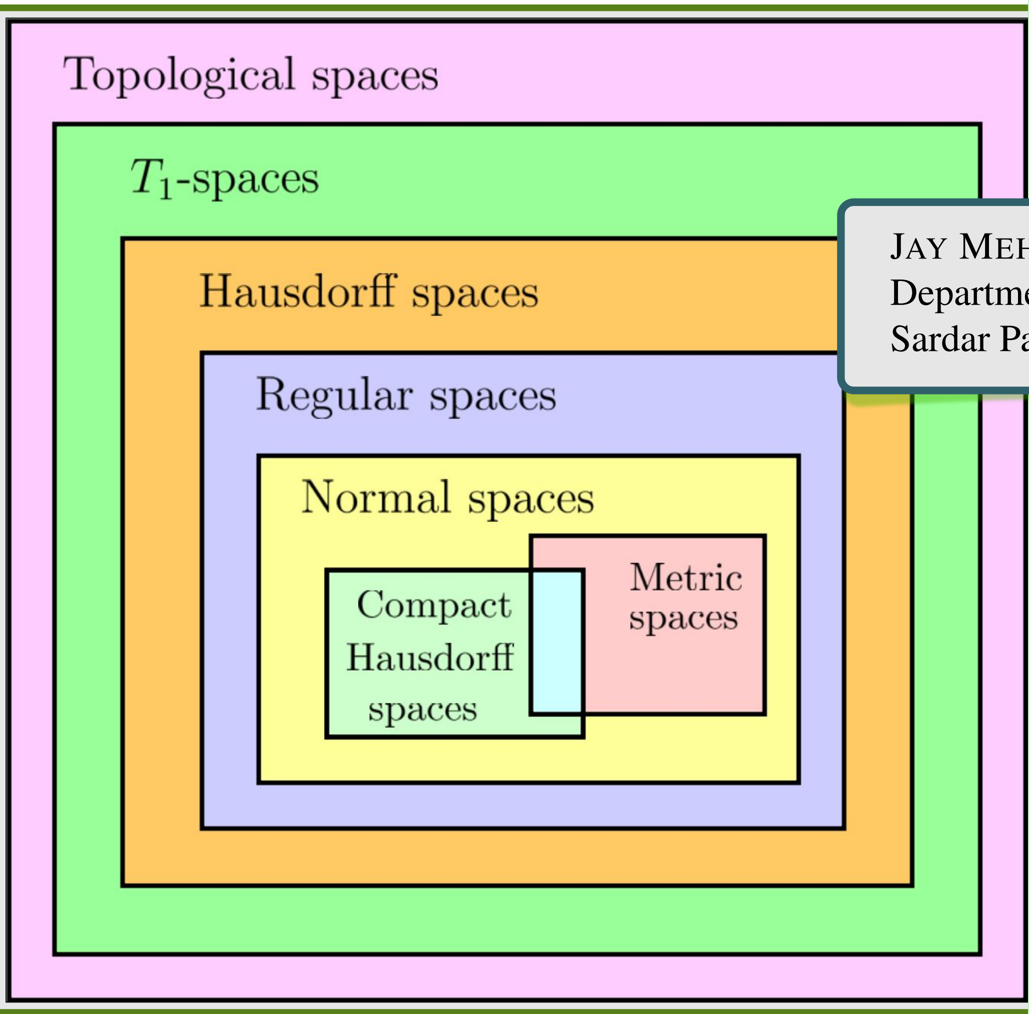
$$(X, \mathcal{T})$$

Lecture notes on

Topology I

PS01CMTH52



$$B_d(x, \varepsilon)$$

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$$X = \bigcup_{i=1}^{n} U_i$$

Semester - I
2022-23

$$\bigcap_{n=1}^{\infty} F_n = \{x\}$$

Contents

Preface and Acknowledgments

This lecture note of the course "Topology I" offered to the M.Sc. (Semester - I) students at Department of Mathematics, Sardar Patel University, 2022-23 is aimed to provide a reading material to the students, in addition to the references mentioned in the university syllabus, so as to save time of the teacher and the students in writing on the board and copying in the notebooks, respectively. These notes are tailor-made for the Topology I (PS01CMTH52) syllabus of M.Sc. (Semester-I) of the University and do not cover all the topics of Topology.

These notes are prepared from the recommended reference books as well as lecture notes of past years, and it is not the original work of the author. We mostly followed the text books by J. R. Munkres and by G. F. Simmons.

Many problems and exercises are listed within the chapters at the end or in between appropriate sections. Students are strongly encouraged to refer and solve these problems in addition to these notes so as to get better understanding and get good problem solving practice, specially when preparing for competitive exams like NET, NBHM, etc. There may be a few errors/typos in this reading material. The students and interested readers are welcomed to give their valuable suggestions, comments or point out errors, if and whenever, they find any.

Acknowledgment

We are indebted to Dr. D. J. Karia for providing his lecture notes on Topology including the source files from which many examples, diagrams, exercises, proofs, etc. are included in this lecture note. Thanks are not enough for his time and effort in pointing out many corrections and suggestions that improved the previous version of this lecture note (old course PS01CMTH22).

JAY MEHTA

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Syllabus

PS01CMTH52: Topology I

Unit I: Topological spaces, basis, subbasis, the product topology on $X \times Y$, the subspace topology, closed sets, closure and interior, limit points, boundary of a set.

Unit II: Hausdorff spaces, convergent sequence, T_1-space, Continuous functions, homeomorphisms, constructing continuous functions, pasting lemma, metric topology, diameter and bounded sets, bounded metric $\overline{d}$ (excluding norm), continuity in metrizable spaces, the sequence lemma, first countability axiom.

Unit III: Connected spaces, connected subspaces of the real line, connected components, compact spaces, finite intersection property, Heine-Borel theorem for real line, second countable spaces, separable spaces.

Unit IV: Regular spaces, Normal spaces, Urysohn's Lemma (statement only), Tietze's Extension Theorem (statement only), Complete metric spaces, Cantor's intersection theorem, Baire's category theorem for complete metric spaces.

References

1. Munkres J., Topology: A First Course, (Second Edition), Prentice Hall of India Pvt. Ltd. New Delhi, 2003.

2. Simmons G.F., Introduction to Topology and Modern Analysis, McGraw-Hill Co., Tokyo, 1963.

3. Willard S., General Topology, Dover Publication, 2004.

4. Kelley J., General Topology, Graduate Texts in Mathematics, Springer-Verlag, 1975.

Online Resources

1. Dinesh Karia, Point Set Topology An Experience of a Teacher, Open Mathematics Notes Series of the American Mathematical Society, November 2020.
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CHAPTER 1

Topological Spaces

1.1 Topological spaces

Definition 1.1.1: Topological space

Let X be a non-empty set and $\mathcal{T}$ be a collection of subsets of X satisfying the following properties:

1. $\emptyset$ and X are in $\mathcal{T}$.
2. The union of elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of any finite subcollection of elements of $\mathcal{T}$ is in $\mathcal{T}$.

Then $\mathcal{T}$ is called a *topology* on X and the pair $(X,\mathcal{T})$ is called a *topological space*.

Thus, a topological space is a pair $(X,\mathcal{T})$. However, at times we just say that X is a topological space without mentioning the topology $\mathcal{T}$ on it whenever there is no scope of confusion or ambiguity.

Definition 1.1.2: Open sets

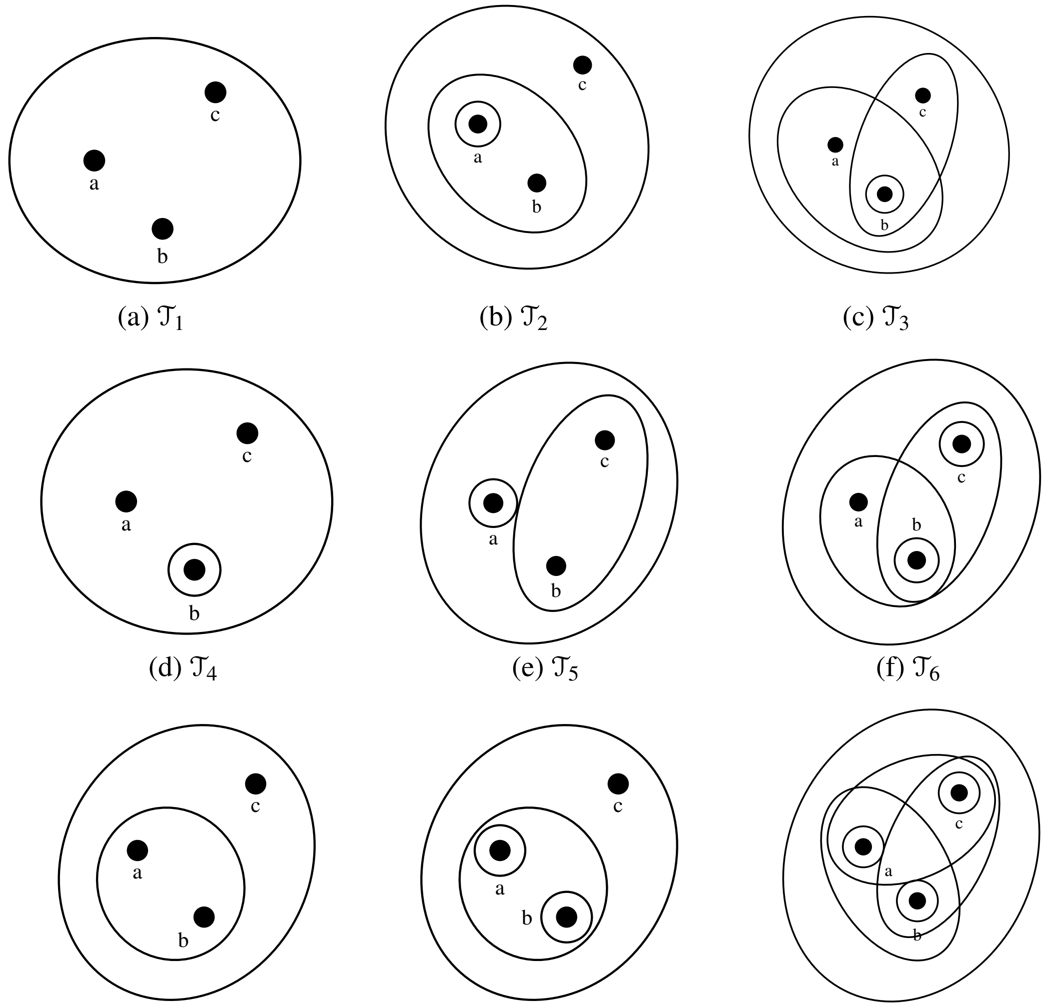
Let $(X,\mathcal{T})$ be a topological space. We say that a subset U of X is an *open set*, if belongs to the collection $\mathcal{T}$.

Thus, a topological space is a non-empty set X together with a collection of subsets of X which are called open sets such that $\emptyset$ and X are open, arbitrary union of open subsets of X is open, and finite intersection of open subsets of X is open.

Example 1.1.3. Let $X = \{a,b,c\}$ be a set of three elements. The following are some of the topologies defined on X which are pictorially demonstrated below.

(a) $\mathcal{T}_1 = \{\emptyset, X\}$ (Figure 1.1a).
(b) $\mathcal{T}_2 = \{\emptyset, \{a\}, \{a,b\}, X\}$ (Figure 1.1b).

(c) $\mathcal{T}_3 = \{\emptyset, \{b\}, \{a,b\}, \{b,c\}, X\}$ (Figure 1.1c).
(d) $\mathcal{T}_4 = \{\emptyset, \{b\}, X\}$ (Figure 1.1d).
(e) $\mathcal{T}_5 = \{\emptyset, \{a\}, \{b,c\}, X\}$ (Figure 1.1e).
(f) $\mathcal{T}_6 = \{\emptyset, \{b\}, \{c\}, \{a,b\}, \{b,c\}, X\}$ (Figure 1.1f).
(g) $\mathcal{T}_7 = \{\emptyset, \{a,b\}, X\}$ (Figure 1.1g).
(h) $\mathcal{T}_8 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$ (Figure 1.1h).
(i) $\mathcal{T}_9 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, X\}$ (Figure 1.1i).



(a) $\mathcal{T}_1$ (b) $\mathcal{T}_2$ (c) $\mathcal{T}_3$ (d) $\mathcal{T}_4$ (e) $\mathcal{T}_5$ (f) $\mathcal{T}_6$ (g) $\mathcal{T}_7$ (h) $\mathcal{T}_8$ (i) $\mathcal{T}_9$

Figure 1.1: Some topologies on the set $X = \{a,b,c\}$

■

Remark 1.1.4. From Example 1.1.3, it can be seen that even a set with just three elements has many different topologies on it. Note that $(X, \mathcal{T}_i)$, $i = 1,2,\ldots,9$ are all different topological spaces though the underlying set X is the same. However, not every collection of subsets of X is a topology on X. For instance, the collections demonstrated below fail to be topology on X.

The collection $\{\emptyset, \{a\}, \{b\}, X\}$ is not a topology on X as $\{a\}, \{b\}$ belong to the collection but their union $\{a,b\}$ is not in the collection.

Also, the collection $\{\emptyset, \{a,b\}, \{b,c\}, X\}$ is not a topology on X as $\{a,b\}$ and $\{b,c\}$ belong to the collection but their intersection $\{b\}$ is not in the collection.

