate difference is that the vectors x,y, and z are linearly dependent (as $z = x + 3y$) while we know that e_1, e_2, e_3 are linearly independent. This brings us back to our question asked earlier:

Question: If $v_1, \ldots, v_n \in V$ are linearly independent then is it true that every element in their span has a unique representation (over F)?

We know that every point in the plane $\mathbb{R}^2$ is uniquely represented using its two coordinates (i.e., the x-coordinate and the y-coordinate). Likewise, every vector in a vector space can be uniquely represented as a linear combination of its basis elements. The following theorem shows that a basis for a vector space represents a coordinate system just like the coordinate system in $\mathbb{R}^2$.

Theorem 1.3.23. *If $v_1, \ldots, v_n \in V$ are linearly independent, then every element in their linear span can be uniquely expressed in the form $\lambda_1 v_1 + \cdots + \lambda_n v_n$ with $\lambda_i \in F$.*

Proof. By definition of linear span, every element in $L(\{v_1, \ldots, v_n\})$ is of the form $\lambda_1 v_1 + \cdots + \lambda_n v_n$. Suppose if possible $v \in L(\{v_1, \ldots, v_n\})$ can be expressed in two different ways, say, $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$ and $v = \mu_1 v_1 + \cdots + \mu_n v_n$. Then

$$\lambda_1 v_1 + \cdots + \lambda_n v_n = v = \mu_1 v_1 + \cdots + \mu_n v_n.$$

This implies,

$$(\lambda_1 - \mu_1)v_1 + (\lambda_2 - \mu_2)v_2 + \cdots + (\lambda_n - \mu_n)v_n = 0.$$

Since, $v_1, v_2, \ldots, v_n$ are given to be linearly independent, we conclude that $\lambda_i - \mu_i = 0$ for all i and hence every element $v \in L(\{v_1, \ldots, v_n\})$ has a unique representation. $\square$

Question 1.3.24. Is the converse true? That is if every element $v \in L(\{v_1, \ldots, v_n\})$ has a unique representation given in above lemma, then can we say that $v_1, v_2, \ldots, v_n$ are linearly independent?

Example 1.3.25 (Two different bases for $\mathbb{R}^3$)**.** Let $S = \{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$ and let $S' = \{v_1, v_2, v_3\}$, where $v_1 = e_1 + e_2 + e_3 = (1, 1, 1)$, $v_2 = e_2 + e_3 = (0, 1, 1)$, and $v_3 = e_3 = (0, 0, 1)$. Then S' is also a basis for $\mathbb{R}^3$ (see Example 1.3.22). It is easy to see that for any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, one can write

$$\begin{aligned} x &= x_1 e_1 + x_2 e_2 + x_3 e_3 \\ &= x_1 v_1 + (x_2 - x_1) v_2 + (x_3 - x_2) v_3. \end{aligned}$$

1.4 Dimensions

The following lemma is one of the most important results in linear algebra which explains the relationship between spanning sets and linearly independent sets.

Lemma 1.4.1. *Let V be a vector space over F and $S = \{x_1, x_2, \ldots, x_m\}$ be a set of m vectors in V.*

(1) If S spans V, then every set of vectors with more than m vectors cannot be linearly independent.

(2) If S is linearly independent, then any set of vectors with less than m vectors cannot span V.

Proof. We are given that $L(S) = V$, i.e., S spans V. Let $S' = \{y_1, y_2, \ldots, y_n\}$ be a set of n vectors in V with $n > m$. We have to show that S' is linearly dependent. Since each vector y_j is a linear combination of the vectors in the spanning set S, we have

$$y_j = \alpha_{1j} x_1 + \alpha_{2j} x_2 + \cdots + \alpha_{mj} x_m = \sum_{i=1}^{m} \alpha_{ij} x_i.$$

Now,

$$\begin{aligned} \beta_1 y_1 + \beta_2 y_2 + \cdots + \beta_n y_n = &\ \beta_1(\alpha_{11} x_1 + \alpha_{21} x_2 + \cdots + \alpha_{m1} x_m) \\ &+ \beta_2(\alpha_{12} x_1 + \alpha_{22} x_2 + \cdots + \alpha_{m2} x_m) \\ &+ \cdots \\ &+ \beta_n(\alpha_{1n} x_1 + \alpha_{2n} x_2 + \cdots + \alpha_{mn} x_m). \end{aligned}$$

Grouping together the coefficients of the same vector gives

$$\begin{aligned} \beta_1 y_1 + \beta_2 y_2 + \cdots + \beta_n y_n = &\ (\alpha_{11}\beta_1 + \alpha_{12}\beta_2 + \cdots + \alpha_{1n}\beta_n) x_1 \\ &+ (\alpha_{21}\beta_1 + \alpha_{22}\beta_2 + \cdots + \alpha_{2n}\beta_n) x_2 \\ &+ \cdots \\ &+ (\alpha_{m1}\beta_1 + \alpha_{m2}\beta_2 + \cdots + \alpha_{mn}\beta_n) x_m. \end{aligned}$$

Thus, since $S = \{x_1, x_2, \ldots, x_m\}$ is linearly independent, the condition

$$\beta_1 y_1 + \beta_2 y_2 + \cdots + \beta_n y_n = 0$$

is equivalent to the following homogeneous system of m linear equations in n unknowns $\beta_1, \beta_2, \ldots, \beta_n$, where $n > m$.

$$\begin{cases} \alpha_{11}\beta_1 + \alpha_{12}\beta_2 + \cdots + \alpha_{1n}\beta_n = 0, \\ \alpha_{21}\beta_1 + \alpha_{22}\beta_2 + \cdots + \alpha_{2n}\beta_n = 0, \\ \qquad \vdots \\ \alpha_{m1}\beta_1 + \alpha_{m2}\beta_2 + \cdots + \alpha_{mn}\beta_n = 0 \end{cases}$$

We know that S' is linearly dependent means $\beta_1 y_1 + \beta_2 y_2 + \cdots + \beta_n y_n = 0$ for some $(\beta_1, \beta_2, \ldots, \beta_n) \neq (0, 0, \ldots, 0)$. Therefore, S' is linearly dependent if and only if the system of linear equations

$$\beta_1 y_1 + \beta_2 y_2 + \cdots + \beta_n y_n = 0$$

has a non-trivial solution $(\beta_1, \beta_2, \ldots, \beta_n)$.

Since $m < n$, by Lemma 1.3.18, the following homogeneous system of linear equations in β_i's

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

must have a nontrivial solution. Hence, S' is linearly dependent.

Now, we prove (2). Let S be linearly independent. Then we have to show that nay set of vectors with less than m vectors cannot span V. Suppose, if possible, S' is a set of vectors in V with $|S'| = n < m = |S|$ such that $L(S') = V$. Then by (1), any set having more than n vectors in V cannot be linearly independent. Thus, S cannot be linearly independent which is a contradiction to the hypothesis. This completes the proof of (2). $\square$

From the above result it is clear that if $S = \{x_1, x_2, \ldots, x_n\}$ of n vectors is a basis for a vector space V over F, then no other set $\{y_1, y_2, \ldots, y_r\}$ of r vectors can be a basis for V if $r \neq n$. Hence, all bases for a vector space V over F must have the same number of vectors, even if there are many different bases for the space. Consequently, we have the following important result.

Theorem 1.4.2. *If a basis for a vector space V over F consists of n vectors, so does every other basis.*

Definition 1.4.3. The *dimension* of a vector space V over a field F is the number, say n, of vectors in a basis for V over F, written by $\dim V = n$. The dimension of V over F is usually denoted by $\dim V$ or sometimes by $\dim_F V$ (to stress that dimension of V is over the field F).

When V has a basis of finite number of vectors, V is said to be *finite dimensional*.

Thus, a vector space V over F is said to be *finite dimensional* (over F) if there is a finite set S such that $L(S) = V$. However, note that $\dim V$ is not necessarily $|S|$. We can say that $\dim V = |S|$ if S is also linearly independent, i.e., S forms a basis for V over F.

Example 1.4.4 (Computing the dimension)**.** The following are clear.

(1) If $V = \{0\}$, then $\dim V = 0$.

(2) If $V = \mathbb{R}^n$ is a vector space over $\mathbb{R}$, then the standard basis $\{e_1, e_2, \dots, e_n\}$ for V implies that $\dim \mathbb{R}^n = n$ over $\mathbb{R}$.

(3) If V is the space of all polynomials of degree less than or equal to n over $\mathbb{R}$, denoted by $P_n(\mathbb{R})$ or $\mathbb{R}_n[x]$, then $\dim \mathbb{R}_n[x] = n + 1$. Note that $\{1, x, x^2, \dots, x^n\}$ forms a basis for $\mathbb{R}_n[x]$ over $\mathbb{R}$.

(4) If $V = M_{m\times n}(\mathbb{R})$ of all $m \times n$ matrices, then $\dim M_{m\times n}(\mathbb{R}) = mn$ since $\{E_{ij} \mid i = 1, \dots, m,\ j = 1, \dots, n\}$ is a basis for V, where E_{ij} is the $m \times n$ matrix whose (i, j)-th entry is 1 and all others are zero.

If $V = C(\mathbb{R})$, the space of all continuous real-valued functions defined on $\mathbb{R}$, then one can show that V is not finite dimensional. A vector space V is *infinite dimensional* if it is not finite dimensional. In this course, we are concerned only with finite-dimensional vector spaces unless otherwise specified.

Theorem 1.4.5. *Let V be a finite dimensional vector space over F.*

(1) Any linearly independent set in V can be extended to a basis by adding more vectors if necessary.

(2) Any set of vectors that spans V can be reduced to a basis by discarding vectors if necessary.

Proof. (1) Let $S = \{x_1, x_2, \dots, x_n\}$ be a linearly independent set in V. If S spans V, then S is a basis for V. If S does not span V, then there exists a vector, say x_{n+1}, in V such that $x_{n+1} \notin L(S)$, i.e., that is it cannot be written as a linear combination of elements of S.