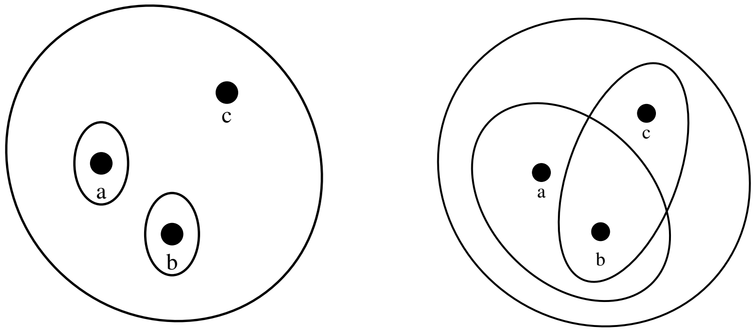


Figure 1.2: Not topologies on $X = \{a,b,c\}$

From the Definition 1.1.1, it is clear that if either $\emptyset$ or X is not in the collection $\mathcal{T}$, then it cannot be a topology on X. However, taking just $\emptyset$ and X in the collection trivially forms a topology on X as given in the following example.

Example 1.1.5. Let X be any non-empty set. The collection containing only X and $\emptyset$ is a topology on X called the *indiscrete topology* or the *trivial topology*.

The collection of all subsets of X, i.e. $\mathscr{P}(X)$, the power set of X is a topology on X and it is called the *discrete topology*. ■

Exercise 1.1

Take a set X with four or five elements, say $X = \{a,b,c,d,e\}$. Construct at least five examples of different topologies on X and two examples of collections which fail to be a topology on X.

Exercise 1.2

Let $X = \{a,b,c\}$.

1. List all the topologies on X containing $\{a\}$.
2. Find a topology on X in which every singleton set is open. Is it the discrete topology?

Example 1.1.6. Let X be a non-empty set. Let $\mathcal{T}_f$ be the collection of all subsets U of X such that either $X \setminus U$ is finite or whole of X, i.e.

$$\mathcal{T}_f = \{U \subset X \mid U = \emptyset \text{ or } X \setminus U \text{ is finite}\}.$$

Then $\mathcal{T}_f$ is a topology on X called the *cofinite topology* or the *finite complement topology*. ■

***Solution*.**

- First we show that $\emptyset, X \in \mathcal{T}_f$.

  From the definition of $\mathcal{T}_f$ it is clear that $\emptyset \in \mathcal{T}_f$. Also, $X \setminus X = \emptyset$, which is a finite set and hence $X \in \mathcal{T}_f$.

- Next, we show that arbitrary union of elements of $\mathcal{T}_f$ is in $\mathcal{T}_f$.

  Let $\{U_\alpha \mid \alpha \in \Lambda\}$ be a collection of nonempty elements of $\mathcal{T}_f$. (Why did we take nonempty elements in the collection?). We want to show that $\bigcup_\alpha U_\alpha \in \mathcal{T}_f$. Clearly $\bigcup_\alpha U_\alpha \subset X$. Now,

$$X \setminus \bigcup_\alpha U_\alpha = \bigcap_\alpha (X \setminus U_\alpha).$$

Since $U_\alpha \in \mathcal{T}_f$ for all α, each $(X \setminus U_\alpha)$ is finite. Therefore $X \setminus \bigcup_\alpha U_\alpha$ is finite and hence $\bigcup_\alpha U_\alpha \in \mathcal{T}_f$.

- Finally, we show that intersection of finite number of members of $\mathcal{T}_f$ is in $\mathcal{T}_f$. Let $U_1, U_2, \ldots, U_n$ be nonempty elements of $\mathcal{T}_f$. We want to show that $\bigcap_{i=1}^{n} U_i \in \mathcal{T}_f$. Clearly $\bigcap_{i=1}^{n} U_i \subset X$. Now,

$$X \setminus \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X \setminus U_i).$$

Since each $U_i \in \mathcal{T}_f$, each $(X \setminus U_i)$ is finite and finite union of finite sets is finite. Therefore $X \setminus \bigcap_{i=1}^{n} U_i$ is finite and hence $\bigcap_{i=1}^{n} U_i \in \mathcal{T}_f$.

Hence, $\mathcal{T}_f$ is a topology on X. □

Similarly one can show (Verify!) that the following is a topological space.

Example 1.1.7. Let X be any non-empty set. Let $\mathcal{T}_c$ be the collection of all subsets U of X such that either $X \setminus U$ is countable or all of X, i.e.

$$\mathcal{T}_c = \{U \subset X \mid U = \emptyset \text{ or } X \setminus U \text{ is countable}\}.$$

Then $\mathcal{T}_c$ is a topology on X called the *cocountable topology*. ■

Exercise 1.3

Let X be a non-empty set. Check which the following collections of subsets of X forms a topology on X.

1. $\mathcal{T}_c = \{U \subset X \mid U = \emptyset \text{ or } X \setminus U \text{ is countable}\}$.
2. $\mathcal{T}_\infty = \{U \subset X \mid X \setminus U \text{ is infinite}\} \cup \{\emptyset, X\}$.
3. $\mathcal{T} = \{U \subset X \mid U \text{ is finite}\}$.
4. $\mathcal{T} = \{U \subset X \mid U \text{ is infinite}\}$.
5. Let $p \in X$ be a fixed element. $\mathcal{T}_p = \{U \subset X \mid U = \emptyset \text{ or } p \in U\}$. This is in fact a topology on X called *VIP (Very Important Point)* topology .
6. Let $p \in X$ be a fixed element. $\mathcal{T} = \{U \subset X \mid U = X \text{ or } p \notin U\}$.
7. Let $A \subset X$ be a fixed nonempty subset of X. $\mathcal{T}_A = \{U \subset X \mid U = \emptyset \text{ or } A \subset U\}$. When is this discrete?
8. Let $A \subset X$ be a fixed nonempty subset of X. $\mathcal{T} = \{U \subset X \mid U = X \text{ or } U \subset A\}$. Can we take A to be empty set?
9. Let $A \subset X$ be a fixed nonempty subset of X. $\mathcal{T} = \{U \subset X \mid U = X \text{ or } A \not\subset U\}$.

Exercise 1.4

Show that the collection $\mathcal{T} = \{U \subset \mathbb{R} \mid U = \mathbb{R} \text{ or } U \cap \mathbb{Q} = \emptyset\}$ is a topology on $\mathbb{R}$. What if we replace $\mathbb{Q}$ by $\mathbb{R} \setminus \mathbb{Q}$?

Exercise 1.5

Let X be any set, $(Y, \mathcal{T}')$ be a topological space and $f : X \to Y$ be a function. Show that

$$\mathcal{T} = \{f^{-1}(U) \subset X \mid U \in \mathcal{T}'\}$$

is a topology on X.

Exercise 1.6

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X. Are $\mathcal{T}_1 \cap \mathcal{T}_2$ and $\mathcal{T}_1 \cup \mathcal{T}_2$ topologies on X? Justify.

Exercise 1.7

Let $\{\mathcal{T}_\alpha \mid \alpha \in \Lambda\}$ be a family of topologies on a set X. Show that $\bigcap\limits_{\alpha \in \Lambda} \mathcal{T}_\alpha$ is a topology on X.

Exercise 1.8

Let $\{\mathcal{T}_\alpha \mid \alpha \in \Lambda\}$ be a family of topologies on a set X. Show that there is a unique smallest topology on X containing each $\mathcal{T}_\alpha$. Also show that there is a unique largest topology on X contained in each $\mathcal{T}_\alpha$.

Exercise 1.9

Show that the cofinite topology on a set X is same as the discrete topology if and only if X is finite. Similarly, show that the cocountable topology and the discrete topology on X coincide if and only if X is countable.

Definition 1.1.8

Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a set X. If $\mathcal{T}' \supset \mathcal{T}$, then we say that $\mathcal{T}'$ is *finer* or *stronger* than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* or *weaker* than $\mathcal{T}'$.

If $\mathcal{T}'$ properly contains $\mathcal{T}$, we say that $\mathcal{T}'$ is strictly finer than $\mathcal{T}$ or $\mathcal{T}$ is strictly coarser (weaker) than $\mathcal{T}'$.

We say that the topologies $\mathcal{T}'$ and $\mathcal{T}$ are *comparable* if either $\mathcal{T}' \subset \mathcal{T}$ or $\mathcal{T} \subset \mathcal{T}'$.

Exercise 1.10

In Example 1.1.3, examine which topologies are comparable. For every pair of comparable topologies investigate which is weaker and which is stronger? Which is the strongest and the weakest of all topologies listed on the set $X = \{a, b, c\}$ in that example?

1.2 Basis for a Topology

Unlike examples considered in the above section, at times, it is not convenient to specify the topology explicitly in an efficient way. Sometimes there is a large number of open sets and hence specifying topology becomes difficult. In such situations, we specify a smaller collection of subsets of X and define topology in terms of this subcollection called basis of the topology.

Definition 1.2.1: Basis

Let X be a nonempty set. A collection $\mathcal{B}$ of subsets of X is called a *basis* for a topology on X if

(1) For each $x \in X$, there exits $B \in \mathcal{B}$ such that $x \in B$.
(2) If $x \in B_1 \cap B_2$, where $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

As mentioned above, the topology on X can be specified in terms of basis and it is given as follows.

Definition 1.2.2: Topology generated by a basis

Let X be a nonempty set and $\mathcal{B}$ be a collection of subsets of X satisfying the properties in Definition 1.2.1. Then the *topology* $\mathcal{T}$ *generated by* $\mathcal{B}$ is defined as:

A subset U of X is said to be open in X (i.e. $U \in \mathcal{T}$) if for each $x \in U$, there exists some $B \in \mathcal{B}$ such that $x \in B \subset U$.

Note that every element of basis itself is a member of $\mathcal{T}$ i.e. $\mathcal{B} \subset \mathcal{T}$. We now verify that the collection $\mathcal{T}$ (generated by $\mathcal{B}$) defined in Definition 1.2.2 is in fact a topology on X.

Proposition 1.2.3

Let X be a nonempty set and $\mathcal{B}$ be a collection of subsets of X satisfying the properties of basis given in Definition 1.2.1. Then the collection

$$\mathcal{T} = \{U \subset X \mid \text{ for each } x \in U, \text{ there exits } B \in \mathcal{B} \text{ such that } x \in B \subset U\}$$

forms a topology on X.

Proof.

- First we show that $\emptyset, X \in \mathcal{T}$.

  $\emptyset$ satisfies the condition in the definition of $\mathcal{T}$ vacuously and hence $\emptyset \in \mathcal{T}$. Let $x \in X$. By property (1) of basis, there exits $B \in \mathcal{B}$ such that $x \in B \subset X$. Thus, $X \in \mathcal{T}$.

- Next we show that arbitrary union of members of $\mathcal{T}$ is in $\mathcal{T}$.

  Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be a collection of members of $\mathcal{T}$. Then we have to show that $U = \bigcup\limits_{\alpha} U_\alpha \in \mathcal{T}$.

  Let $x \in U$. Then $x \in U_\alpha$ for some α. Since U_α is open in X (i.e. $U_\alpha \in \mathcal{T}$), there exists a basis element $B \in \mathcal{B}$ such that $x \in B \subset U_\alpha$. But then $x \in B \subset U$ and hence $U \in \mathcal{T}$.

- Now we show that intersection of finitely many members of $\mathcal{T}$ is in $\mathcal{T}$. Let $U_1, U_2, \ldots, U_n \in \mathcal{T}$. We want to show that $\bigcap\limits_{i=1}^{n} U_i \in \mathcal{T}$. First we show this for two elements of $\mathcal{T}$.

  Let U_1 and U_2 be elements of $\mathcal{T}$. Then we show that $U_1 \cap U_2 \in \mathcal{T}$. Let $x \in U_1 \cap U_2$. Since $U_1 \in \mathcal{T}$, there exists $B_1 \in \mathcal{B}$ such that $x \in B_1 \subset U_1$. Similarly, since $U_2 \in \mathcal{T}$, there exists $B_2 \in \mathcal{B}$ such that $x \in B_2 \subset U_2$. By property (2) of basis, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. Then $x \in B_3 \subset U_1 \cap U_2$ and hence $U_1 \cap U_2 \in \mathcal{T}$.

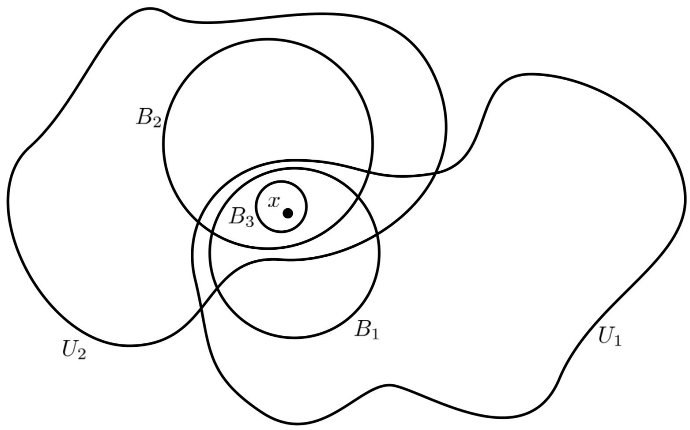


Figure 1.3: Topology generated by basis

Finally, by induction, we show that $U_1 \cap \cdots \cap U_n \in \mathcal{T}$. For $n = 1$ the result is trivial Assume that the result is true for $n-1$. Now,

$$U_1 \cap \cdots \cap U_n = (U_1 \cap \cdots \cap U_{n-1}) \cap U_n.$$

By induction hypothesis, $U_1 \cap \cdots \cap U_{n-1} \in \mathcal{T}$. Also since $U_n \in \mathscr{T}$ and since we have proved the result for $n = 2$ above, it follows that $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

Hence, $\mathcal{T}$ is a topology on X. □

Let us consider some examples of bases.