Claim. $\{x_1, x_2, \dots, x_n, x_{n+1}\}$ is linearly independent.
Let $\alpha_1 x_1 + \cdots + \alpha_n x_n + \alpha_{n+1} x_{n+1} = 0$.

Case 1. $\alpha_{n+1} = 0$.
Then we have $\alpha_1 x_1 + \cdots + \alpha_n x_n + 0x_{n+1} = 0$. Since $x_1, x_2, \dots, x_n$ are linearly independent, $\alpha_i = 0$ for all $i = 1, 2, \dots, n$.
Case 2. $\alpha_{n+1} \neq 0$.
Then $x_{n+1} = -\frac{\alpha_1}{\alpha_{n+1}} x_1 - \frac{\alpha_2}{\alpha_{n+1}} x_2 - \cdots - \frac{\alpha_n}{\alpha_{n+1}} x_n$. Then $x_{n+1} \in L(S)$ which is a contradiction.
Hence, the claim. That is, $S \cup \{x_{n+1}\}$ is linearly independent. Now, if it spans V, then it is a basis for V. If it does not span V, then the same procedure can be repeated giving a linearly independent set that spans V, i.e., a basis for V. Once we get a basis (i.e., a spanning set), this process stops by Lemma 1.4.1 (1) as any further larger set cannot be linearly independent. Since V is finite dimensional, by definition of basis, this process must stop in a finite number of steps.
This completes the proof of the first part.

(2) Exercise.

□

Remark 1.4.6. Above theorem shows that a basis for a vector space V over F is a set of vectors in V which is *maximally independent* and *minimally spanning*. That is,

$$\text{Card(L.I. set)} \leq \text{Card(Basis)} \leq \text{Card(Span)}.$$

In particular, if W is a subspace of V, then any basis for W is a linearly independent set in W and hence also in V. Therefore, it can be extended to a basis for V (by the above theorem). Consequently, $\dim W \leq \dim V$.

Corollary 1.4.7. *Let V be a vector space over F of dimension n. Then*

(1) any set of n vectors that spans V is a basis for V, and

(2) any set of n linearly independent vectors in V is a basis for V.

Proof. If a spanning set of n vectors is not linearly independent, then by (2) of the above theorem, it can be reduced to a basis for V by discarding some vectors. Then the resultant set has less than n elements which forms a basis for V. This is a contradiction to the hypothesis that $\dim V = n$. This proves (1).

If a set of n linearly independent vectors does not span V, then by (1) of the above theorem, it can be extended to a basis by adding more vectors. Then the resultant set has more than n elements which forms a basis for V. Again this is a contradiction to $\dim V = n$. This proves (2). □

Above corollary says that if it is known that $\dim V = n$ and if a set has n vectors which is either linearly independent or spans V, then it is already a basis for V. That is, we need to check only one condition for the basis if the dimension of the space is known.

Example 1.4.8 (Constructing a basis)**.** Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$x_1 = (1, -2, 5, -3),\ x_2 = (0, 1, 1, 4),\ x_3 = (1, 0, 1, 0).$$

Find a basis for W and extend it to a basis for $\mathbb{R}^4$.

Solution. Since W is spanned by 3 vectors, $\dim W \leq 3$.

We start with the 3×4 matrix A whose rows are x_1, x_2, and x_3.

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

We reduce A into a row-echelon form as follows. Eliminate the leading entry in the first column of row 3 using the operation $R_3 \leftarrow R_3 - R_1$:

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 2 & -4 & 3 \end{bmatrix}$$

Eliminate the entry in the second column of row 3 using the operation $R_3 \leftarrow R_3 - 2R_2$:

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -6 & -5 \end{bmatrix}$$

Normalize the third row by multiplying by $-\frac{1}{6}$ (i.e., $R_3 \leftarrow -\frac{1}{6}R_3$) to get the final row-echelon Form matrix U:

$$U = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & \frac{5}{6} \end{bmatrix}$$

The three non-zero rows of the matrix U are clearly linearly independent (Note that the LI can be checked by taking a linear combination also). Moreover, they also span W as the vectors x_1, x_2, and x_3 can be expressed as a linear combination of these three non-zero row vectors of U. Therefore, the three non-zero rows of U forms a basis for W and so does the set $\{x_1, x_2, x_3\}$ (by Corollary 1.4.7).

Now, we know that $\dim \mathbb{R}^4$ over $\mathbb{R}$ is 4 as $\{e_1, e_2, e_3, e_4\}$ is the standard basis for $\mathbb{R}^4$. So we need to add one more vector to $\{x_1, x_2, x_3\}$ or to the rows of U to extend the basis for W to a basis for V. Note that $x_4 = (0, 0, 0, 1) \notin L(\{x_1, x_2, x_3\})$. Thus, $\{x_1, x_2, x_3, x_4\}$ (or adding x_4 to the non-zero rows of U) gives a basis for $\mathbb{R}^4$ extended from the basis for W. $\square$

Exercise 1.4.9. Let V be a vector space over F and W be a subspace of V. Show that if $\dim W = \dim V$, then $W = V$.

Exercise 1.4.10. Find a basis and dimension of each of the following subspaces of $M_n(\mathbb{R})$ (or $M_{n\times n}(\mathbb{R})$) of all the $n \times n$ matrices over $\mathbb{R}$.

(1) The space of all $n \times n$ diagonal matrices whose traces are zero.
(2) The space of all $n \times n$ symmetric matrices.
(3) The space of all $n \times n$ skew-symmetric matrices.

Corollary 1.4.11. *Let V be a vector space over F. For any subspace U of V, there is a subspace W of V such that $V = U \oplus W$.*

Proof. Choose a basis $\{u_1, u_2, \ldots, u_k\}$ for U, and extend it to a basis $\{u_1, \ldots, u_k, u_{k+1}, \ldots, u_n\}$ for V. Let W be the subspace spanned by $\{u_{k+1}, \ldots, u_n\}$, i.e.,

$$W = L(\{u_{k+1}, \ldots, u_n\}).$$

Then we show that $V = U \oplus W$. That is, $V = U + W$ and $U \cap V = \{0\}$.

Let $v \in V$. Since every element in V can be written as a linear combination of basis elements, we have

$$v = \alpha_1 u_1 + \cdots + \alpha_k u_k + \alpha_{k+1} u_{k+1} + \cdots + \alpha_n u_n.$$

Since $\{u_1, \ldots, u_k\}$ is a basis for U and $\{u_{k+1}, \cdots, u_n\}$ is a basis for W, the elements $u = \alpha_1 u_1 + \cdots + \alpha_k u_k$ and $\alpha_{k+1} u_{k+1} + \cdots + \alpha_n u_n$ belong to U and W respectively. Therefore, $v = u + v$ and so $V = U + W$. Now, let $x \in U \cap W$. Then

$$x = \beta_1 u_1 + \cdots + \beta_k u_k \in U$$

and

$$x = \beta_{k+1}u_{k+1} + \cdots + \beta_n u_n \in W$$

for some $\beta_i \in F$, $i = 1, 2, \ldots, n$. Therefore,

$$\beta_1 u_1 + \cdots + \beta_k u_k - \beta_{k+1}u_{k+1} - \cdots - \beta_n u_n = 0.$$

Since $\{u_1, \ldots, u_k, u_{k+1}, \ldots, u_n\}$ is a basis for V, $\beta_i = 0$ for all i. Therefore, $x = 0$. Hence, $U \cap W = \{0\}$. This completes the proof. □

Exercise 1.4.12. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for a vector space V and let $W_i = \{rv_i : r \in \mathbb{R}\}$ be the subspace of V spanned by v_i. Show that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$.

1.5 Row and column spaces

Let A be $m \times n$ matrix with rows and columns abbreviated as follows:

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} \\ &= \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}, \end{aligned}$$

where r_i is the i-th row vector of A in $\mathbb{R}^n$ and c_j is the j-th column vector of A in $\mathbb{R}^m$

Definition 1.5.1. Let A be a $m \times n$ matrix with row vectors $\{r_1, r_2, \ldots, r_m\}$ and column vectors $\{c_1, c_2, \ldots, c_n\}$.

(1) The *row space* of A is the subspace in $\mathbb{R}^n$ spanned by the row vectors $\{r_1, r_2, \ldots, r_m\}$, denoted by $\mathcal{R}(A)$.

(2) The *column space* of A is the subspace in $\mathbb{R}^m$ spanned by the column vectors $\{c_1, c_2, \ldots, c_n\}$, denoted by $\mathcal{C}(A)$.

(3) The solution set of the homogeneous equation $Ax = 0$ is called the *null space* of A, denoted by $\mathcal{N}(A)$.

The null space $\mathcal{N}(A)$ is a subspace of the n-space $\mathbb{R}^n$. Its dimension is called the *nullity* of A. Since the row vectors of A are just the column vectors of its transpose A^T and the column vectors of A are the row vectors of A^T, the row (column) space of A is just the column (row) space of A^T, i.e.,

$$\mathcal{R}(A) = \mathcal{C}(A^T) \quad \text{and} \quad \mathcal{C}(A) = \mathcal{R}(A^T).$$

Note that $\mathcal{C}(A) = x_1c_1 + x_2c_2 + \cdots + x_nc_n$, for $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Thus,

$$\mathcal{C}(A) = \{Ax \mid x \in \mathbb{R}^n\},$$

where $x = (x_1, x_2, \ldots, x_n)$. Thus, give a vector $b \in \mathbb{R}^m$, the equation $Ax = b$ has a solution if and only if $b \in \mathcal{C}(A) \subseteq \mathbb{R}^m$. In other words, the column space $\mathcal{C}(A)$ is the set of vectors $b \in \mathbb{R}^m$ for which $Ax = b$ has a solution.

Example 1.5.2 (Find a basis for the null space)**.** Let U be in the reduced row-echelon form given by

$$U = \begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly the first three rows containing the leading 1's are linearly independent and they for a basis for $\mathcal{R}(U)$. So, $\dim \mathcal{R}(U) = 3$ (since the last two is the zero vector). Likewise, the first three columns containing leading 1's are also linearly independent (by Theorem 1.3.17). However, the last two columns of U can be written as a linear combination of these three columns. So, we have $\dim \mathcal{C}(U) = 3$ as the first three columns form a basis for $\mathcal{C}U)$.

Now, to obtain a basis for the null space $\mathcal{N}(U)$, we first solve the system $Ux = 0$ to get

$$\begin{aligned} 1(x_1) + 0(x_2) + 0(x_3) + 2(x_4) + 2(x_5) &= 0 \\ 0(x_1) + 1(x_2) + 0(x_3) - 1(x_4) + 3(x_5) &= 0 \\ 0(x_1) + 0(x_2) + 1(x_3) + 4(x_4) - 1(x_5) &= 0. \end{aligned}$$

We have five variables and three equations. Taking the free variables $x_4 = s$ and $x_5 = t$, rewriting each equation to isolate the basic variables (x_1, x_2, x_3) in terms of the free parameters (s, t), we have $x_1 = -2s - 2t$ $x_2 = s - 3t$ $x_3 = -4s + t$.

Writing all the five variables back into a single column vector x, we get

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - 2t \\ s - 3t \\ -4s + t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ -4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = sn_s + tn_t.$$

Thus, the two vectors $n_s = (-2, 1, -4, 1, 0)$ and $n_t = (-2, -3, 1, 0, 1)$ span the null space $\mathcal{N}(U)$, where s and t are arbitrary values of the free variables x_4 and x_5 respectively. Looking at their last two coordinates, we can say that they are linearly independent also. Therefore, $\{n_s, n_t\}$ form a basis for the null space $\mathcal{N}(U)$ and hence $\dim \mathcal{N}(U) = 2$.

Theorem 1.5.3. *Let U be a (reduced) row-echelon form of a matrix A. Then*

$$\mathcal{R}(A) = \mathcal{R}(U) \text{ and } \mathcal{N}(A) = \mathcal{N}(U).$$

Moreover, if U has r nonzero row vectors containing leading 1's, then they form a basis for the row space $\mathcal{R}(A)$, so that the dimension of $\mathcal{R}(A)$ is r.

Proof. Given any matrix A, we first investigate the row space $\mathcal{R}(A)$ and the null space $\mathcal{N}(A)$ of A by comparing then with the row space of $\mathcal{R}(U)$ and the null space $\mathcal{N}(U)$ of its reduced row-echelon form U. Since the solution set of $Ax = 0$ and $Ux = 0$ is the same (because elementary row operations do not alter the solution set of a system of equations), we have $\mathcal{N}(A) = \mathcal{N}(U)$.

Let $\{r_1, r_2, \ldots, r_m\}$ be the row vectors of an $m \times n$ matrix. The following three types of elementary row operations change A into the matrices A_i, $i = 1, 2, 3$.

$$A_1 = \begin{bmatrix} r_1 \\ \vdots \\ kr_i \\ \vdots \\ r_m \end{bmatrix} \text{ for } k \neq 0; \quad A_2 = \begin{bmatrix} \vdots \\ r_j \\ \vdots \\ r_i \\ \vdots \end{bmatrix} \text{ for } i < j; \quad A_3 = \begin{bmatrix} r_1 \\ \vdots \\ r_i + kr_j \\ \vdots \\ r_m \end{bmatrix}.$$

Clearly, the row vectors of the three matrices A_1, A_2, and A_3 are linear combinations of the row vectors of A. On the other hand, by the inverse elementary row operations, these matrices can be again changed into A. Thus, the row vectors of A can also be written as linear combinations of those of A_i's. This means that if matrices A and B are row equivalent, then their row spaces must be equal, i.e., $\mathcal{R}(A) = \mathcal{R}(B)$.

Now, the nonzero row vectors in the reduced row-echelon form U are always linearly independent and span the row space of U (see Theorem 1.3.17). Thus, they form a basis for the row space $\mathcal{R}(A)$ of A. This completes the proof. □

Example 1.5.4 (Find a basis for the row space and the column space of A)**.** Let A be a matrix given by

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}.$$

Find bases for the row space $\mathcal{R}(A)$, the null space $\mathcal{N}(A)$, and the column space $\mathcal{C}(A)$ of A.

Solution. **Basis for the row space $\mathcal{R}(A)$ of A.**

We use the Gauss-Jordan elimination on A to get the reduced row-echelon form U of A. The pivot in Row 1, Column 1 is 1. We eliminate the non-zero entries below it using:

- $R_2 \leftarrow R_2 + 2R_1$
- $R_4 \leftarrow R_4 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & -3 & 3 & 4 & 1 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix}$$

Multiply Row 2 by -1 ($R_2 \leftarrow -1 \cdot R_2$):

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & -3 & 3 & 4 & 1 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix}.$$

Eliminating the non-zero entries above and below Row 2, Column 2:

- $R_1 \leftarrow R_1 - 2R_2$
- $R_3 \leftarrow R_3 + 3R_2$

$$\begin{bmatrix} 1 & 0 & 2 & 8 & 9 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix}.$$

Column 3 has no non-zero pivot candidates below Row 2, so we move to Column 4. Dividing Row 3 by -5 ($R_3 \leftarrow -\frac{1}{5}R_3$), we get

$$\begin{bmatrix} 1 & 0 & 2 & 8 & 9 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix}.$$

Eliminating the remaining entries in Column 4, we get

- $R_1 \leftarrow R_1 - 8R_3$
- $R_2 \leftarrow R_2 + 3R_3$
- $R_4 \leftarrow R_4 + 13R_3$

$$U = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since elementary row operations preserve the row space, $\mathcal{R}(A) = \mathcal{R}(U)$. The non-zero rows

$$\mathbf{v}_1 = (1, 0, 2, 0, 1), \quad \mathbf{v}_2 = (0, 1, -1, 0, 1), \quad \mathbf{v}_3 = (0, 0, 0, 1, 1)$$

of U form a basis for $\mathcal{R}(A)$ as they are linearly independent. So,

$$\dim \mathcal{R}(A) = 3.$$

Note. Note that, the elementary row operations did not involve interchanging of rows. Thus, the first three rows of U were obtained from the first three rows r_1, r_2, r_3 of A. Consequently, the first three rows of A also form a basis for the row space $\mathcal{R}(A)$ of A.

Basis for the null space $\mathcal{N}(A)$ of A

Since $\mathcal{N}(A) = \mathcal{N}(U)$, it is enough to solve the homogeneous system $Ux = 0$. Since the first, second and the fourth columns of U contain the leading 1's, the basis (or pivot) variables are x_1, x_2, and x_4 while the free variables are $x_3 = s, x_5 = t$. Thus, $Ux = 0$ gives

$$\begin{aligned} x_1 + 2x_3 + x_5 = 0 &\implies x_1 = -2s - t \\ x_2 - x_3 + x_5 = 0 &\implies x_2 = s - t \\ x_4 + x_5 = 0 &\implies x_4 = -t \end{aligned}$$

In vector form, we have

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, = sn_s + tn_t$$

where $n_s = (-2, 1, 1, 0, 0)$ and $n_t = (-1, -1, 0, -1, 1)$. The two vectors n_s and n_t are solutions when $(n_s, n_t) = (s, t)$ is $(1, 0)$ and $(n_s, n_t) = (s, t)$ is $(0, 1)$. Also, they are linearly independent (looking at their 3rd and 5th coordinate). Thus, the set $\{n_s, n_t\}$ forms a basis for $\mathcal{N}(A)$, and so $\dim \mathcal{N}(A) = 2$.

Basis for the column space $\mathcal{C}(A)$ of A

Let c_1, c_2, c_3, c_4, c_5 denote the column vectors of matrix A. Since these column vectors span $\mathcal{C}(A)$, we only need to discard the columns that can be expressed as linear combinations of the other column vectors.

The linear dependence equation is given by

$$x_1 c_1 + x_2 c_2 + x_3 c_3 + x_4 c_4 + x_5 c_5 = 0, \quad \text{i.e., } Ax = \mathbf{0}$$

This holds if and only if $x = (x_1, \ldots, x_5)^T \in \mathcal{N}(A)$. By taking the basis vectors of $\mathcal{N}(A)$:

$$n_s = (-2, 1, 1, 0, 0)^T \quad \text{and} \quad n_t = (-1, -1, 0, -1, 1)^T$$

we obtain two non-trivial linear dependence relations among the columns of A:

$$\begin{aligned} -2c_1 + c_2 + c_3 &= 0 \\ -c_1 - c_2 - c_4 + c_5 &= 0 \end{aligned}$$

Hence, the column vectors c_3 and c_5 (corresponding to the free variables in $Ax = 0$) can be expressed as:

$$\begin{aligned} c_3 &= 2c_1 - c_2 \\ c_5 &= c_1 + c_2 + c_4 \end{aligned}$$

Since c_3 and c_5 are linear combinations of c_1, c_2, c_4, the reduced set $\{c_1, c_2, c_4\}$ still spans $\mathcal{C}(A)$.

To show that $\{c_1, c_2, c_4\}$ is linearly independent, define the submatrices:

$$\tilde{A} = [c_1\ c_2\ c_4] = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -5 & -1 \\ 0 & -3 & 4 \\ 3 & 6 & -7 \end{bmatrix}, \quad \tilde{U} = [u_1\ u_2\ u_4] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\mathbf{u}_j$ is the j-th column vector of U.