Example 1.2.4. Let $\mathcal{B}$ be the collection of all circular regions (interior of circles) in the plane $\mathbb{R}^2$. Then $\mathcal{B}$ satisfies both the conditions of basis in Definition 1.2.1. Condition (2) is shown in the figure below, i.e. if $x \in \mathbb{R}^2$ is in the intersection of two circular regions B_1 and B_2, then there is a small circular region B_3 containing x which is contained in $B_1 \cap B_2$.

Similarly, let $\mathcal{B}'$ be the collection of all rectangular regions (interior of rectangles) in the plane $\mathbb{R}^2$, where the rectangles have sides parallel to the axes. Then $\mathcal{B}'$ is a basis for a topology on $\mathbb{R}^2$ as it satisfies both the conditions of basis. Note that in this case, condition (2) is trivially satisfied as intersection of two rectangular regions is also a rectangular region, i.e. we can take $B'_3 = B'_1 \cap B'_2$.

Both the above examples are illustrated in the figure given below.

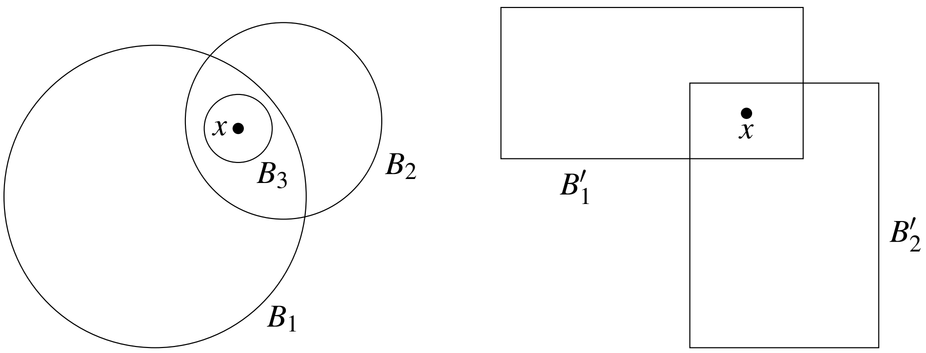


Figure 1.4: Examples of Bases in $\mathbb{R}^2$

■

Example 1.2.5. Let X be a nonempty set. Then the collection of all one-point (i.e. singleton) subsets of X is a basis for the discrete topology on X. ■

The following lemma yields another way of describing topology $\mathcal{T}$ generated by a basis $\mathcal{B}$ which states that an open set (i.e. an element of $\mathcal{T}$) is the union of basis elements.

Lemma 1.2.6

Let X be a set and $\mathcal{B}$ a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.

Proof. Let $\mathcal{T}'$ be the collection of of all unions of members of $\mathcal{B}$. We want to show that $\mathcal{T} = \mathcal{T}'$, where $\mathcal{T}$ is the topology generated by the basis $\mathcal{B}$.

Let $U \in \mathcal{T}$ and $x \in U$. Since $\mathcal{B}$ is a basis for the topology $\mathcal{T}$, there exists a $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. Therefore,

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B_x \subseteq U$$

Thus $U = \bigcup\limits_{x \in U} B_x$ and so $U \in \mathcal{T}'$. Note that, if $U = \emptyset$, then U can be written as the union of empty family of members of $\mathcal{B}$, i.e. $\emptyset = \bigcup\limits_{\alpha \in \emptyset} B_\alpha \in \mathcal{T}'$. Therefore, $\mathcal{T} \subset \mathcal{T}'$.

Conversely, let $U \in \mathcal{T}'$. Then $U = \bigcup\limits_{\alpha} B_\alpha$, where $B_\alpha \in \mathcal{B}$. Since (by the definition of basis) $\mathcal{B} \subset \mathcal{T}$, we have $B_\alpha \in \mathcal{T}$ for all α. Since $\mathcal{T}$ is a topology, $U = \bigcup\limits_{\alpha} B_\alpha \in \mathcal{T}$. Therefore $\mathcal{T}' \subset \mathcal{T}$. □

Remark 1.2.7.

1. What we proved in the above lemma is that every open set U in X can be written as a union of elements of basis and vice versa.

   Thus, in the case where the topology $\mathcal{T}$ on X cannot be specified efficiently, one can describe the basis $\mathcal{B}$ for the topology $\mathcal{T}$ and then the open sets (i.e. members of $\mathcal{T}$) in X can be described as all possible unions of members of the basis $\mathcal{B}$.

2. We have described two different ways of how a topology can be generated from a given basis. What about the converse? Suppose we start with a topological space $(X, \mathcal{T})$. Can we describe a basis which generates the given topology $\mathcal{T}$ on X?

   Recall Definition 1.2.2 of how a topology $\mathcal{T}$ is generated by a basis $\mathcal{B}$. We say that a subset U of X is open if for each $x \in U$, there exists some $B \in \mathcal{B}$ such that $x \in B \subset U$. Now, suppose we have a collection $\mathcal{C}$ of **open** subsets of X satisfying the same condition as $\mathcal{B}$ stated above. Will $\mathcal{C}$ be a basis for the topology $\mathcal{T}$ on X? The answer is affirmative and this is precisely proved in the following lemma.

Lemma 1.2.8

Let $(X, \mathcal{T})$ be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology $\mathcal{T}$ on X.

Proof. First we show that $\mathcal{C}$ is a basis for some topology on X, i.e. $\mathcal{C}$ satisfies the conditions of basis.

1. Let $x \in X$, since X is open, by hypothesis there exists an element C of $\mathcal{C}$ such that $x \in C \subset X$.
2. Let $x \in C_1 \cap C_2$, where $C_1, C_2 \in \mathcal{C}$. Then C_1 and C_2 are open sets of X and hence $C_1 \cap C_2$ is also open. Therefore, by hypothesis there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.

Thus, $\mathcal{C}$ is a basis for some topology on X. Let $\mathcal{T}'$ be the topology on X generated by $\mathcal{C}$. We now show that $\mathcal{T}' = \mathcal{T}$.

- Let $U \in \mathcal{T}$ and $x \in U$. By the hypothesis, there exists $C \in \mathcal{C}$ such that $x \in C \subset U$. Then (by Definition 1.2.2) it follows that $U \in \mathcal{T}'$. Thus, $\mathcal{T} \subset \mathcal{T}'$.
- Let $U \in \mathcal{T}'$. Since $\mathcal{T}'$ is the topology generated by basis $\mathcal{C}$, by previous lemma, $U = \bigcup_{\alpha} C_\alpha$, where $C_\alpha \in \mathcal{C}$. By definition of $\mathcal{C}$, C_α are open subsets of X, i.e. $C_\alpha \in \mathcal{T}$ for all α. Therefore, $U \in \mathcal{T}$ and hence $\mathcal{T}' \subset \mathcal{T}$.

Hence, $\mathcal{C}$ is a basis for topology $\mathcal{T}$ on X. □

Exercise 1.11

Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{C}$ be a collection of open sets of X such that every $U \in \mathcal{T}$ can be written as a union of elements of $\mathcal{C}$. Show that $\mathcal{C}$ is a basis for topology $\mathcal{T}$ on X.

The following result helps to determine which topology is finer or weaker on a set when the topologies are generated by the bases and are comparable.

Lemma 1.2.9

Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on a set X. Then the following are equivalent.

(1) $\mathcal{T}'$ is finer than $\mathcal{T}$, i.e. $\mathcal{T} \subset \mathcal{T}'$.

(2) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

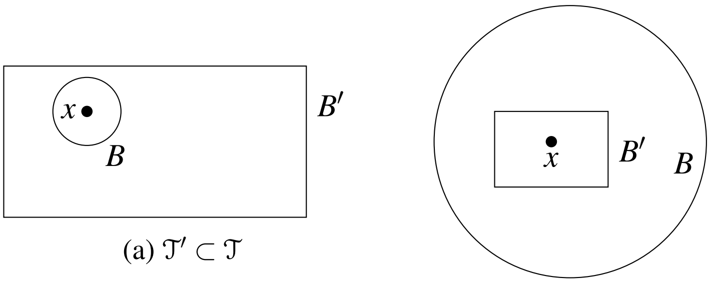
Proof. (1) $\Rightarrow$ (2). Suppose $\mathcal{T}'$ is finer than $\mathcal{T}$, i.e. $\mathcal{T}' \supset \mathcal{T}$.

Let $x \in X$ and $x \in B$ for some $B \in \mathcal{B}$. By definition of the basis, $\mathcal{B} \subset \mathcal{T}$ and $\mathcal{T} \subset \mathcal{T}'$. Therefore, we have $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is generated by the basis $\mathcal{B}'$, there exists some $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

(2) $\Rightarrow$ (1). Assume that (2) holds. We want to show that $\mathcal{T} \subset \mathcal{T}'$.

Let $U \in \mathcal{T}$ and let $x \in U$. Since $\mathcal{B}$ generates the topology $\mathcal{T}$, there exists some $B \in \mathcal{B}$ such that $x \in B \subset U$. By Condition (2), there exists $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. Therefore, $x \in B' \subset U$ and hence by definition, $U \in \mathcal{T}'$. □

Example 1.2.10. Let $\mathcal{B}$ be the collection of all the circular regions (interior of circles) in the plane $\mathbb{R}^2$ and let $\mathcal{B}'$ be the collection of all rectangular regions (interior of rectangles) in the plane. In Example 1.2.4, we have seen that both $\mathcal{B}$ and $\mathcal{B}'$ are bases for some topology on $\mathbb{R}^2$. In fact, they generate the same topology which follows from above lemma and is illustrated in the following figure.



(a) $\mathcal{T}' \subset \mathcal{T}$

(b) $\mathcal{T} \subset \mathcal{T}'$

Figure 1.5: The same topology generated by $\mathcal{B}$ and $\mathcal{B}'$ on $\mathbb{R}^2$

■

Now we define some interesting topologies on the real line $\mathbb{R}$.

Definition 1.2.11: Usual or Standard topology

Let $\mathcal{B} = \{(a,b) \mid a,b \in \mathbb{R}, a < b\}$. Then $\mathcal{B}$ is basis for a topology on $\mathbb{R}$ which is called the *usual* or the *standard topology* on $\mathbb{R}$.

Whenever we consider $\mathbb{R}$ as a topological space, we assume it with the usual topology unless specified. We now verify that $\mathcal{B}$ is a basis.

Example 1.2.12. Let $\mathcal{B}$ be the collection of all open intervals in the real line,

$$(a,b) = \{x \mid a < x < b\}.$$

Then $\mathcal{B}$ is a basis for the standard topology on $\mathbb{R}$. ■

***Solution*.** We verify that $\mathcal{B}$ satisfies both the properties given in Definition 1.2.1.

1. Let $x \in \mathbb{R}$, then $x \in (x-1, x+1)$ and $(x-1, x+1) \in \mathcal{B}$.
2. Let $x \in B_1 \cap B_2$, where $B_1 = (a,b), B_2 = (c,d) \in \mathcal{B}$. Then we have to find some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

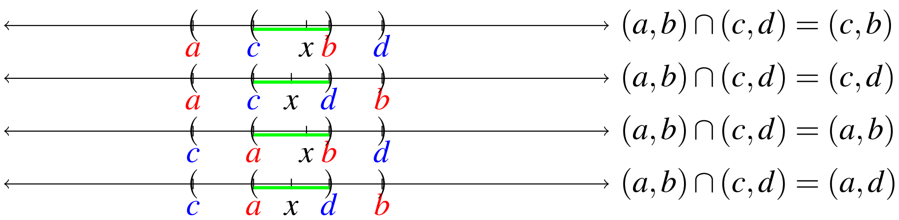


Figure 1.6: Intersection of two intervals.

Take $e = \max\{a,c\}$ and $f = \min\{b,d\}$. Then $a,c \leq e < x < f \leq b,d$. Thus, taking $B_3 = (e,f)$, we have

$$x \in (e,f) \subset (a,b) \cap (c,d) = B_1 \cap B_2.$$

Hence, $\mathcal{B}$ is a basis. □

Definition 1.2.13: Lower limit and Upper limit topology

Let $\mathcal{B}_\ell = \{[a,b) \mid a,b \in \mathbb{R},\ a<b\}$. Then $\mathcal{B}_\ell$ is basis for a topology on $\mathbb{R}$ and the topology generated by $\mathcal{B}_\ell$ is called the *lower limit topology* .

When $\mathbb{R}$ is considered with the lower limit topology, we denote it by $\mathbb{R}_\ell$.

Let $\mathcal{B}_u$ be the collection of half-open intervals of the form

$$(a,b] = \{x \mid a < x \leq b\}.$$

That is, $\mathcal{B}_u = \{(a,b] \mid a,b \in \mathbb{R},\ a<b\}$. Then the topology on $\mathbb{R}$ generated by $\mathcal{B}_u$ is called the *upper limit topology*.

When $\mathbb{R}$ is considered with the upper limit topology, we denote it by $\mathbb{R}_u$.

Definition 1.2.14: K-topology

Let $K = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. Let $\mathcal{B}_K$ be the collection of all open intervals (a,b), along with the sets of the form $(a,b) \setminus K$. The topology generated by $\mathcal{B}_K$ is called the *K-topology* on $\mathbb{R}$.

When $\mathbb{R}$ is considered with the K-topology, we denote it by $\mathbb{R}_K$.

Exercise 1.12

Show that $\mathcal{B}_\ell$, $\mathcal{B}_u$, and $\mathcal{B}_K$ are bases for $\mathbb{R}_\ell$, $\mathbb{R}_u$, and $\mathbb{R}_K$ respectively.

Exercise 1.13

Show that the intersection of two basis elements is either empty or another basis element.

The following lemma gives some relations between the topologies we saw above.

Lemma 1.2.15

The topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are both strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.

Proof. Let $\mathcal{T}$, $\mathcal{T}_\ell$, and $\mathcal{T}_K$ be the topologies of $\mathbb{R}$, $\mathbb{R}_\ell$, and $\mathbb{R}_K$ respectively, where $\mathcal{T}$ is standard topology on $\mathbb{R}$ with basis $\mathcal{B} = \{(a,b) \mid a,b \in \mathbb{R},\ a<b\}$.

Let $(a,b) \in \mathcal{B}$ and $x \in (a,b)$. Then $[x,b) \in \mathcal{B}_\ell$ and we have

$$x \in [x,b) \subset (a,b).$$

Therefore, by Lemma 1.2.9, we have $\mathcal{T} \subset \mathcal{T}_\ell$.On the other hand, let $[x,d)$ be a basis element of $\mathbb{R}_\ell$ containing x. Then there does not exists (a,b) such that $x \in (a,b) \subset [x,d)$ **(Verify!)**. Hence, $\mathcal{T}_\ell$ is strictly finer than $\mathcal{T}$.

Similarly, let (a,b) be a basis element for $\mathcal{T}$ and $x \in (a,b)$. Then the same element (a,b) is in $\mathcal{T}_K$ and we have

$$x \in (a,b) \subseteq (a,b).$$

Therefore, $\mathcal{T} \subset \mathcal{T}_K$. On the other hand, let $B = (-1,1) \setminus K \in \mathcal{T}_K$ but not in $\mathcal{T}$ because there is no open interval (a,b) such that $0 \in (a,b) \subset B$. Therefore, $\mathcal{T}_K$ is strictly finer than $\mathcal{T}$.

Now, we show that $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable. For this, first let $[x,b)$ be a basis element for $\mathcal{T}_\ell$ containing x. Then there does not exists any basis element B for $\mathcal{T}_K$ such that $x \in B \subset [x,b)$. Therefore, $\mathcal{T}_\ell \not\subset \mathcal{T}_K$.

On the other hand, $B = (-1,1) \setminus K$ be the basis element containing 0 for $\mathcal{T}_K$. Then there does not exists any interval of the form $[a,b)$ in $\mathcal{B}_\ell$ such that $0 \in [a,b) \subset B$. Therefore, $\mathcal{T}_\ell \not\subset \mathcal{T}_K$. Hence, $\mathbb{R}_\ell$ and $\mathbb{T}_K$ are not comparable topological spaces. □

1.2.1 Subbasis

We have seen that sometimes instead of specifying the topology $\mathcal{T}$ directly, we describe a smaller collection $\mathcal{B}$ called basis and the topology generated by the basis is the the collection of arbitrary unions of the members of $\mathcal{B}$. It is possible to specify the topology by even a smaller subcollection called subbasis for the topology. More precisely, it is defined as follows.

Definition 1.2.16: Subbasis

Let X be a nonempty set. A collection $\mathcal{S}$ of subsets of X is called a *subbasis* for a topology on X if the union of elements of $\mathcal{S}$ is equal to X, i.e. $\bigcup\limits_{S\in\mathcal{S}} S = X$.

The *topology* $\mathcal{T}$ *generated by the subbasis* $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of members of $\mathcal{S}$.

We show that the collection $\mathcal{T}$ defined by subbasis by taking all unions of finite intersections of elements of $\mathcal{S}$ is a topology on X. Let $\mathcal{B}$ be the collection of finite intersections of elements of $\mathcal{S}$. Then it suffices to show that $\mathcal{B}$ is a basis for for some topology.

Proposition 1.2.17

Let X be a set and $\mathcal{S}$ be a subbasis for a topology on X. Let

$$\mathcal{B} = \{B \subset X \mid B \text{ is intersection of finitely many elements of } \mathcal{S}\}.$$

Then $\mathcal{B}$ is a basis for a topology on X.

Proof. Observe that $\mathcal{S} \subset \mathcal{B}$ as every element $S \in \mathcal{S}$ can be seen as (a finite) intersection with itself, $S = S \cap S$. Hence, $\mathcal{S} \subset \mathcal{B}$.

Now we verify that $\mathcal{B}$, defined above, is a basis.

1. Let $x \in X$. Then by definition of subbasis, $x \in \bigcup\limits_{S\in\mathcal{S}} S$. Therefore there exists some $S \in \mathcal{S}$ such that $x \in S$. But then $S \in \mathcal{S} \subset \mathcal{B}$ and $x \in S$. Thus, the first condition of basis is satisfied.
2. Let $B_1, B_2 \in \mathcal{B}$. Then

$$B_1 = S_1 \cap \cdots \cap S_n \text{ and } B_2 = S'_1 \cap \cdots \cap S'_m$$

for some $m, n \in \mathbb{N}$ and $S_i, S'_j \in \mathcal{S}$, where $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,m$. Then

$$B_1 \cap B_2 = (S_1 \cap \cdots \cap S_n) \cap (S'_1 \cap \cdots \cap S'_m)$$

is also a finite intersection of members of $\mathcal{S}$ and hence $B_1 \cap B_2 \in \mathcal{B}$. Therefore, for $x \in B_1 \cap B_2$ taking $B_3 = B_1 \cap B_2 \in \mathcal{B}$, we have

$$x \in B_3 \subseteq B_1 \cap B_2.$$

Hence, $\mathcal{B}$ is a basis for a topology on X. □

Remark 1.2.18. Summarizing the ways of specifying a topology on a set we have seen so far, we can say that a topology $\mathcal{T}$ on a set X can be described in three ways.

First is specifying the collection $\mathcal{T}$ itself explicitly. The second method is to specify a smaller collection $\mathcal{B}$, called basis for the topology $\mathcal{T}$, from which $\mathcal{T}$ is generated by taking arbitrary union of members of $\mathcal{B}$. Finally, we saw that we can specify an even smaller subcollection $\mathcal{S}$, called the subbasis for the topology $\mathcal{T}$ on X, from which $\mathcal{T}$ can be generated by taking arbitrary unions of finite intersections of members of $\mathcal{S}$. Also, the basis $\mathcal{B}$ for $\mathcal{T}$ can be defined from $\mathcal{S}$ as finite intersections of members of $\mathcal{S}$.

Exercise 1.14

Give an example with proper justification to show that arbitrary intersection of open sets need not be open in the following topologies on $\mathbb{R}$.

1. $\mathcal{T}_c$ = cocountable topology
2. $\mathcal{T}_\ell$ = lower (or upper) limit topology
3. $\mathcal{T}$ = usual topology

Exercise 1.15

Let $\mathcal{T} = \mathcal{T}_\ell \cap \mathcal{T}_u$. Show that $\mathcal{T}$ is the usual topology on $\mathbb{R}$.

Exercise 1.16

Let X be a set and $\mathcal{B}$ be a basis for a topology on X. Show that the topology generated by the basis $\mathcal{B}$ is the smallest (weakest) topology containing $\mathcal{B}$ and it is the intersection of all topologies on X containing $\mathcal{B}$.

Prove the same for a subbasis $\mathcal{S}$.

Exercise 1.17

Let $\mathcal{B}_1 = \{(a,\infty) \mid a \in \mathbb{R}\}$. Show that $\mathcal{B}_1$ is a topology on $\mathbb{R}$. Will $\mathcal{B}_1$ be still a basis if we replace "$a \in \mathbb{R}$" by "$a \in \mathbb{Z}$" or "$a \in \mathbb{Q}$" or "$a \notin \mathbb{Z}$" or "$a \notin \mathbb{Q}$". If so is the case, then will any of them generate the same topology?

Exercise 1.18

Check which of the following collections of subsets of $\mathbb{R}$ forms a basis for a topology on $\mathbb{R}$. If yes, do they generate the usual topology on $\mathbb{R}$?

1. $\mathcal{B}_2 = \{B \subset \mathbb{R} \mid B \cap \mathbb{Q} \neq \emptyset\}$.
2. $\mathcal{B}_3 = \{(a,b) \mid a,b \in \mathbb{Q},\ a < b\}$.
3. $\mathcal{B}_4 = \{(-n,n) \mid n \in \mathbb{N}\}$.
4. $\mathcal{B}_5 = \{(0,n) \mid n \in \mathbb{N}\} \cup \{(-n,0) \mid n \in \mathbb{N}\}$.

Exercise 1.19

Consider the following topologies on $\mathbb{R}$ and compare them with each other. In each of the cases, determine which of it contains any of the others, if they are comparable.

1. $\mathcal{T}$ = standard topology
2. $\mathcal{T}_K$ = K-topology
3. $\mathcal{T}_f$ = cofinite topology
4. $\mathcal{T}_u$ = upper limit topology
5. $\mathcal{T}_\ell$ = lower limit topology
6. $\mathcal{T}_c$ = cocountable topology
7. $\mathcal{T}'$ having basis $\mathcal{B}' = \{(-\infty, a) \mid a \in \mathbb{R}\}$.

Exercise 1.20

Show that the collection

$$\mathcal{C} = \{[a,b) \mid a < b,\ a,b \in \mathbb{Q}\}$$

is a basis for a topology on $\mathbb{R}$ which is **different from the lower limit topology** on $\mathbb{R}$.

Exercise 1.21

Let X be a set. Show that the following are subbases for the discrete topology on X.

1. The collection of all subsets of X having exactly three elements, where $|X| \geq 4$.
2. The collection of all countable subsets of X.
3. The collection of all finite subsets of X.

Exercise 1.22

Let X be a set. Show that the collection

$$\mathcal{S} = \{S \subset X \mid X \setminus S \text{ is singleton}\}$$

is a subbasis for the cofinite topology $\mathcal{T}_f$ on X.

Exercise 1.23

Show that every topology is a basis and every basis is a subbasis.

1.3 Product Topology on $X \times Y$

Let X and Y be two nonempty sets. Then their Cartesian product $X \times Y$ is defined as

$$X \times Y = \{(x,y) \mid x \in X,\ y \in Y\}.$$

Now, suppose X and Y are topological spaces. The natural question would be can we define a topology on $X \times Y$, making it a topological space, in terms of the topology on X and Y. The answer is yes. We sure can define a topology on $X \times Y$ in the following way.

Definition 1.3.1: Product topology on $X \times Y$

Let X and Y be topological spaces. Let $\mathcal{B}$ be the collection of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y. Then $\mathcal{B}$ is a basis for a topology on $X \times Y$ and the topology generated by $\mathcal{B}$ is called the *product topology* on $X \times Y$.

We now verify that $\mathcal{B}$ defined above is a basis.

Proposition 1.3.2

Let X and Y be topological spaces. Let

$$\mathcal{B} = \{U \times V \subset X \times Y \mid U \text{ is open in } X,\ V \text{ is open in } Y\}.$$

Then the collection $\mathcal{B}$ is a basis for a topology on $X \times Y$ called the product topology.

Proof. We show that $\mathcal{B}$ satisfies both the conditions in Definition 1.2.1.

1. Since X is open in X and Y is open in Y, $X \times Y \in \mathcal{B}$. Thereforethe first condition of basis holds trivially.
2. The second conditions is also satisfied as the intersection of any two members of $\mathcal{B}$ is also a member of $\mathcal{B}$. For this, let $U_1 \times V_1$ and $U_2 \times V_2$ be members of $\mathcal{B}$, where U_1, U_2 are open in X and V_1, V_2 are open in Y. Then $U_1 \cap U_2$ is open in X and $V_1 \cap V_2$ is open in Y. Therefore,

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}.$$

Hence, $\mathcal{B}$ is a basis. □

Remark 1.3.3. Note that the collection $\mathcal{B}$ is a basis but not a topology itself. This is because $(U_1 \times V_1) \cup (U_2 \times V_2)$ cannot be written as $U_3 \times V_3$. That is, union of two basis elements is open but it need not be a basis element, i.e. it might not be written as a product of open sets of X and Y. This is pictorially shown in the following figure, where we can see that union of two rectangles is not a rectangle.