- Clearly, $\tilde{U}$ is the reduced row-echelon form of $\tilde{A}$, so $\mathcal{N}(\tilde{A}) = \mathcal{N}(\tilde{U})$.
- The columns u_1, u_2, u_4 contain the leading 1's and are clearly linearly independent. Thus, $\tilde{U}x = 0$ has only the trivial solution.
- This implies $\tilde{A}x = 0$ also has only the trivial solution, proving that $\{c_1, c_2, c_4\}$ is linearly independent.

Therefore, $\{c_1, c_2, c_4\}$ is a basis for $\mathcal{C}(A)$ and so $\dim \mathcal{C}(A) = 3 =$ number of basic variables.

That is, the column vectors of A corresponding to the basic variables in $Ux = 0$ form a basis for the column space $\mathcal{C}(A)$. □

Summarizing, given a matrix A, we first find its reduced row-echelon form U of by Gauss-Jordan elimination. Then

- a basis for $\mathcal{R}(A) = \mathcal{R}(U)$ is the set of nonzero row vectors of U,
- a basis for $\mathcal{N}(A) = \mathcal{N}(U)$ can be found by solving $Ux = 0$, Note that, in general, $\mathcal{C}(U) \neq \mathcal{C}(A)$, since the column space of A is not preserved by Gauss-Jordan elimination (see Exercise 1.5.7) However, we have $\dim \mathcal{C}(A) = \dim \mathcal{C}(U)$, and a basis for $\mathcal{C}(A)$ can be formed by selecting the columns in A, not in U, which correspond to the basic variables (or the leading 1's in U).

Alternatively, a basis for the column space $\mathcal{C}(A)$ can also be found with the elementary column operation, equivalent to finding a basis for the row space $\mathcal{R}(A^T)$ for A^T.

Exercise 1.5.5. Find bases for $\mathcal{R}(A)$ and $\mathcal{N}(A)$ of the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}.$$

Also, find a basis for $\mathcal{C}(A)$ by finding a basis for $\mathcal{R}(A^T)$.

Exercise 1.5.6. Let A and B be two $n \times n$ matrices. Show that $AB = 0$ if and only if the column space of B is a subspace of the null space of A.

Exercise 1.5.7. Find an example of a matrix A and its row-echelon form U such that $\mathcal{C}(A) \neq \mathcal{C}(U)$. What is wrong in $\mathcal{C}(A) = \mathcal{R}(A^T) = \mathcal{R}(U^T) = \mathcal{C}(U)$?

1.6 Rank and nullity

Theorem 1.6.1 (The fundamental theorem)**.** *For any $m \times n$ matrix A, the row space and the column space of A have the same dimension; that is,* $\dim \mathcal{R}(A) = \dim \mathcal{C}(A)$.

Proof. Let $\dim \mathcal{R}(A) = r$ and let U be the reduced row-echelon form of A. Then the number of non-zero rows (or columns) of U containing leading 1's is r, which is equal to the number of basic variables in $Ux = 0$ or $Ax = 0$. We shall prove that $\dim \mathcal{C}(A) = r = \dim \mathcal{R}(A)$ by showing that the r columns of A corresponding to the leading 1's (or basic variables) form a basis for $\mathcal{C}(A)$.

(1) They are linearly independent.
Let $\tilde{A}$ denote the the submatrix of A whose columns are those of A corresponding to the r basic variables (or leading 1's) in U, and let $\tilde{U}$ denote the submatrix of U consisting of the r columns containing leading 1's. Then, clearly, $\tilde{U}$ is the reduced row-echelon form of $\tilde{A}$. Therefore, $\tilde{A}x = 0$ if and only if $\tilde{U}x = 0$. But $\tilde{U}x = 0$ has only trivial solution since the columns of U containing the leading 1's are linearly independent (by Theorem 1.3.17). Thus, $\tilde{A}x = 0$ also has only trivial solution. So, the columns of $\tilde{A}$ are linearly independent.

(2) They span $\mathcal{C}(A)$.
The goal is to show that every column of A can be written using only the pivot columns (i.e., the columns in $\tilde{A}$ which corresponds to the basic variables).

Let the columns of A corresponding to free variables be $\{\mathbf{c}_{i_1}, \mathbf{c}_{i_2}, \dots, \mathbf{c}_{i_k}\}$. Choose one specific free variable column, say c_{i_j}. The corresponding free variable is x_{i_j}. Consider the homogeneous system of equations:

$$Ax = x_1c_1 + x_2c_2 + \cdots + x_nc_n = 0$$

Now, assign specific values to the free variables as follows.

- Set $x_{i_j} = 1$ (for our chosen free column).
- Set all other free variables equal to 0.

Since all other free variables are set to 0, their corresponding terms vanish from $Ax = 0$. The remaining equation contains only the following

- The chosen free column c_{i_j} (with coefficient 1).
- The basic columns (in $\tilde{A}$) with their calculated non-zero coefficient values.

Thus, $Ax = 0$ simplifies to

$$\mathbf{c}_{i_j} + (\text{linear combination of columns of } \tilde{A}) = \mathbf{0}$$

Rearranging the terms, we get

$$\mathbf{c}_{i_j} = -(\text{linear combination of columns of } \tilde{A})$$

This proves that $\mathbf{c}_{i_j}$ is a linear combination of the columns of $\tilde{A}$. Since this argument holds for every free column $\mathbf{c}_{i_j}$ ($j = 1, 2, \dots, k$), all free columns are redundant for spanning $\mathcal{C}(A)$. Therefore, the columns of $\tilde{A}$ span $\mathcal{C}(A)$. This completes the proof.

□

Remark 1.6.2. Suppose $\dim \mathcal{R}(A) = r$. Then there are r linearly independent rows. Then in the reduced row-echelon form U of A, there are r non-zero rows. There are r leading 1's and so there are r pivot-columns (in the reduced row-echelon form U of A), i.e., columns corresponding to basic variables. These r columns are linearly independent (by Theorem 1.3.17). Thus, $\dim \mathcal{C}(A) \geq r$.

On the other hand, since the inequality holds for arbitrary matrices, using A^T in place of A, we get $\dim \mathcal{C}(A^T) \geq \dim \mathcal{R}(A^T)$. But $\mathcal{C}(A^) = \mathcal{R}(A)$ and $\mathcal{R}(A^T) = \mathcal{C}(A)$. This gives $\dim \mathcal{R}(A) \geq \dim \mathcal{C}(A)$. Hence, $\dim \mathcal{R}(A) = \dim \mathcal{C}(A)$.

In summary, Theorems 1.5.3 and 1.6.1 establish the fundamental relationships among the row space, column space, null space, rank, and nullity of a matrix. These results are collected below for our convenient reference.

$$\begin{aligned}
\dim \mathcal{N}(A) &= \dim \mathcal{N}(U) \\
&= \text{the number of free variables in } Ux = 0, \\
\dim \mathcal{R}(A) &= \dim \mathcal{R}(U) \\
&= \text{the number of nonzero row vectors of } U \\
&= \text{the maximal number of linearly independent row vectors of } A \\
&= \text{the number of basic variables in } Ux = 0 \\
&= \text{the maximal number of linearly independent 1column vectors of } A \\
&= \dim \mathcal{C}(A).
\end{aligned}$$

The first group of equalities shows that elementary row operations preserve the null space dimension, so the nullity can be computed directly from a row-echelon form. The second group shows that the dimensions of the row and column spaces are equal, and this common dimension is called the *rank* of the matrix.

Definition 1.6.3. For an $m \times n$ matrix A, the *rank* of A is defined as the dimension of its row space (or its column space), and is denoted by $\operatorname{rank} A$.

Since, $\dim \mathcal{R}(A) = \dim \mathcal{C}(A)$, either space may be used to compute the rank. Clearly, $\operatorname{rank}(I_n) = n$, and $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.

Corollary 1.6.4. *If A is an $m \times n$ matrix, then* $\operatorname{rank}(A) \leq \min\{m, n\}$.

Proof. For an $m \times n$ matrix A, since $\dim \mathcal{R}(A) \leq m$, and $\dim \mathcal{C}(A) \leq n$, the result follows immediately. □

Since
$$\dim \mathcal{R}(A) = \dim \mathcal{C}(A) = \operatorname{rank}(A),$$
and the nullity of A equals the number of free variables in the homogeneous system $Ax = 0$, we obtain one of the most important results in linear algebra.

Theorem 1.6.5 (Rank-Nullity Theorem)**.** *For any $m \times n$ matrix A,*

$$\begin{aligned} \dim \mathcal{R}(A) + \dim \mathcal{N}(A) &= \operatorname{rank}(A) + \operatorname{nullity}(A) = n. \\ \dim \mathcal{C}(A) + \dim \mathcal{N}(A^T) &= \operatorname{rank}(A) + \operatorname{nullity}(A^T) = m. \end{aligned}$$

Proof. Since
$$\dim \mathcal{R}(A) = \dim \mathcal{C}(A) = \operatorname{rank}(A),$$
the rank of A equals the number of pivot (basic) variables in the homogeneous system $Ax = 0$. Likewise, $\dim \mathcal{N}(A) = \operatorname{nullity}(A)$ is the number of free variables.

Every variable in the system is either a basic variable or a free variable. Since there are n variables altogether,
$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n.$$
Replacing $\operatorname{rank}(A)$ by $\dim \mathcal{R}(A)$ gives
$$\dim \mathcal{R}(A) + \dim \mathcal{N}(A) = n.$$

Applying the same argument to the transpose matrix A^T, which has m columns, yields
$$\operatorname{rank}(A^T) + \operatorname{nullity}(A^T) = m.$$
Since $\operatorname{rank}(A^T) = \operatorname{rank}(A) = \dim \mathcal{C}(A)$, we obtain
$$\dim \mathcal{C}(A) + \dim \mathcal{N}(A^T) = m.$$
This completes the proof. □

An important consequence of the Rank–Nullity Theorem occurs when the null space is trivial, which is given by the following corollary.

Corollary 1.6.6. *Let A be an $n \times n$ square matrix. Then A is invertible if and only if* $\operatorname{rank} A = n$.