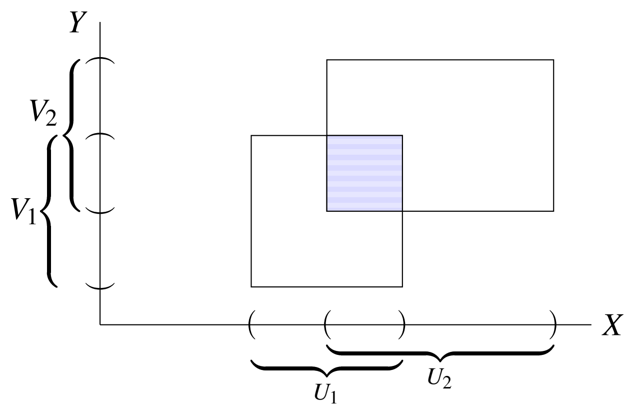


Figure 1.7: Union of basic open sets

As seen above, given topologies on X and Y, we can specify the product topology on $X \times Y$ by describing its basis in terms of topologies on X and Y. What if topologies on X and Y were itself specified using some basis, can we still specify the basis for product topology on $X \times Y$? In other words, suppose topologies on X and Y are generated by bases $\mathcal{B}$ and $\mathcal{C}$ respectively. How do we specify basis for the product topology on $X \times Y$ in terms of $\mathcal{B}$ and $\mathcal{C}$? The following theorem gives an answer to this.

Theorem 1.3.4

If $\mathcal{B}$ is a basis for the topology on X and $\mathcal{C}$ is a basis for the topology on Y, then the collection

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the product topology on $X \times Y$.

Proof. Let W be an open subset of $X \times Y$ and $(x,y) \in W$. Then by definition of the product topology, there exists a basis element $U \times V$ such that

$$(x,y) \in U \times V \subset W,$$

where U is open in X and V is open in Y. Since $\mathcal{B}$ and $\mathcal{C}$ are bases for topology on X and Y respectively, there exists $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $x \in B \subset U$ and $y \in C \subset V$. Then by definition, $B \times C \in \mathcal{D}$ and

$$(x,y) \in B \times C \subset W.$$

Therefore, by the Lemma 1.2.8, $\mathcal{D}$ is a basis for the product topology on $X \times Y$. □

Example 1.3.5. We have seen the standard (i.e. the usual) topology on $\mathbb{R}$. The product of this topology with itself gives the product topology on $\mathbb{R}^2$. By Proposition 1.3.2, it has a basis consisting of all product $U \times V$, where U, V open in $\mathbb{R}$. Due to Theorem 1.3.4, we can give even a smaller subcollection as its basis, which is given by products $(a,b) \times (c,d)$ of all open intervals in $\mathbb{R}$.

Each basis element for the product topology of $\mathbb{R}^2$ can be pictured as a rectangular region, i.e. interior of a rectangle as seen in Example 1.2.4. Also, as seen in Example 1.2.10, the basis consisting of all circular regions (i.e. interior of circular region or open balls) in $\mathbb{R}^2$ also generates the (same) product topology on $\mathbb{R}^2$. ■

We have seen that a topology on a set can be specified in three different ways. We can specify the topology itself or a smaller subcollection called basis which generates the topology or an even smaller subcollection called subbasis whose finite intersection generates the basis for the topology. So far we have described the product topology on $X \times Y$ in terms of its basis. Next, we shall do the same in terms of subbasis. For this, first we define certain maps on $X \times Y$ called projections.

Definition 1.3.6: Projections

Let X and Y be two sets. Let $\pi_1 : X \times Y \to X$ be defined by

$$\pi_1(x,y) = x$$

and let $\pi_2 : X \times Y \to Y$ be defined by

$$\pi_2(x,y) = y.$$

The maps π_1 and π_2 are called the *projections* of $X \times Y$ onto its first and second factor respectively.

Observe that projections are onto maps.

Theorem 1.3.7

Let X and Y be two sets. The collection

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

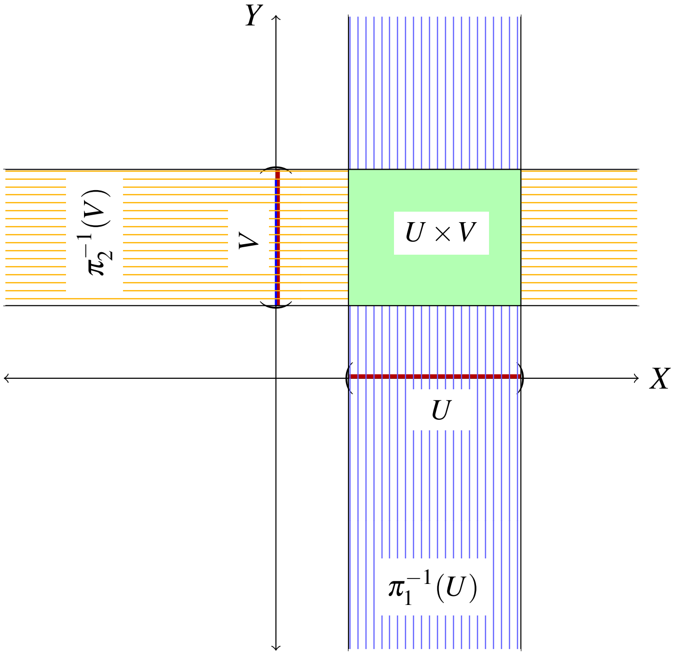


Figure 1.8: $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$

Proof. Since X is open in X and $\pi_1^{-1}(X) = X \times Y$, we have $X \times Y \in \mathcal{S}$. So, the union of elements of $\mathcal{S}$ is $X \times Y$. Hence, $\mathcal{S}$ is a subbasis for a topology on $X \times Y$.

Now, we show that $\mathcal{S}$ generates the product topology on $X \times Y$. Let $\mathcal{T}$ denote the product topology on $X \times Y$ and $\mathcal{T}'$ denote the topology generated by $\mathcal{S}$. We have to show that $\mathcal{T} = \mathcal{T}'$.

If $U \subset X$ is open, then $\pi_1^{-1}(U) = U \times Y$ is open in $X \times Y$ with respect to $\mathcal{T}'$. Similarly, if V is an open subset of Y, then $\pi_1^{-1}(V) = X \times V$ is open in $X \times Y$. Thus, every element of $\mathcal{S}$ is an element of $\mathcal{T}$. Since $\mathcal{T}$ is a topology, the arbitrary unions of finite intersections of elements of $\mathcal{S}$ also belong to $\mathcal{T}$ and so $\mathcal{T}' \subset \mathcal{T}$.

On the other hand, let $U \times V$ be a basis element for the product topology $\mathcal{T}$ on $X \times Y$. Then

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V).$$

Thus, $U \times V$ is a finite intersection of elements of $\mathcal{S}$ and so $U \times V \in \mathcal{T}'$. Therefore, $\mathcal{T} \subset \mathcal{T}'$. $\square$

Exercise 1.24

Let $X_1 = \{1,2,3\}$, $\mathcal{T}_1 = \{\emptyset, X_1, \{1\}, \{1,2\}\}$, $X_2 = \{a,b,c\}$, and $\mathcal{T}_2 = \{\emptyset, X_2, \{a,b\}, \{b,c\}, \{b\}\}$.

Determine the product topology on $X_1 \times X_2$, by listing the basis elements and by listing all the open sets.

Exercise 1.25

Show that

1. Product of two discrete topological spaces is a discrete topological space.
2. Product of two indiscrete topological spaces is an indiscrete topological space.
3. Product of two cofinite topological spaces need not be a cofinite topological space.

1.4 The Subspace Topology

Let $(X,\mathcal{T})$ be a topological space and Y be some subset of X. We can make Y a topological space by defining topology on Y in terms of the topology $\mathcal{T}$ on X. Such an inherited topology is called the subspace topology. More precisely,

Definition 1.4.1: Subspace topology

Let X be a topological space with a topology $\mathcal{T}$ and let Y be a subset of X. Let

$$\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}.$$

The collection $\mathcal{T}_Y$ is a topology on Y, called the *subspace topology* and Y is called a *subspace* of X. Thus, the open sets of Y are all intersections of open sets of X with Y.

We now show that the above collection $\mathcal{T}_Y$ is in fact a topology on Y.

Proposition 1.4.2

Let $(X,\mathcal{T})$ be a topological space and $Y \subset X$. Then the collection

$$\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}$$

is a topology on Y.

Proof.

- Clearly $\emptyset, Y \in \mathcal{T}_Y$ since $\emptyset, X \in \mathcal{T}$ and $\emptyset = \emptyset \cap Y$ and $Y = X \cap Y$.
- Let $\{G_\alpha \mid \alpha \in \Lambda\}$ be a family of elements of $\mathcal{T}_Y$. Then by the definition of $\mathcal{T}_Y$, for every α, there exits $U_\alpha \in \mathcal{T}$ such that $G_\alpha = U_\alpha \cap Y$. Now,

$$\bigcup_{\alpha\in\Lambda} G_\alpha = \bigcup_{\alpha\in\Lambda} (U_\alpha \cap Y) = \left(\bigcup_{\alpha\in\Lambda} U_\alpha\right) \cap Y.$$

  Since $\mathcal{T}$ is a topology on X, $\bigcup\limits_{\alpha\in\Lambda} U_\alpha \in \mathcal{T}$. Therefore, it follows that $\bigcup\limits_{\alpha\in\Lambda} G_\alpha \in \mathcal{T}_Y$. Thus, $\mathcal{T}_Y$ is closed under arbitrary union.
- Let $G_1, G_2, \ldots, G_n \in \mathcal{T}_Y$. Then there exist $U_1, U_2, \ldots, U_n \in \mathcal{T}$ such that $G_i = U_i \cap Y$ for all $i = 1, 2, \ldots, n$. Also,

$$\bigcap_{i=1}^{n} G_i = \bigcap_{i=1}^{n} (U_i \cap Y) = \left(\bigcap_{i=1}^{n} U_i\right) \cap Y.$$

Since $\mathcal{T}$ is a topology on X, $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$. Therefore, it follows that $\bigcap_{i=1}^{n} G_i \in \mathcal{T}_Y$. Thus, $\mathcal{T}_Y$ is closed under finite intersection.

Hence, $\mathcal{T}_Y$ is a topology on Y. □

Thus, the elements of topology on Y are obtained as intersection of elements in the topology on X with Y. The same is true for the basis too, i.e. given a basis $\mathcal{B}$ for a topology $\mathcal{T}$ on X, the elements of the basis for the subspace topology $\mathcal{T}_Y$ on Y are obtained as intersection of elements of $\mathcal{B}$ with Y. We have the following lemma.

Lemma 1.4.3

Let X be a topological space and Y be a subset of X. If $\mathcal{B}$ is a basis for the topology of X, then the collection

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof. Let G be a subset in Y which is open in the subspace topology on Y and $y \in G$. Then there exists an open subset U of X such that $G = U \cap Y$. Now, $y \in U \cap Y$ and so there exists $B \in \mathcal{B}$ such that $y \in B \subset U$. Therefore,

$$y \in B \cap Y \subset U \cap Y = G,$$

where $B \cap Y \in \mathcal{B}_Y$. Hence, by the Lemma 1.2.8, $\mathcal{B}_Y$ is a basis for the subspace topology on Y. □

Given a topological space X and its subspace Y, when we use the term "an open set", we have to be more specific in the context that is in the topology of Y or in the topology of X. Thus, if Y is a subspace of X, we shall use the term a set U is open in Y if it belongs to the topology of Y and we say that U is open in X if U belongs to the topology on X.

It is not always true that a set which is open in (the subspace topology on) Y is also open in X. Let us consider couple of examples of the subspace topology to understand this.

Example 1.4.4. Let $X = \mathbb{R}$ with standard topology and $Y = [0,1]$ be subspace of $\mathbb{R}$ with the subspace topology. The basis elements of the subspace topology on Y are of the form $(a,b) \cap Y$, where (a,b) is an open interval (basis element) in $\mathbb{R}$. Such a basis element is one of the following type:

$$(a,b) \cap Y = \begin{cases} (a,b) & \text{if } a \text{ and } b \text{ are in } Y, \\ [0,b) & \text{if only } b \text{ is in } Y, \\ (a,1] & \text{if only } a \text{ is in } Y, \\ Y \text{ or } \emptyset & \text{if neither } a \text{ nor } b \text{ is in } Y. \end{cases}$$

By definition, each of the above sets is open in Y. Note that the sets of the type $[0,b)$ and $(a,1]$ are not open in $X = \mathbb{R}$ as they cannot be written as union of basis elements (open intervals) of $\mathbb{R}$ with usual topology. ■

Example 1.4.5. Consider $\mathbb{R}$ with the standard topology and let $Y = [0,1) \cup \{2\}$ be the subset of $\mathbb{R}$ with the subspace topology. Then the set $\{2\}$ is open in Y as it is the intersection of the open set $\left(\frac{3}{2}, \frac{5}{2}\right)$ in $\mathbb{R}$ with Y. But $\{2\}$ is not open in $\mathbb{R}$. ■

Thus, from the above two examples, it is clear that a set which is open in Y need not be open in X. However, in a special situation if a set is open in Y, then it is open in X too. Consider the following lemma.

Lemma 1.4.6

Let X be a topological space and Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof. Since U is open in Y, there is a set V open in X such that $U = V \cap Y$. Since Y is open in X and V is open in X, $U = V \cap Y$ is open in X. □

Suppose X and Y are topological spaces and $A \subset X$, $B \subset Y$. Then A and B are also topological spaces with the subspace topology. Therefore, $A \times B$ also becomes a topological space with product topology which is given by product of subspace topology on A with the subspace topology on B.