Proof.

$$\begin{aligned} A \text{ is invertible } &\Leftrightarrow Ax = 0 \text{ has only trivial solution } x = 0 && \text{(by a theorem)} \\ &\Leftrightarrow \mathcal{N}(A) = \{0\} \\ &\Leftrightarrow \dim \mathcal{N}(A) = \operatorname{nullity}(A) = 0 \\ &\Leftrightarrow \dim \mathcal{R}(A) = \operatorname{rank}(A) = n && \text{(by the Rank Theorem)}. \end{aligned}$$

□

Example 1.6.7. Find the rank and nullity of

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & 2 \\ 1 & 2 & -2 & -4 & 3 \end{bmatrix}.$$

Solution. To determine the rank and nullity of A, we reduce it to row echelon form using elementary row operations. Since the first entry of the first row is already a pivot, eliminate the entries below it.

$$\begin{aligned} R_2 &\leftarrow R_2 + R_1, \\ R_3 &\leftarrow R_3 - R_1, \\ R_4 &\leftarrow R_4 - R_1. \end{aligned}$$

This gives

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -3 & -9 & 1 \\ 0 & 0 & -2 & -6 & 2 \end{bmatrix}.$$

Now use the pivot in Row 2 (column 3). First, $R_3 \leftarrow R_3 + 3R_2$, which yields $R_3 = (0, 0, 0, 0, 4)$. Next, $R_4 \leftarrow R_4 + 2R_2$, giving $R_4 = (0, 0, 0, 0, 4)$. Hence, the matrix take the form

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

Finally, $R_4 \leftarrow R_4 - R_3$, to obtain

$$U = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For convenience, divide the third row by 4, to get

$$U = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Observe that there are three nonzero rows. Therefore, $\operatorname{rank} A = 3$. The pivot columns are columns $1,\ 3,\ 5$. Thus the corresponding columns of the original matrix form a basis for the column space. By the Rank-Nullity Theorem,

$$\dim(\mathcal{N}(A)) = 5 - \dim(\mathcal{R}(A)) = 5 - 3 = 2.$$

Therefore,

$$\boxed{\operatorname{rank} A = \dim \mathcal{R}(A) = \dim \mathcal{C}(A) = 3, \quad \operatorname{nullity} A = \dim \mathcal{N}(A) = 2.}$$

□

Theorem 1.6.8. *For any two matrices A and B for which the product AB is defined,*

(1) $\mathcal{N}(AB) \supseteq \mathcal{N}(B)$.
(2) $\mathcal{N}((AB)^T) \supseteq \mathcal{N}(A^T)$.
(3) $\mathcal{C}(AB) \subseteq \mathcal{C}(A)$.
(4) $\mathcal{R}(AB) \subseteq \mathcal{R}(B)$.

Proof. Suppose $A \in \mathbb{R}^{m\times n},\ B \in \mathbb{R}^{n\times p}$. Then $AB \in \mathbb{R}^{m\times p}$.
(1) $\mathcal{N}(AB) \supseteq \mathcal{N}(B)$.

$$\begin{aligned} x \in \mathcal{N}(B) &\Rightarrow Bx = 0 \\ &\Rightarrow A(Bx) = (AB)x = 0 \\ &\Rightarrow x \in \mathcal{N}(AB). \end{aligned}$$

Hence, $\boxed{\mathcal{N}(B) \subseteq \mathcal{N}(AB).}$
(2) $\mathcal{N}((AB)^T) \supseteq \mathcal{N}(A^T)$.

$$\begin{aligned} y \in \mathcal{N}(A^T) &\Rightarrow A^T y = 0 \\ &\Rightarrow B^T(A^T y) = (B^T A^T)y = 0 \\ &\Rightarrow (AB)^T y = 0 && (\because (AB)^{=} B^T A^T) \\ &\Rightarrow y \in \mathcal{N}((AB)^T). \end{aligned}$$

Hence, $\boxed{\mathcal{N}(A^T) \subseteq \mathcal{N}((AB)^T).}$
(3) $\mathcal{C}(AB) \subseteq \mathcal{C}(A)$.

$$\begin{aligned} y \in \mathcal{C}(AB) &\Rightarrow y = (AB)x \quad \text{for some vector } x && \text{(by definition)} \\ &\Rightarrow y = A(Bx) && \text{(by associativity)} \\ &\Rightarrow y = Au && \text{(taking u = Bx)} \end{aligned}$$

$$\Rightarrow y \in \mathcal{C}(A).$$

Hence, $\boxed{\mathcal{C}(AB) \subseteq \mathcal{C}(A).}$

(4) $\mathcal{R}(AB) \subseteq \mathcal{R}(B)$.

$$\begin{aligned} \mathcal{R}(AB) &= \mathcal{C}((AB)^T) \\ &= \mathcal{C}(B^T A^T) && (\because (AB)^T = B^T A^T) \\ &\subseteq \mathcal{C}(B^T) && (\text{by } (3)) \\ &= \mathcal{R}(B). \end{aligned}$$

Hence, $\boxed{\mathcal{R}(AB) \subseteq \mathcal{R}(B).}$ □

Corollary 1.6.9. *For any matrices A and B for which the product AB is defined,*

$$\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}.$$

Proof. We know that $\operatorname{rank}(A) = \dim \mathcal{R}(A) = \dim \mathcal{C}(A)$. By (3), we have

$$\mathcal{C}(AB) \subseteq \mathcal{C}(A) \Rightarrow \operatorname{rank}(AB) = \dim \mathcal{C}(AB) \leq \dim \mathcal{C}(A) = \operatorname{rank}(A).$$

Similarly, by (4), we have

$$\mathcal{R}(AB) \subseteq \mathcal{R}(B) \Rightarrow \operatorname{rank}(AB) = \dim \mathcal{R}(AB) \leq \dim \mathcal{R}(B) = \operatorname{rank}(B).$$

Combining the two inequalities gives

$$\boxed{\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}.}$$

□

In some particular situations, equality holds. For example, if A is a square matrix and invertible, then multiplication by A does not change the rank of another matrix. The following exercise explains this result.

Exercise 1.6.10. Let A be an invertible square matrix. Show that, for any matrix A,

$$\operatorname{rank}(AB) = \operatorname{rank}(B) = \operatorname{rank}(BA).$$

1.7 Bases for subspaces

Let V and W be two subspaces of $\mathbb{R}^n$. In this section, we shall see two ways of finding bases for the subspaces $V + W$ and $V \cap W$ and hence deduce an important relation between the dimension of these spaces in terms of the dimensions of V and W.

Let $S = \{v_1, v_2, \ldots, v_k\}$ and $S' = \{w_1, w_2, \ldots, w_l\}$ be bases for V and W respectively. Let Q be the $n \times (k + l)$ matrix whose columns are those basis vectors, i.e.,

$$Q = \begin{bmatrix} v_1 & \cdots & v_k & w_1 & \cdots & w_l \end{bmatrix}_{n \times (k+l)}.$$

Theorem 1.7.1. *Let V and W be two subspaces of $\mathbb{R}^n$, and Q be the matrix defined above.*

(1) $\mathcal{C}(Q) = V + W$, so that a basis for the column space $\mathcal{C}(Q)$ is a basis for $V + W$.
(2) $\mathcal{N}(Q)$ can be identified with $V \cap W$ so that $\dim(V \cap W) = \dim \mathcal{N}(Q)$.

Proof. (1) From the definition of the matrix Q it is clear that $\mathcal{C}(Q) = V + W$.

(2) Let $x = (a_1, \dots, a_k, b_1, \dots, b_l) \in \mathcal{N}(Q) \subseteq \mathbb{R}^{k+l}$. Then $Qx = 0$, which gives

$$a_1v_1 + \cdots + a_kv_k + b_1w_1 + \cdots + b_lw_l = 0.$$

Then we get

$$a_1v_1 + \cdots + a_kv_k = -(b_1w_1 + \cdots + b_lw_l).$$

Let $a_1v_1 + \cdots + a_kv_k = y = -(b_1w_1 + \cdots + b_lw_l)$. Since $a_1v_1 + \cdots + a_kv_k \in V$ (being linear combination of basis elements of V) and $b_1w_1 + \cdots + b_lw_l \in W$ (begin linear combination of basis elements of W), $y \in V \cap W$. Thus, given any vector $x \in \mathcal{N}(Q)$, there is a vector $y \in V \cap W$.

Conversely, let $y \in V \cap W$. Then y can be written as a linear combinations of the bases of V and W independently as follows.

$$y = a_1v_1 + \cdots + a_kv_k,$$
$$y = b_1w_1 + \cdots + b_lw_l,$$

for some $a_1, \dots, a_k$ and $b_1, \dots, b_l$ in $\mathbb{R}$. Then

$$a_1v_1 + \cdots + a_kv_k - b_1w_1 + \cdots - b_lw_l = 0.$$

Then $Qx = 0$ for $x = (a_1, \dots, a_k, -b_1, \dots, -b_l) \in \mathbb{R}^{k+l}$. That is, $x \in \mathcal{N}(Q)$. Thus, given any vector $y \in V \cap W \subseteq \mathbb{R}^n$, there is a corresponding $x \in \mathcal{N}(Q) \subseteq \mathbb{R}^{k+l}$.

Hence, the function $T : \mathcal{N}(Q) \to V \cap W$ defined by $T(x) = y$ is a one-one correspondence between the sets $\mathcal{N}(Q)$ and $V \cap W$ (Why is T one-one? Check!), where $x = (a_1, \dots, a_k, b_2, \dots, b_l)$ and $y = a_1v_1 + \cdots + a_kv_k = -(b_1w_1 + \cdots + b_lw_l)$ as defined before.