On the other hand, $X \times Y$ is a topological space with the product topology and $A \times B \subset X \times Y$. Therefore, $A \times B$ becomes a topological space with the subspace topology inherited from the product topology of $X \times Y$.

A natural question here is: are these two topologies on $A \times B$ the same? In fact, they are the same and we have the following theorem.

Theorem 1.4.7

Let X and Y be topological spaces, A be a subspace of X, and B be a subspace of Y. Then the product topology on $A \times B$ is the same as the topology which $A \times B$ inherits as a subspace of $X \times Y$.

Proof. We prove the result by showing that the basis for the subspace topology on $A \times B$ is the same as the basis for the product topology on $A \times B$.

Let $U \times V$ be the general basis element of product topology on $X \times Y$, where U is open in X and V is open in Y. Therefore, $(U \times V) \cap (A \times B)$ is the general basis element for the subspace topology on $A \times B$. Now,

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B).$$

Since $U \cap A$ and $V \cap B$ are general open sets for the subspace topology on A and B respectively, the set $(U \cap A) \times (V \cap B)$ is the general basis element for the product topology on $A \times B$.

Hence, the bases for the product topology and for the subspace topology on $A \times B$ are same. Therefore, the topologies are the same. □

Exercise 1.26

Let Y be a subspace of a topological space X and $Z \subset Y$. Show that the topology Z inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Exercise 1.27

If $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on X such that $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, and $Y \subset X$, then what can be said about the corresponding subspace topologies on Y?

Exercise 1.28

Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y and which of them are open in $\mathbb{R}$?

$$\begin{aligned} A &= \left\{x \mid \tfrac{1}{2} < |x| < 1\right\}, \\ B &= \left\{x \mid \tfrac{1}{2} < |x| \leq 1\right\}, \\ C &= \left\{x \mid \tfrac{1}{2} \leq |x| < 1\right\}, \\ D &= \left\{x \mid \tfrac{1}{2} \leq |x| \leq 1\right\}, \\ E &= \left\{x \mid 0 < |x| < 1 \text{ and } \tfrac{1}{x} \notin \mathbb{N}\right\}. \end{aligned}$$

Exercise 1.29

Let X and Y be topological spaces. A map $f : X \to Y$ is said to be an *open map* if for every open set U of X, the set $f(U)$ is open in Y. Show that the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

Exercise 1.30

Show that the countable collection

$$\mathcal{B} = \{(a,b) \times (c,d) \mid a < b \text{ and } c < d, \text{ and } a,b,c,d \in \mathbb{Q}\}$$

is a basis for $\mathbb{R}^2$. (On $\mathbb{R}^n$, we assume the product of usual topology is nothing is mentioned).

1.5 Closed Sets and Limit Points

1.5.1 Closed Sets

Definition 1.5.1: Closed set

Let $(X, \mathcal{T})$ be a topological space. A subset A of X is said to be *closed* if the set $X \smallsetminus A$ is open i.e., if $X \smallsetminus A \in \mathcal{T}$.

Example 1.5.2. The set $[a,b]$ is a closed subset of $\mathbb{R}$ because its complement

$$\mathbb{R} \smallsetminus [a,b] = (-\infty, a) \cup (b, \infty)$$

is open. Similarly, $[a, \infty)$ is closed as its complement $(-\infty, a)$ is open.

Note that the set $[a,b)$ is neither open nor closed. ■

Example 1.5.3. The set $\{(x,y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } y \geq 0\}$ is closed as its complement

$$((-\infty, 0) \times \mathbb{R}) \cup (\mathbb{R} \times (-\infty, 0))$$

is open in $\mathbb{R}^2$. The above set is open because it is the union of two open sets. Each of these two sets are open as they are the product of open sets in $\mathbb{R}$, i.e. $(-\infty, 0)$ and $\mathbb{R}$. ■

Example 1.5.4. The closed subsets of X with cofinite topology are X itself and all the finite subsets of X. ■

Example 1.5.5. Let X be a set with the discrete topology. Then we know that every subset of X is open. Hence, every subset of X is closed. ■

Example 1.5.6. Consider the set $Y = [0,1] \cup (2,3)$ with subspace topology of $\mathbb{R}$. Since $\left(-\frac{1}{2}, \frac{3}{2}\right)$ is open in $\mathbb{R}$ and $[0,1] = \left(-\frac{1}{2}, \frac{3}{2}\right) \cap Y$, it follows that $[0,1]$ is open in Y. Similarly, $(2,3)$ is open in Y.

Note that, $[0,1]$ and $(2,3)$ are complements of each other in Y. Thus, both $[0,1]$ and $(2,3)$ are closed in Y (as well as open). ■

Example 1.5.7. Any finite subset of $\mathbb{R}$ is closed.

Let $A = \{a_1, a_2, \ldots, a_n\}$ be subset of $\mathbb{R}$. Without loss of generality, we may assume that $a_1 < a_2 < \cdots < a_n$. Then the set A is closed because its complement

$$\mathbb{R} \setminus A = (-\infty, a_1) \cup (a_1, a_2) \cup \cdots \cup (a_{n-1}, a_n) \cup (a_n, \infty)$$

is open. ■

From the above examples, it is clear that a set can be open, or closed, or both open as well closed, or neither open nor closed.

We called a set open if it belongs to the topology and hence we saw that open sets satisfy the properties of a topology, i.e. $\emptyset$, X are open; arbitrary union of open sets is open; and intersection of finitely many open sets is an open set. Closed subsets of a topological space satisfy similar properties.

Theorem 1.5.8

Let $(X, \mathcal{T})$ be a topological space. Then the following conditions hold:

(1) $\emptyset$ and X are closed.
(2) Arbitrary intersection of closed set is closed.
(3) Finite union of closed set is closed.

Proof. (1) $\emptyset$ and X are closed because they are complements of open sets X and $\emptyset$ respectively.

(2) Let $\{A_\alpha \mid \alpha \in \Lambda\}$ be a family of closed subsets of X. We want to show that $\bigcap\limits_{\alpha \in \Lambda} A_\alpha$ is closed. By De Morgan's law,

$$X \setminus \bigcap_{\alpha \in \Lambda} A_\alpha = \bigcup_{\alpha \in \Lambda} (X \setminus A_\alpha).$$

Since A_α are closed sets, $X \setminus A_\alpha$ are open. Arbitrary union of open sets being open, by above, $X \setminus \bigcap\limits_{\alpha \in \Lambda} A_\alpha$ is open and hence $\bigcap\limits_{\alpha \in \Lambda} A_\alpha$ is closed.

(3) Suppose $A_1, A_2, \ldots, A_n$ are closed. We want to show that $\bigcup\limits_{i=1}^{n} A_i$ is closed. We have

$$X \setminus \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X \setminus A_i).$$

Since A_i are closed, $X \setminus A_i$ are open sets. Finite intersection of open sets being open, $X \setminus \bigcup_{i=1}^{n} A_i$ is open and hence $\bigcup_{i=1}^{n} A_i$ is closed.

□

Exercise 1.31

Show by giving an example that arbitrary union of closed sets need not be a closed.

We can also define topology on a set by means of closed sets.Then we can define open sets to be the complement of the closed sets.

Let X be a topological space and Y be a subspace of X with the subspace topology. Clearly a subset A of Y is said to be *closed in* Y if and only if its complement $Y \setminus A$ is open in the subspace topology on Y. Recall that open sets in Y with the subspace topology were defined to be the intersection of open sets of X with Y. The same is true for the closed sets and we have the following lemma.

Theorem 1.5.9

Let X be a topological space, Y be a subspace of X and $A \subset Y$. The set A is closed in Y if and only if it is the intersection of a closed set of X with Y.

Proof. Assume that A is intersection of a closed subset of X with Y, i.e. suppose that C is closed in X and $A = C \cap Y$. Then $X \setminus C$ is open in X. Therefore, by the definition of subspace topology, $(X \setminus C) \cap Y$ is open in Y. But

$$Y \setminus A = Y \setminus (C \cap Y) = (X \setminus C) \cap Y.$$

Hence $Y \setminus A$ is open in Y and so A is closed in Y.

Conversely, suppose A is closed in Y. Then $Y \setminus A$ is open in Y and so by definition, there is an open set U of X such that

$$Y \setminus A = U \cap Y.$$

But then $A = (X \setminus U) \cap Y$ and $X \setminus U$ is closed in X. Thus, A is intersection of a closed subset of X with Y. □

Let X be a topological space and Y be a subspace of X. From the examples we have seen earlier, a set A that is closed in Y need not be closed in X. For instance, in Example 1.5.6 the set $A = (2,3)$ is closed in $Y = [0,1] \cup (2,3)$ but it is not closed in $X = \mathbb{R}$. As in the case of open sets, there is a similar criterion for A to be closed in X.

Theorem 1.5.10

Let X be a topological space, Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

Proof. Since A is closed in Y, by above theorem, $A = C \cap Y$ for some closed set C of X. Since C is closed in X and Y is closed in X, $A = C \cap Y$ is closed in X. □

Alternative proof. Alternatively, since A is closed in Y, the set $Y \setminus A$ is open in Y. Therefore, by the definition of subspace topology, $Y \setminus A = U \cap Y$ for some open set U of X. Then

$$A = Y \setminus (U \cap Y) = (X \setminus U) \cap Y.$$

Since U is open in X, $X \setminus U$ is closed in X. Also since Y is closed in X, from the above expression, we say that A is closed in X. □

Exercise 1.32

Determine which of the following sets are either open or closed in the respective topologies. If yes, are they the only proper closed sets?

1. Finite subsets of a cofinite topological space.
2. Finite subsets of $\mathbb{R}_\ell$.
3. $(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ in $\mathbb{Q}$.
4. $(2,3) \cap \mathbb{Q}$ in $\mathbb{Q}$.
5. Any subset of $\mathbb{Z}$ with the subspace topology.
6. (a) $\mathbb{Z}$ (b) $\mathbb{N}$ (c) $\mathbb{Q}$ (d) $[0,1)$

   in $\mathbb{R}$ with the standard topology and in $\mathbb{R}$ with lower limit topology.

Exercise 1.33

Give an example of a topological space in which not all one point sets are closed, i.e. some singleton subsets are not closed.

Exercise 1.34

Give an example of an open subset A of $[0,1]$ which is not open in $\mathbb{R}$ (with usual topology). Give an example of a closed subset B of $(0,1)$ which is not closed in $\mathbb{R}$.

1.5.2 Closure and Interior of a Set

Definition 1.5.11: Interior and Closure

Let X be a topological space and $A \subset X$. The *interior* of A is denoted by A° or $\operatorname{Int}(A)$ and is defined as the union of all open sets contained in A.

The *closure* of A is denoted by $\overline{A}$ or $\operatorname{cl}(A)$ and is defined as the intersection of all closed sets containing A.

Remark 1.5.12. From the definition of interior and closure, the following are clear and immediate observations.

1. A° is an open set and $\overline{A}$ is a closed set.
2. $A^\circ \subset A \subset \overline{A}$.
3. A° is the largest open set in X contained in A. Similarly, $\overline{A}$ is the smallest closed set in X containing A.

4. If A is open, then $A^\circ = A$. Similarly, if A is closed, then $\overline{A} = A$.

Exercise 1.35

(Prove Remark 1.5.12 (4)). Let X be a topological space and $A \subset X$. Prove that A is open if and only if $A^\circ = A$. Also prove that A is closed if and only if $\overline{A} = A$.

Note that whenever X is a topological space, Y is a subspace of X and $A \subset Y$, we have to be specific when we say closure of A. This is because the closure of A in X (i.e. as a subset of X) need not be same as the closure of A in Y (i.e. as a subset of Y). We use the notation $\overline{A}$ for the closure of A in the larger space X. The following theorem shows that closure of A in Y can be expressed in terms of $\overline{A}$.

Theorem 1.5.13

Let X be a topological space and Y be a subspace of X. Let A be a subset of Y and $\overline{A}$ denote the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.

Proof. Let B denote the closure of A in Y.

- Since $\overline{A}$ is closed in X, the set $\overline{A} \cap Y$ is closed in Y. Since $A \subset Y$, $\overline{A} \cap Y$ contains A. But by definition B is the intersection of all closed subsets of Y containing A. Therefore, we must have
$$B \subset \overline{A} \cap Y.$$
- On the other hand, B is closed in Y. Therefore, $B = C \cap Y$ for some set C closed in X. Also $A \subset B = C \cap Y$. Thus, C is a closed subset of X containing A. By definition, $\overline{A}$ is the intersection of all closed subsets of X containing A. Therefore, $\overline{A} \subset C$ and hence, we must have
$$\overline{A} \cap Y \subset C \cap Y = B.$$

Hence, $B = \overline{A} \cap Y$. in other words, closure of A in Y is closure of A in X intersection with Y. □

The above theorem is not true in case of interior of a set. Attempt the following exercise.

Exercise 1.36

Let Y be a subspace of a topological space X and $A \subset Y$. Show by giving an example that interior of A in Y need not be equal to interior of A in X intersection with Y, i.e. $A^\circ \cap Y$.

The definition of closure of a set is not much useful in finding the closures of sets as it is the intersection of all closed sets containing A, which may be a very big number. The following theorem describes closure of a set in another way, which involves open sets and basis for the topology.

Theorem 1.5.14

Let X be a topological space and $A \subset X$.

(1) $x \in \overline{A}$ if and only if every open set U containing x intersects A.

(2) If the topology of X is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.

Proof. (1) We shall prove that

$$x \notin \overline{A} \Leftrightarrow \text{there exists an open set } U \text{ containing } x \text{ that does not intersect } A.$$

Let $x \notin \overline{A}$. Take $U = X \setminus \overline{A}$. Then $x \in U$ and since $\overline{A}$ is closed, U is open. Also, $U \cap A = \emptyset$.

Conversely, suppose there exists an open set U such that $x \in U$ and $U \cap A = \emptyset$. Then $X \setminus U$ is a closed set containing A and $x \notin (X \setminus U)$. But since $\overline{A}$ is the intersection of all closed sets of X containing A, we must have $\overline{A} \subset (X \setminus U)$. Therefore, $x \notin \overline{A}$.

(2) Let $x \in \overline{A}$. Then as above, every open set containing x intersects A. Since every basis element is an open set, every basis element containing x intersects A.

Conversely, assume that every basis element containing x intersects A. Let U be an open set and $x \in U$. Then there exists a basis element B such that $x \in B \subset U$. Then $U \cap A \neq \emptyset$ since $B \cap A \neq \emptyset$. Thus every open set containing x intersects A. So, by (1), $x \in \overline{A}$.

□

As it is usually preferred, instead of saying "U is an open set containing x", we would use the phrase "U is a neighborhood of x". A neighborhood of an element means and open set containing that element. In this terminology, the above result can be restated as follows.

Let X be a topological space and $A \subset X$. Then

$$x \in \overline{A} \text{ if and only if every neighborhood of } x \text{ intersects } A.$$

Example 1.5.15. Let $X = \mathbb{R}$ and $A = (0,1]$ be a subset of $\mathbb{R}$. Since every neighborhood of 0 intersects A and every point of $[0,1]$ has a neighborhood intersecting A, we conclude that $\overline{A} = [0,1]$. By a similar argument, we have

$$B = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \qquad \overline{B} = \{0\} \cup B$$
$$C = \{0\} \cup (1,2) \qquad \overline{C} = \{0\} \cup [1,2]$$
$$D = \mathbb{Q} \qquad \overline{D} = \mathbb{R}$$
$$E = \mathbb{N} \qquad \overline{E} = \mathbb{N}$$
$$F = \mathbb{R}_+ \qquad \overline{F} = \mathbb{R}_+ \cup \{0\}$$

■

Example 1.5.16. Let $X = \mathbb{R}$ and $Y = (0,1]$ be a subspace of X. The set $A = \left(0, \frac{1}{2}\right)$ is a subset of Y. Then $\overline{A}$ (in $\mathbb{R}$) is the set $\left[0, \frac{1}{2}\right]$ and the closure of A in Y is the set $\left[0, \frac{1}{2}\right] \cap Y = \left(0, \frac{1}{2}\right]$. ■

Exercise 1.37

Let $X = \{1,2,3,4\}$, $\mathcal{T} = \{\emptyset, X, \{1\}, \{2,3\}, \{1,2,3\}\}$. Find the interior and closure of $A = \{1,4\}$ and $B = \{1,2\}$.

Exercise 1.38

Find the closures and interiors of $[0,1], (0,1), [0,1), (0,1], \mathbb{Z}, \mathbb{Q}$, and $\{\frac{1}{n} : n \in \mathbb{N}\}$ in $\mathbb{R}$ with respect to each of the topologies:

1. cofinite,
2. cocountable,
3. usual
4. upper limit.

Exercise 1.39

Let X be a topological space and $A, B \subset X$. Answer the following.

1. If $A \subset B$, then show that $A^\circ \subset B^\circ$. Does the converse hold?
2. If $A \subset B$, then show that $\overline{A} \subset \overline{B}$. Does the converse hold?
3. Show that $(A \cap B)^\circ = A^\circ \cap B^\circ$ and $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
4. $X \setminus \overline{A} = (X \setminus A)^\circ$ and $X \setminus A^\circ = \overline{X \setminus A}$.
5. Prove or disprove: $(A \cup B)^\circ = A^\circ \cup B^\circ$.
6. Prove or disprove: $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
7. Prove or disprove: $\overline{A \setminus B} = \overline{A} \setminus \overline{B}$.

Exercise 1.40

Let X and Y be topological spaces, and $A \subset X$ and $B \subset Y$. In the space $X \times Y$ with the product topology, show that

$$\overline{A \times B} = \overline{A} \times \overline{B} \quad \text{and} \quad (A \times B)^\circ = A^\circ \times B^\circ.$$

Exercise 1.41

Let X and Y be topological spaces, and $A \subset X$ and $B \subset Y$. If A is closed in X and B is closed in Y, then show that $A \times B$ is closed in the space $X \times Y$ with the product topology. The converse holds for nonempty sets A and B.

1.5.3 Limit Points

There is one more way to define the closure of a set which involves the notion of a limit point.

Definition 1.5.17: Limit Point

Let X be a topological space, $A \subset X$ and $x \in X$. We say that x is a *limit point* (or *cluster point* or *point of accumulation*) of A if every neighborhood of x intersects A in some point other than x.

The set of all limit points of A is denote by A'.

In other words, x is a limit point of A if it is in the closure of $A \setminus \{x\}$, i.e. $x \in \overline{A \setminus \{x\}}$. Note that the point x may or may not belong to A.

Example 1.5.18. Let $X = \mathbb{R}$ and $A = (0,1]$ be a subset of X. Since every neighborhood of 0 contains a point of A other than 0, we can say that 0 is a limit point of A. Similarly, $\frac{1}{2}$ is a limit point of A. In fact, every point of $[0,1]$ is a limit point of A and no other point is a limit point of A (**Check!**). Hence, $A' = [0,1]$.

Let $B = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. Then 0 is a limit point of B because every neighborhood of 0 contains points $\frac{1}{n}$ for sufficiently large n. Note that no other point of B is a limit point of B. This is because

$$\frac{1}{n} \in \left(\frac{1}{n+1}, \frac{1}{n-1}\right) \quad \text{and} \quad \left(\frac{1}{n+1}, \frac{1}{n-1}\right) \bigcap \left(B \setminus \left\{\frac{1}{n}\right\}\right) = \emptyset.$$

As above, no other point of $\mathbb{R}$ is a limit point of B (**Check!**). Thus, 0 is the only limit point of B and so $B' = \{0\}$. Similarly, we have

$$\begin{aligned} C &= \{0\} \cup (1,2) & C' &= [1,2] \\ D &= \mathbb{Q} & D' &= \mathbb{R} \\ E &= \mathbb{N} & E' &= \emptyset \\ F &= \mathbb{R}_+ & F' &= \mathbb{R}_+ \cup \{0\} \end{aligned}$$

■

It can be observed from Examples 1.5.15 and 1.5.18 that there is a relation between the closure of a set and the set of limit points of the set. The relation is given by the following result.

Theorem 1.5.19

Let X be a topological space, and $A \subset X$. Then

$$\overline{A} = A \cup A'.$$

Proof. If $x \in A'$, then every neighborhood of x intersects A in a point other than x. So, by Theorem 1.5.14, $x \in \overline{A}$. Thus, $A' \subset \overline{A}$. By definition, $A \subset \overline{A}$. Thus,

$$A \cup A' \subset \overline{A}.$$

Conversely, let $x \in \overline{A}$. Then we have to show that $x \in A \cup A'$, i.e. either $x \in A$ or $x \in A'$. If $x \in A$, then we are done. Suppose $x \notin A$. Since $x \in \overline{A}$, every neighborhood U of x intersect A. But $x \notin A$. Therefore, U intersects A in a point different from x. Thus, $x \in A'$ and so we have

$$\overline{A} \subset A \cup A'.$$

□

Corollary 1.5.20

A subspace of a topological space X is closed if and only if it contains all its limit points.

Proof. The set A is closed if and only if $A = \bar{A}$, and this holds if and only if $A' \subset A$. □

Exercise 1.42

Find the limit points of $(0,1)$, $[0,1]$, $[0,1)$, $\mathbb{N}$, $\mathbb{Q}$ in $\mathbb{R}$ with the following topologies on $\mathbb{R}$.

1. usual topology
2. lower limit topology
3. cocountable topology
4. cofinite topology
5. $\mathcal{T} = \{G \subset X : G = \emptyset, G = \mathbb{R} \text{ or } G = (a,\infty), a \in \mathbb{R}\}$.

Exercise 1.43

In each of the following find A' in the given topology. $A = \{1,2\}$ in $X = \{1,2,3,4\}$ with the topology $\mathcal{T} = \{\emptyset, X, \{1\}, \{1,2\}, \{1,2,3\}\}$.

1. usual topology,
2. lower limit topology,
3. cocountable topology,
4. cofinite topology
5. $\mathcal{T}_K$.

Exercise 1.44

Give examples of the following subsets of $\mathbb{R}$.

1. A countable set without any limit point.
2. A countable closed set having finitely many limit points.
3. A countable closed set having infinitely many limit points.
4. A countable set having countably many limit points which is not closed.
5. A countable set having uncountably many limit points.

Exercise 1.45

Give examples of the following subsets A of some topological space in which $A' \neq \emptyset$.

1. $A' \subset X \setminus A$
2. $A' \subset A$
3. $A' \not\subset A, A' \not\subset X \setminus A$.

1.5.4 Boundary of a Set

Definition 1.5.21: Boundary of a set

Let X be a topological space and $A \subset X$. The *boundary* of A is denote by $\partial(A)$ or $\operatorname{bd}(A)$ or $\operatorname{Bd}(A)$ and is defined to be the set $\overline{A} \cap \overline{(X \setminus A)}$, i.e.

$$\operatorname{Bd}(A) = \overline{A} \cap \overline{(X \setminus A)}.$$

From the definition of boundary, it is clear that $\operatorname{Bd}(A) = \operatorname{Bd}(X \setminus A)$. Since $\overline{A}$ and $\overline{X \setminus A}$ are closed sets, it follows that $\operatorname{Bd}(A)$ is closed.

Proposition 1.5.22

Let X be a topological space and $A \subset X$. Then A is closed if and only if $\operatorname{Bd}(A) \subset A$.

Proof. Suppose A is closed. Then $A = \overline{A}$ by definition of closure. Then

$$\operatorname{Bd}(A) = \overline{A} \cap \overline{(X \setminus A)} = A \cap \overline{(X \setminus A)} \subset A.$$

Conversely, suppose $\operatorname{Bd}(A) \subset A$. We have to show that $A = \overline{A}$. Clearly $A \subset \overline{A}$. Now, let $x \in \overline{A}$ such that $x \notin A$. Then

$$x \in \overline{A} \cap (X \setminus A) \subset \overline{A} \cap \overline{(X \setminus A)} = \operatorname{Bd}(A) \subset A.$$

This is a contradiction since $x \notin A$. Therefore, we must have $A = \overline{A}$ and hence A is closed. □

Exercise 1.46

Let X be a topological space and $A \subset X$. Show that

1. A° and $\mathrm{Bd}(A)$ are disjoint, and $\overline{A} = A^\circ \cup \mathrm{Bd}(A)$.
2. $\mathrm{Bd}(A) = \emptyset \iff A$ is both open and closed.
3. U is open $\iff \mathrm{Bd}(U) = \overline{U} \setminus U$.
4. If A is open, then is it true that $A = (\overline{A})^\circ$? Justify.

Exercise 1.47

Let X be a set with two topologies $\mathcal{T}_1 \subset \mathcal{T}_2$. For a subset A of X, let $A_i^\circ, \overline{A}_i, A_i', \mathrm{Bd}_i(A)$ denote the interior, closure, derived set and boundary of A with respect to $\mathcal{T}_i$, $i = 1, 2$.

1. Show that

   (i) $A_1^\circ \subset A_2^\circ$.
   (ii) $\overline{A}_2 \subset \overline{A}_1$.
   (iii) $A_2' \subset A_1'$.
   (iv) $\mathrm{Bd}(A_2) \subset \mathrm{Bd}(A_1)$.

2. Give examples to show that the equality need not hold in the above cases.

Exercise 1.48

Find the boundary and interior of each of the following subsets of $\mathbb{R}^2$.

1. $A = \{x \times y \mid y = 0\}$
2. $B = \{x \times y \mid x > 0 \text{ and } y \neq 0\}$
3. $C = A \cup B$
4. $D = \{x \times y \mid x \in \mathbb{Q}\}$
5. $E = \{x \times y \mid 0 < x^2 - y^2 \leq 1\}$
6. $F = \{x \times y \mid x \neq 0,\ y \leq \frac{1}{x}\}$

where the notation $x \times y \in X \times Y$ means $x \in X$ and $y \in Y$. One can also use the notion (x, y) instead of $x \times y$.

CHAPTER 2

Separation Axioms, Continuous Functions, and the Metric Topology

2.1 The Separation Axioms

2.1.1 Hausdorff Space

Definition 2.1.1: Hausdorff space (T_2-space)

A topological space X is called a *Hausdorff space* or a *T_2-space* if for every $x, y \in X$, $x \neq y$, there exist disjoint neighborhoods U and V of x and y respectively, i.e.

U, V are open in X, $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Theorem 2.1.2

Every finite point set in a Hausdorff space X is closed.

Proof. Since finite union of closed sets is closed, it suffices to show that every one-point set $\{x_0\}$ of X is closed.