Further, if $T(x_1) = y_1$ and $T(x_2) = y_2$, then we shall show that $T(x_1 + x_2) = y_1 + y_2$ and $T(\alpha x_1) = \alpha y_1$ for any $\alpha \in \mathbb{R}$. This proves that the spaces $\mathcal{N}(Q)$ and $V \cap W$ can be identified as vector spaces (we shall see that this identification T is called isomorphism in Section **??**).

To prove this, let

$$x_1 = (a_{11}, \dots, a_{1k}, b_{11}, \dots, b_{1l})$$

and

$$x_2 = (a_{21}, \dots, a_{2k}, b_{21}, \dots, b_{2l})$$

be two vectors in $N(Q)$, and suppose they correspond to

$$y_1 = a_{11}v_1 + \cdots + a_{1k}v_k$$

and

$$y_2 = a_{21}v_1 + \cdots + a_{2k}v_k.$$

Now,

$$x_1 + x_2 = (a_{11} + a_{21}, \dots, a_{1k} + a_{2k}, b_{11} + b_{21}, \dots, b_{1l} + b_{2l}).$$

The vector corresponding to $x_1 + x_2$ is therefore

$$\begin{aligned} y &= (a_{11} + a_{21})v_1 + \cdots + (a_{1k} + a_{2k})v_k \\ &= (a_{11}v_1 + \cdots + a_{1k}v_k) + (a_{21}v_1 + \cdots + a_{2k}v_k) \\ &= y_1 + y_2. \end{aligned}$$

Hence $x_1 + x_2$ corresponds to $y_1 + y_2$. Similarly, for any scalar α,

$$\alpha x_1 = (\alpha a_{11}, \ldots, \alpha a_{1k}, \alpha b_{11}, \ldots, \alpha b_{1l}).$$

The corresponding vector is

$$\begin{aligned} y &= \alpha a_{11}v_1 + \cdots + \alpha a_{1k}v_k \\ &= \alpha(a_{11}v_1 + \cdots + a_{1k}v_k) \\ &= \alpha y_1. \end{aligned}$$

In particular, for a basis for $\mathcal{N}(Q)$, the corresponding set in $V \cap W$ is a basis for $V \cap W$. That is, if the set of vectors

$$\begin{cases} x_1 = (a_{11}, \ldots, a_{1k}, b_{11}, \ldots, b_{1l}), \\ \quad\vdots \\ x_s = (a_{s1}, \ldots, a_{sk}, b_{s1}, \ldots, b_{sl}) \end{cases}$$

is a basis for $\mathcal{N}(Q)$, then the set of vectors

$$\begin{cases} y_1 = a_{11}v_1 + \cdots + a_{1k}v_k, \\ \quad\vdots \\ y_s = a_{s1}v_1 + \cdots + a_{sk}v_k \end{cases} \quad \text{or} \quad \begin{cases} y_1 = -(b_{11}w_1 + \cdots + b_{1l}w_l), \\ \quad\vdots \\ y_s = -(b_{s1}w_1 + \cdots + b_{sl}w_l) \end{cases}$$

is a basis for $V \cap W$, and vice-versa. Therefore,

$$\dim N(Q) = \dim(V \cap W).$$

□

In general,

$$\dim(V + W) \neq \dim V + \dim W.$$

The reason is that the spaces V and W may have some vectors in common. These common vectors are counted twice in $\dim V + \dim W$, and the next theorem will give the precise relation between these dimensions.

Theorem 1.7.2. *For any subspace V and W of the n-space $\mathbb{R}^n$,*

$$\dim(V + W) = \dim(V \cap W) = \dim V + \dim W.$$

Proof. Let $S = \{v_1, v_2, \ldots, v_k\}$ and $S' = \{w_1, w_2, \ldots, w_l\}$ be bases for V and W respectively. Let Q be the $n \times (k + l)$ matrix whose columns are those basis vectors, i.e.,

$$Q = \begin{bmatrix} v_1 & \cdots & v_k & w_1 & \cdots & w_l \end{bmatrix}_{n \times (k+l)}.$$

Then by the Rank Theorem (see Theorem 1.6.5) and the above theorem (Theorem 1.7.1), we have

$$\dim V + \dim W = k + l = \dim \mathcal{C}(Q) + \dim \mathcal{N}(Q) = \dim(V + W) + \dim(V \cap W).$$

$\square$

In particular, $\dim(V + W) = \dim V + \dim W$ if and only if $V \cap W = \{0\}$. That is, $\dim(V + W) = \dim V + \dim W$ if and only if $V + W = V \oplus W$ or we can say that

$$\dim(V \oplus W) = \dim V + \dim W.$$

Example 1.7.3 (Find a basis for subspace)**.** Let V and W be two subspaces of $\mathbb{R}^5$ with bases

$$\begin{cases} v_1 = (1, 3, -2, 2, 3), \\ v_2 = (1, 4, -3, 4, 2), \\ v_3 = (1, 3, 0, 2, 3), \end{cases} \text{ and } \quad \begin{cases} w_1 = (2, 3, -1, -2, 9), \\ w_2 = (1, 5, -6, 6, 1), \\ w_3 = (2, 4, 4, 2, 8), \end{cases}$$

respectively. Find bases for $V + W$ and $V \cap W$.

Since $V + W = \operatorname{span}\{v_1, v_2, v_3, w_1, w_2, w_3\}$, the matrix Q is of the form

$$Q = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 2 \\ 3 & -4 & 0 & 3 & 5 & 2 \\ -4 & 3 & 3 & -1 & -6 & 4 \\ -2 & 4 & 2 & -6 & 6 & 2 \\ 3 & 2 & 3 & 9 & 1 & 8 \end{bmatrix}.$$

To find the independent columns, we derive the reduced row-echelon form of Q. First eliminate the entries below the first pivot using the operations

$$\begin{aligned} R_2 &\leftarrow R_2 - 3R_1, \\ R_3 &\leftarrow R_3 + 4R_1, \\ R_4 &\leftarrow R_4 + 2R_1, \\ R_5 &\leftarrow R_5 - 3R_1. \end{aligned}$$

to get

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 2 \\ 0 & -7 & -3 & -3 & 2 & -4 \\ 0 & 7 & 7 & 7 & -2 & 12 \\ 0 & 6 & 4 & -2 & 8 & 6 \\ 0 & -1 & 0 & 3 & -2 & 2 \end{bmatrix}.$$

Making the second pivot equal to 1 using

$$R_2 \leftarrow -\frac{1}{7} R_2,$$

we get

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & \frac{3}{7} & \frac{3}{7} & -\frac{2}{7} & \frac{4}{7} \\ 0 & 7 & 7 & 7 & -2 & 12 \\ 0 & 6 & 4 & -2 & 8 & 6 \\ 0 & -1 & 0 & 3 & -2 & 2 \end{bmatrix}.$$

Eliminate the remaining entries in column 2 by applying

$$\begin{aligned}R_1 &\leftarrow R_1 - R_2,\\ R_3 &\leftarrow R_3 - 7R_2,\\ R_4 &\leftarrow R_4 - 6R_2,\\ R_5 &\leftarrow R_5 + R_2.\end{aligned}$$

Continue the Gauss–Jordan elimination until all pivots become 1 and all other entries in pivot columns become zero. The final reduced row-echelon form is

$$\mathrm{RREF}(Q) = U = \begin{bmatrix} 1 & 0 & 0 & 5 & 0 & 0\\ 0 & 1 & 0 & -3 & 2 & 0\\ 0 & 0 & 1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

The basic variable columns (leading 1's) in $Qx = 0$ are located in columns 1, 2, 3, and 6. Therefore, the corresponding original columns of Q form a basis for $V + W$. Thus,

$$\boxed{\mathcal{B}_{V+W} = \{v_1, v_2, v_3, w_3\}}$$

and so the dimension is:

$$\dim(V + W) = 4.$$

To find a basis for $V \cap W$, we solve $Ux = 0$. Solving $Ux = 0$ yields the basic variables in terms of free variables x_4 and x_5 as follows.

$$\begin{aligned}x_1 &= -5x_4\\ x_2 &= 3x_4 - 2x_5\\ x_3 &= x_5\\ x_6 &= 0\end{aligned}$$

Choosing $(x_4, x_5) = (1, 0)$ and $(x_4, x_5) = (0, 1)$ gives two linearly independent vectors in $\mathcal{N}(Q) \subset \mathbb{R}^6$ as

$$x_1 = \begin{pmatrix}-5 & 3 & 0 & 1 & 0 & 0\end{pmatrix}, \qquad x_2 = \begin{pmatrix}0 & -2 & 1 & 0 & 1 & 0\end{pmatrix}$$

We translate these back to vectors in $\mathbb{R}^5$ for $V \cap W$. From $Qx_i = 0$, $i = 1, 2$, we get

$$\begin{aligned}-5v_1 + 3v_2 + w_1 &= 0\\ -2v_2 + v_3 + w_2 &= 0\end{aligned}$$

Therefore, (as in the proof of the above theorem), $\{y_1, y_2\}$ is a basis for $V \cap W$, where

$$y_1 = 5v_1 - 3v_2 = w_1 = \begin{pmatrix}2\\3\\-1\\-2\\9\end{pmatrix}, \qquad y_2 = 2v_2 - v_3 = w_2 = \begin{pmatrix}1\\5\\-6\\6\\1\end{pmatrix}$$

Therefore,

$$\boxed{\mathcal{B}_{V\cap W} = \left\{\begin{bmatrix}2 & 3 & -1 & -2 & 9\end{bmatrix}, \begin{bmatrix}1 & 5 & -6 & 6 & 1\end{bmatrix}\right\}}$$

is a basis for $V \cap W$, and $\dim(V \cap W) = 2$.

We can verify our result using the dimension theorem for subspaces:

$$\dim(V + W) + \dim(V \cap W) = 4 + 2 = 6 = 3 + 3 = \dim V + \dim W.$$

Remark 1.7.4 (Another method for finding bases)**.** Another method for finding their bases by constructing a matrix Q whose rows are basis vectors for V and basis vectors for W. In this case, clearly $V + W = \mathcal{R}(Q)$. By finding a basis for the row space $\mathcal{R}(Q)$, one can get a basis for $V + W$.

A basis for $V \cap W$ can be found as follows: Let A be the $k \times n$ matrix whose rows are basis vectors for V, and B the $\ell \times n$ matrix whose rows are basis vectors for W. Then $V = \mathcal{R}(A)$ and $W = \mathcal{R}(B)$. Let $\bar{A}$ denote the matrix A with an additional unknown vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ attached as the bottom row, i.e.,

$$\bar{A} = \begin{bmatrix} A \\ x \end{bmatrix},$$

and the matrix $\bar{B}$ is defined similarly. Then it is clear that $\mathcal{R}(A) = \mathcal{R}(\bar{A})$ and $\mathcal{R}(B) = \mathcal{R}(\bar{B})$ if and only if $x \in V \cap W = \mathcal{R}(A) \cap \mathcal{R}(B)$. This means that the row-echelon form of A and that of $\bar{A}$ should be the same via the same Gaussian elimination. Thus, by comparing the row vectors of the row-echelon form of A with those of $\bar{A}$, one can obtain a system of linear equations for $x = (x_1, x_2, \ldots, x_n)$. By the same argument applied to B and $\bar{B}$, one gets another system of linear equations for the same $x = (x_1, x_2, \ldots, x_n)$. Common solutions of these two systems together will provide us a basis for $V \cap W$.

Example 1.7.5 (Find a basis for a subspace)**.** Let V be the subspace of $\mathbb{R}^5$ spanned by

$$\begin{aligned} v_1 &= (1, 3, -2, 2, 3), \\ v_2 &= (1, 4, -3, 4, 2), \\ v_3 &= (2, 3, -1, -2, 10), \end{aligned}$$

and W the subspace spanned by

$$\begin{aligned} w_1 &= (1, 3, 0, 2, 1), \\ w_2 &= (1, 5, -6, 6, 3), \\ w_3 &= (2, 5, 3, 2, 1). \end{aligned}$$

Find a basis for $V + W$ and for $V \cap W$.

Solution. Form the matrix whose rows are the generators of V:

$$A = \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 1 & 4 & -3 & 4 & 2 \\ 2 & 3 & -1 & -2 & 10 \end{bmatrix}.$$

Applying elementary row operations,

$$\begin{aligned} R_2 &\leftarrow R_2 - R_1, \\ R_3 &\leftarrow R_3 - 2R_1, \end{aligned}$$

gives

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & -3 & 3 & -6 & 4 \end{bmatrix}.$$

Next, $R_3 \leftarrow R_3 + 3R_2$, yielding its reduced row-echelon form as

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

There are three nonzero rows. Hence, $\boxed{\dim V = 3.}$ Similarly, the matrix B whose row vectors are w_j's is reduced to a row-echelon form:

$$\begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 2 & -6 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so that $\boxed{\dim W = 2}$.

Since $V + W = \operatorname{span}\{v_1, v_2, v_3, w_1, w_2, w_3\}$, consider the 6×5 matrix

$$Q = \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 1 & 4 & -3 & 4 & 2 \\ 2 & 3 & -1 & -2 & 10 \\ 1 & 3 & 0 & 2 & 1 \\ 1 & 5 & -6 & 6 & 3 \\ 2 & 5 & 3 & 2 & 1 \end{bmatrix}.$$

whose row vectors are v_i's and w_j's, then $V + W = \mathcal{R}(Q)$, the row space of Q. By Gaussian elimination, Q is reduced to a row-echelon form, excluding (the last two) zero rows:

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hence, the four nonzero rows form a basis of $V + W$:

$$\Big\{(1, 3, -2, 2, 3), \quad (0, 1, -1, 2, -1), (0, 0, 1, 0, -1), \quad (0, 0, 0, 0, 1)\Big\}.$$

Therefore, $\boxed{\dim(V + W) = 4.}$

We now find a basis for $V \cap W$. A vector $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ is contained in $V \cap W$ if and only if x is contained in both the row space of A and that of B.

Let $\bar{A}$ be A with x attached as the last row:

$$\bar{A} = \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 1 & 4 & -3 & 4 & 2 \\ 2 & 3 & -1 & -2 & 10 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}.$$

Perform exactly the same row operations used to reduce A:

$$R_2 \leftarrow R_2 - R_1, \quad R_3 \leftarrow R_3 - 2R_1, \quad R_4 \leftarrow R_4 - x_1 R_1,$$

followed by

$$R_3 \leftarrow R_3 + 3R_2, \quad R_4 \leftarrow R_4 - (x_2 - 3x_1)R_2.$$

to get

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -x_1 + x_2 + x_3 & 4x_1 - 2x_2 + x_4 & 0 \end{bmatrix}.$$

The last row becomes $(0, 0, -x_1 + x_2 + x_3, 4x_1 - 2x_2 + x_4, 0)$.

Therefore, $x \in \mathcal{R}(A) = V$ if and only if $\mathcal{R}(A) = \mathcal{R}(\bar{A})$. By comparing the row vectors of the row-echelon form of A with those of $\bar{A}$, one can say that $x \in \mathcal{R}(A)$ if and only if the last row vector of the row-echelon form of $\bar{A}$ is the zero vector, that is, x is a solution of the homogeneous system of equations:

$$\begin{cases} -x_1 + x_2 + x_3 = 0, \\ 4x_1 - 2x_2 + x_4 = 0. \end{cases}$$

Similarly, attaching x to B and carrying out the same row operations gives

$$\begin{cases} -9x_1 + 3x_2 + x_3 = 0, \\ 4x_1 - 2x_2 + x_4 = 0, \\ 2x_1 - x_2 + x_5 = 0. \end{cases}$$

Therefore (solving these two homogeneous systems together), $x \in V \cap W$ if and only if

$$\begin{cases} -x_1 + x_2 + x_3 = 0, \\ -9x_1 + 3x_2 + x_3 = 0, \\ 4x_1 - 2x_2 + x_4 = 0, \\ 2x_1 - x_2 + x_5 = 0. \end{cases}$$

Subtracting the first equation from the second, $-8x_1 + 2x_2 = 0$, so that $x_2 = 4x_1$. Substituting into the remaining equations,

$$\begin{aligned} x_3 &= x_1 - x_2 = -3x_1, \\ x_4 &= -4x_1 + 2x_2 = 4x_1, \\ x_5 &= -2x_1 + x_2 = 2x_1. \end{aligned}$$

Letting $x_1 = t$, we obtain $x = t(1, 4, -3, 4, 2)$. Thus,

$$\boxed{V \cap W = \operatorname{span}\{(1, 4, -3, 4, 2)\} = \{t(1, 4, -3, 4, 2) : t \in \mathbb{R}\}.}$$

Hence a basis for $V \cap W$ is $\boxed{\{(1, 4, -3, 4, 2)\}}$, and $\boxed{\dim(V \cap W) = 1.}$ Finally,

$$\dim(V + W) = \dim V + \dim W - \dim(V \cap W) = 3 + 2 - 1 = 4,$$

which agrees with the result obtained above. $\square$

1.8 Invertibility

We know that non-square matrix A may have only one-sided inverse (if exists), i.e., a left inverse or a right inverse. In this section, we shall show that the existence of such a one-sided inverse of A implies the existence of the uniqueness of the solutions of a system $Ax = b$.

Theorem 1.8.1. *Let A be a $m \times n$ matrix. Then the following are equivalent.*

(1) For each $b \in \mathbb{R}^m$, $Ax = b$ has at least one solution in $\mathbb{R}^n$.
(2) The column vectors of A span $\mathbb{R}^m$, i.e., $\mathcal{C}(A) = \mathbb{R}^m$.
(3) $\operatorname{rank}(A) = m$ (hence $m \le n$).
(4) A has a right inverse (i.e., B such that $AB = I_n$).

Proof.

(1) $\Leftrightarrow$ (2) Assume (1). We have to show that $\mathcal{C}(A) = \mathbb{R}^m$. Clearly, $\mathcal{C}(A) \subseteq \mathbb{R}^m$. For the reverse inclusion, let $b \in \mathbb{R}^m$. By hypothesis (i.e., by our assumption), there is $x \in \mathbb{R}^n$ such that $Ax = b$. This means that b can be written as a linear combination of column vectors of A. Thus, $b \in \mathcal{C}(A)$.
Conversely, assume that (2) holds. We have to show that for each $b \in \mathbb{R}^m$, there is at least one $x \in \mathbb{R}^n$ such that $Ax = b$. So, let $b \in \mathbb{R}^m$. By hypothesis (i.e., (2)), $b \in \mathcal{C}(A)$. Thus, b can be written as a linear combination of n columns of A with coefficients, say $x_1, x_2, \ldots, x_n$ in $\mathbb{R}$. Then $Ax = b$ has a solution $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^m$.

(2) $\Leftrightarrow$ (3) We know that $\dim \mathcal{C}(A) = \operatorname{rank} A = \dim \mathcal{R}(A) \le \min\{m, n\}$. Now,

$$\begin{aligned} \mathcal{C}(A) = \mathbb{R}^m &\Leftrightarrow \dim \mathcal{C}(A) = m \\ &\Leftrightarrow \operatorname{rank} A = m \text{ (and so } m \le \min\{m, n\}). \end{aligned}$$

Consequently, $m \le n$.

(1) $\Rightarrow$ (4) Assume (1). Let $e_1, e_2, \ldots, e_m$ be the standard basis for $\mathbb{R}^m$. By our assumption, there exist $x_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$, such that $Ax_i = e_i$. Let $B = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}$ be the $n \times m$ matrix with columns $x_1, x_2, \ldots, x_m$. Then by matrix multiplication, we have

$$AB = A\begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \cdots & e_m \end{bmatrix} = I_m.$$

(4) $\Rightarrow$ (1) Assume that B is a right inverse of A. Then $AB = I_m$. Then for any $b \in \mathbb{R}^m$, we have $ABb = I_m b = b$. Taking $x = Bb$ proves (1).

□

Theorem 1.8.2. *Let A be a $m \times n$ matrix. Then the following statements are equivalent.*

(1) For each $b \in \mathbb{R}^m$, $Ax = b$ has at most one solution $x \in \mathbb{R}^n$.
(2) The column vectors of A are linearly independent.
(3) $\dim \mathcal{C}(A) = \operatorname{rank} A = n$ (hence $n \le m$).
(4) $\mathcal{R}(A) = \mathbb{R}^n$.
(5) $\mathcal{N}(A) = \{0\}$.
(6) A has a left inverse (i.e., C such that $CA = I_n$).

Proof.

(1) ⇒ (2) Note that the column vectors of A are linearly independent if and only if the homogeneous equation $Ax = 0$ has only trivial solution. However, $Ax = 0$ always has a trivial solution $x = 0$. By (1), it is the unique solution. Thus, if a linear combination of columns of A is zero, then all its coefficients are zero (as $x = 0$ is the only solution). This proves (2).

(2) ⇔ (3)

$$\begin{aligned}\text{Columns of } A \text{ are LI} &\Leftrightarrow \text{the columns of } A \text{ form a basis for } \mathcal{C}(A)\\ &\Leftrightarrow \dim \mathcal{C}(A) = n\\ &\Leftrightarrow \operatorname{rank} A = \dim \mathcal{C}(A) = n \le \min\{m, n\}.\end{aligned}$$

Consequently $n \le m$.

(3) ⇔ (4) Clearly, $\mathcal{R}(A) \subseteq \mathbb{R}^n$. Now,

$$\begin{aligned}\dim \mathcal{C}(A) = \operatorname{rank} A = n &\Leftrightarrow \dim \mathcal{R}(A) = n && (\because \dim \mathcal{R}(A) = \operatorname{rank} A)\\ &\Leftrightarrow \mathcal{R}(A) = \mathbb{R}^n && (\because \mathcal{R}(A) \subseteq \mathbb{R}^n \text{ and } \dim \text{ is same}).\end{aligned}$$

(4) ⇔ (5) By the Rank theorem (Theorem 1.6.5), we know that

$$\dim \mathcal{R}(A) + \dim \mathcal{N}(A) = n$$

$$\begin{aligned}\mathcal{R}(A) = \mathbb{R}^n &\Leftrightarrow \operatorname{rank} A = \dim \mathcal{R}(A) = n\\ &\Leftrightarrow \dim \mathcal{N}(A) = 0 && \text{(by Rank-Nullity theorem)}\\ &\Leftrightarrow \mathcal{N}(A) = \{0\}.\end{aligned}$$

(2) ⇒ (6) Assume (2), i.e., the column vectors of A are linearly independent. Then $\operatorname{rank} A = n$. Extend these column vectors of A to a basis for $\mathbb{R}^m$ by adding $m - n$ additional independent vectors to them. Construct an $m \times m$ matrix S with those basis vectors in its columns. Then the matrix S has rank m, and hence it is invertible.
Let C be the $n \times m$ matrix obtained from S^{-1} by discarding the last $m - n$ rows. Sine the first n columns of S constitute the matrix A, we have $CA = I_n$.

$$I_m = S^{-1}S = \left[\begin{array}{ccc} & C_{n\times m} & \\ \hdashline * & * & * \end{array}\right]\left[\begin{array}{c:c} & * \\ A_{m\times n} & * \\ & * \end{array}\right] = \left[\begin{array}{c:c} I_n & 0 \\ \hdashline 0 & I \end{array}\right].$$

(6) ⇒ (1) Let C be a left inverse of A. We have to show that for each $b \in \mathbb{R}^m$, $Ax = b$ has at most one solution $x \in \mathbb{R}^n$.
Let $b \in \mathbb{R}^n$. If $Ax = b$ has no solution, then we are done. Suppose $Ax = b$ has two solutions x_1 and x_2. Then

$$x_1 = CAx_1 = Cb = CAx_2 = x_2.$$

This proves (6).

□

Remark 1.8.3. (1) In the above two results, we proved that an $m \times n$ matrix A has a right inverse if and only if $\operatorname{rank} A = m$, while A has a left inverse if and only if $\operatorname{rank} A = n$. Therefore, if $m \neq n$, then A cannot have both left and right inverses.

(2) If $m = n$, then A is a square matrix. Then A has a right inverse (and a left inverse) if and only if $\operatorname{rank} A = m = n$. Furthermore, in this case, the inverses are same. Therefore, a square matrix A has rank n if and only if A is invertible.
This means that for a square matrix, "Existence = Uniqueness" and the ten statements listed in Theorems 1.8.1 and 1.8.2 are all equivalent. In particular, for invertibility of a square matrix it is sufficient to show the existence of a one-sided inverse.

1.9 Applications

1.9.1 The Wronskian

Let V be the vector space of all infinitely many times differentiable functions on $\mathbb{R}$. Then clearly V is an infinite dimensional vector space.

Let $f_1, f_2, \ldots, f_n$ be n functions in V. Let

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = \mathbf{0},$$

where $\mathbf{0}$ is the zero function taking the value zero at all the points in the domain. Then the n functions are linearly independent in V if $c_i = 0$ for all i. Differentiating $n-1$ times, one can obtain n equations as:

$$c_1 f_1^{(i)}(x) + c_2 f_2^{(i)}(x) + \cdots + c_n f_n^{(i)}(x) = 0, \quad 0 \le i \le n-1,$$

for all $x \in \mathbb{R}$. In the matrix form, we get

$$\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The determinant of the coefficient matrix is called the *Wronskian* for $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ and denoted by $W(x)$. Therefore, if there is a point $x_0 \in \mathbb{R}$ such that $W(x_0) \neq 0$, then the coefficient matrix is nonsingular (i.e., invertible) at $x = x_0$, and so all $c_i = 0$. Therefore,

> if the Wronskian $W(x) \neq 0$ for at least on $x \in \mathbb{R}$,
> then $f_1, f_2, \ldots, f_n$ are linearly independent.

However, the Wronskian $W(x) = 0$ for all x does not imply the linear dependence of the given functions f_i's. In fact, $W(x) = 0$ means that the functions are linearly dependent at each point $x \in \mathbb{R}$, but the constants c_i's giving nontrivial linear dependence may vary as x varies in the domain.

Example 1.9.1 (Test the linear independence of functions by Wronskian). (1) For the sets of functions

$$F_1 = \{x, \cos x, \sin x\}, \qquad F_2 = \{x, e^x, e^{-x}\},$$

the Wronskians are

$$W_1(x) = \begin{vmatrix} x & \cos x & \sin x \\ 1 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}.$$

Expanding along the first row,

$$\begin{aligned}
W_1(x) &= x\begin{vmatrix}-\sin x & \cos x\\ -\cos x & -\sin x\end{vmatrix} - \cos x\begin{vmatrix}1 & \cos x\\ 0 & -\sin x\end{vmatrix} + \sin x\begin{vmatrix}1 & -\sin x\\ 0 & -\cos x\end{vmatrix}\\
&= x\left[(-\sin x)(-\sin x) - (\cos x)(-\cos x)\right] - \cos x(-\sin x) + \sin x(-\cos x)\\
&= x(\sin^2 x + \cos^2 x) + \sin x\cos x - \sin x\cos x\\
&= x.
\end{aligned}$$

Hence,

$$\boxed{W_1(x) = x.}$$

Similarly,

$$W_2(x) = \begin{vmatrix} x & e^x & e^{-x}\\ 1 & e^x & -e^{-x}\\ 0 & e^x & e^{-x}\end{vmatrix}.$$

Expanding along the first row,

$$\begin{aligned}
W_2(x) &= x\begin{vmatrix}e^x & -e^{-x}\\ e^x & e^{-x}\end{vmatrix} - e^x\begin{vmatrix}1 & -e^{-x}\\ 0 & e^{-x}\end{vmatrix} + e^{-x}\begin{vmatrix}1 & e^x\\ 0 & e^x\end{vmatrix}\\
&= x\left(e^x e^{-x} + e^x e^{-x}\right) - e^x(e^{-x}) + e^{-x}(e^x)\\
&= 2x - 1 + 1\\
&= 2x.
\end{aligned}$$

Hence,

$$\boxed{W_2(x) = 2x.}$$

Since

$$W_1(x) \neq 0, \qquad W_2(x) \neq 0$$

for every $x \neq 0$, both sets

$$F_1 = \{x, \cos x, \sin x\}, \qquad F_2 = \{x, e^x, e^{-x}\}$$

are linearly independent.

(2) Consider the functions

$$\{x|x|,\ x^2\}.$$

The Wronskian is

$$W(x) = \begin{vmatrix} x|x| & x^2\\ 2|x| & 2x\end{vmatrix}.$$

Since

$$\frac{d}{dx}(x|x|) = 2|x|,$$

we have

$$\begin{aligned}
W(x) &= (x|x|)(2x) - x^2(2|x|)\\
&= 2x^2|x| - 2x^2|x|\\
&= 0.
\end{aligned}$$

Therefore,

$$\boxed{W(x) = 0.}$$

Notice that

$$x|x| = \begin{cases} -x^2, & x < 0, \\ x^2, & x \geq 0. \end{cases}$$

Thus,

- on $(-\infty, 0)$, $x|x| = -x^2$, so the functions are linearly dependent;
- on $[0, \infty)$, $x|x| = x^2$, so the functions are also linearly dependent.

However, there is no single constant c such that

$$x|x| = cx^2$$

for every real number x. Therefore the functions are linearly independent on the whole real line, even though their Wronskian is identically zero.

Thus,

$$\boxed{\text{The Wronskian being zero does not always imply linear dependence.}}$$