We prove this by showing that no other point of X is in the closure of $\{x_0\}$. If $x \in X$ such that $x \neq x_0$, then since X is a Hausdorff space, there exist neighborhoods U and V such that $x \in U$, $x_0 \in V$, and $U \cap V = \emptyset$. Thus, U is a neighborhood of x which does not intersect $\{x_0\}$ and hence x cannot be in the closure of $\{x_0\}$. Therefore $\{x_0\}$ is closed. □

The condition that finite point sets be closed is called T_1 axiom which is weaker than the Hausdorff condition (or T_2 axiom). For this see Exercise 2.1 but first we give the definition of the T_1 axiom.

Definition 2.1.3: (T_1-space)

A topological space X is called a T_1-*space* if for every $x, y \in X$, $x \neq y$, there exist neighborhoods U and V of x and y respectively such that $x \notin V$ and $y \notin U$.

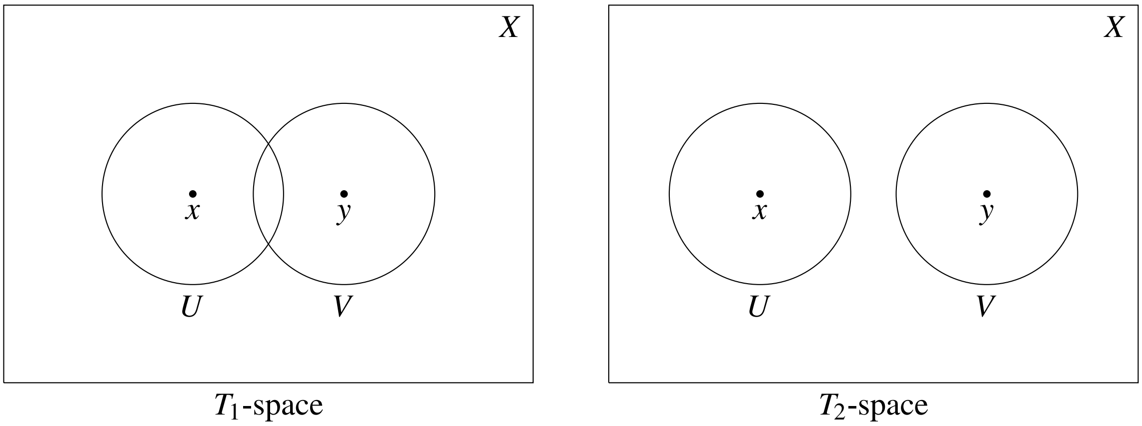


Figure 2.1: Schematic T_1 and T_2-spaces

Proposition 2.1.4

A topological space X is T_1 if and only if every singleton subset of X is closed in X.

Proof. Let X be a T_1-space and $A = \{x\}$. Suppose $y \in X$ and $y \neq x$. Since X is T_1, we get two open sets U, V such that $x \in U$, $y \in V$, $x \notin V$, $y \notin U$. Thus V is a neighbourhood y not intersecting A. So, $x \notin \overline{A}$. Thus $\overline{A} \subset A$. Hence A is closed, that is $\{x\}$ is closed in X.

Conversely, suppose every singleton subset of X is closed in X. Let $x, y \in X$ be two distinct points. Then $\{x\}, \{y\}$ are closed. Hence $U = X \setminus \{y\}$ and $V = X \setminus \{x\}$ are open sets. Also, $x \in U$, $y \in V$, $x \notin V$, $y \notin U$. Thus X is T_1. □

Exercise 2.1

Show that a Hausdorff space is T_1. Is the converse true? Justify.

Exercise 2.2

Show that a subspace of T_1 space is T_1 and a subspace of Hausdorff space is Hausdorff.

Exercise 2.3

Show that a topological space $(X, \mathfrak{T})$ a T_1-space if and only if the cofinite topology $\mathfrak{T}_f$ on X is weaker than $\mathfrak{T}$.

Theorem 2.1.5

Let X be a T_1 space and $A \subset X$. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

Proof. Suppose every neighborhood of x contains infinitely many points of A. Then it contains some point of A other than x itself. By definition, this means that, x is a limit point of A.

Conversely, suppose that x is a limit point of A. Suppose that some neighborhood U of x intersects A in only finitely many points other than x, say $x_1, x_2, \ldots, x_m$, i.e.

$$U \cap (A \setminus \{x\}) = \{x_1, x_2, \ldots, x_m\}.$$

Since X is a T_1 space, every finite point set is closed and so the set $\{x_1, x_2, \ldots, x_m\}$ is closed. Therefore, $X \setminus \{x_1, x_2, \ldots, x_m\}$ is an open set of X. Then

$$U \cap (X \setminus \{x_1, x_2, \ldots, x_m\})$$

is a neighborhood of x that does not intersect $A \setminus \{x\}$. This is contradiction to our assumption that x is a limit point of A. So U must contain infinitely many points of A. □

Exercise 2.4

Determine which of the following topologies makes $\mathbb{R}$ a T_1-space or a Hausdorff space.

(i) usual topology,
(ii) lower limit topology,
(iii) cocountable topology,
(iv) cofinite topology
(v) $\mathcal{T}_K$.

Exercise 2.5

Let $\mathcal{T}_1, \mathcal{T}_2$ be two topologies on a set X such that $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

(i) If $\mathcal{T}_1$ is T_1, then show that $\mathcal{T}_2$ is also T_1.
(ii) If $\mathcal{T}_1$ is T_2, then show that $\mathcal{T}_2$ is also T_2.

Definition 2.1.6: Convergent sequence

Let X be a topological space. We say that a sequence $\{x_n\}$ in X is convergent to some x in X if for every neighborhood U of x, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. It is denoted by $x_n \to x$ and the point x of X is called *limit of the sequence* $\{x_n\}$.

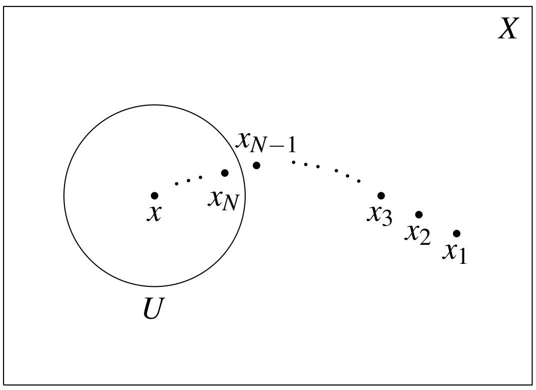


Figure 2.2: Convergent sequence

Theorem 2.1.7

Let X be a topological space. If X is Hausdorff, then a sequence of points of X converges to at most one point of X.

In other words, a convergent sequence in a T_2 space has a unique limit.

Proof. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ does not converge to any point of X, then we are done.

Suppose if possible there exist $x, y \in X$, $x \neq y$ such that $x_n \to x$ and $x_n \to y$. Since X is Hausdorff, there exist neighborhoods U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Since $x_n \to x$, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. Similarly since $x_n \to y$, there exists $M \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq M$. Then for all $n \geq N+M$, $x_n \in U \cap V = \emptyset$ which is a contradiction. Hence a sequence $\{x_n\}$ in X cannot converge to more than one point of X. □

Exercise 2.6

Determine the convergence of the following sequences in

(a) usual topology,
(b) lower limit topology,
(c) cocountable topology,
(d) cofinite topology,
(e) $\mathfrak{T} = \{G \subset X : G = \emptyset, \mathbb{R} \text{ or } (a, \infty), a \in \mathbb{R}\}$
(f) $\mathfrak{T}_K$.

(i) $\{\frac{1}{n}\}$,
(ii) $\{\frac{-1}{n}\}$,
(iii) $\{\frac{(-1)^n}{n}\}$,
(iv) $\{(-1)^n\}$,
(v) $\{n\}$.

Exercise 2.7

Show that a topological space X is Hausdorff if and only if the *diagonal* $\triangle = \{(x,x) : x \in X\}$ is closed in $X \times X$.

2.2 Continuous Functions

Definition 2.2.1: Continuous functions

Let X and Y be topological spaces. A function $f : X \to Y$ is said to be *continuous* if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

$f^{-1}(V)$ is the set of all points x of X for which $f(x) \in V$. It is the empty set if V does not intersect $f(X)$. Note that continuity of a function depends not only on the function but also on the topologies of its domain and range. For example, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -x$ is continuous in usual topology but not continuous in lower limit topology.

If the topology of the space Y is given in terms of the basis $\mathcal{B}$, then the continuity of a function $f : X \to Y$ can be defined as in the following lemma.

Lemma 2.2.2

Let X and Y be topological spaces and $\mathcal{B}$ be the basis for the topology on Y. A function $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is open for every $B \in \mathcal{B}$.

Proof. Suppose f is continuous. Since $B \in \mathcal{B}$, B is open in Y. Therefore by the definition of continuity of a function, $f^{-1}(B)$ is open in X.

Conversely, assume that inverse image of every basis element is open. Let V be an open subset of Y. Since $\mathcal{B}$ is a basis for the topology on Y, we have

$$V = \bigcup_{\alpha \in \Lambda} B_\alpha.$$

Then

$$f^{-1}(V) = f^{-1}\left(\bigcup_{\alpha \in \Lambda} B_\alpha\right) = \bigcup_{\alpha \in \Lambda} f^{-1}(B_\alpha).$$

Since $f^{-1}(B_\alpha)$ is open for all α, if follows that $f^{-1}(V)$ is open in X. So, f is continuous. □

In case the topology on the space Y is given by a subbasis $\mathcal{S}$, then the following lemma characterizes continuity of f in terms of subbasis.

Lemma 2.2.3

Let X and Y be topological spaces and $\mathcal{S}$ be the subbasis for the topology on Y. A function $f : X \to Y$ is continuous if and only if $f^{-1}(S)$ is open for every $S \in \mathcal{S}$.

Proof. Let f be continuous and $S \in \mathcal{S}$. Then S is open in Y and hence $f^{-1}(S)$ is open in X.

Conversely, assume that inverse image of every member of $\mathcal{S}$ is open. We know that any basis element B for the topology of Y can be written as a finite intersection $S_1 \cap S_2 \cap \cdots \cap S_n$ of members of $\mathcal{S}$. Then

$$f^{-1}(B) = f^{-1}(S_1 \cap S_2 \cap \cdots \cap S_n) = f^{-1}(S_1) \cap f^{-1}(S_2) \cap \cdots \cap f^{-1}(S_n).$$

Since $f^{-1}(S)$ is open for every $S \in \mathcal{S}$, inverse image of every basis element is open. Therefore by the above lemma, f is continuous. □

Example 2.2.4. Let $\mathbb{R}$ denote the set of real numbers with usual topology and $\mathbb{R}_\ell$ denote the set with lower limit topology. Consider the identity function $f : \mathbb{R} \to \mathbb{R}_\ell$ defined by $f(x) = x$ for all $x \in \mathbb{R}$. Then f is not continuous as the inverse image of open set $[0,1)$ in $\mathbb{R}_\ell$ under f is not open in $\mathbb{R}$ (with usual topology).

On the other hand, its inverse, the identity function $g : \mathbb{R}_\ell \to \mathbb{R}$ defined by $g(x) = x$, $(x \in \mathbb{R})$, is continuous since inverse image of open set (a,b) of $\mathbb{R}$ is itself which is open in $\mathbb{R}_\ell$. ■

Example 2.2.5. Every function on a discrete space is continuous. Every function to an indiscrete space is continuous.

Let X be a discrete space, Y be any topological space and $f : X \to Y$ be a function. Then for any open set V in Y, $f^{-1}(V) \subset X$. Hence $f^{-1}(V)$ is open in X. So, f is continuous.

Let X be any topological space, Y be an indiscrete topological space and $f : X \to Y$ be a function. Then the only open sets in Y are $\emptyset$ and Y. Also, $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$ which are open in X. So, f is continuous. ■

Theorem 2.2.6

Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following are equivalent.

(1) f is continuous.
(2) For every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$.
(3) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

Proof. (1) $\Rightarrow$ (2). Assume that f is continuous.
Let $w \in f(\overline{A})$. Then there is $x \in \overline{A}$ such that $w = f(x)$. We have to show that $w \in \overline{f(A)}$, i.e. every neighborhood of w intersects $f(A)$.

Let V be a neighborhood of $w(= f(x))$. Then $f^{-1}(V)$ is a neighborhood of x. Since $x \in \overline{A}$, $f^{-1}(V) \cap A \neq \emptyset$. Let $y \in f^{-1}(V) \cap A$. Then $f(y) \in V \cap f(A)$, i.e. $V \cap f(A) \neq \emptyset$. Thus, every neighborhood of w intersects $f(A)$. Therefore, $w \in \overline{f(A)}$. So $\boxed{f(\overline{A}) \subset \overline{f(A)}}$.

(2) $\Rightarrow$ (3). Let B be a closed subset of Y and $A = f^{-1}(B)$. We want to show that A is closed set of X. We shall prove this by showing $A = \overline{A}$. Now,

$$A = f^{-1}(B) \Rightarrow f(A) = f(f^{-1}(B)) \subset B.$$

Let $x \in \overline{A}$. Then since B is closed, we have

$$f(x) \in f(\overline{A}) \subset \overline{f(A)} \subset \overline{B} = B.$$

Therefore, $x \in f^{-1}(B) = A$. Thus, $\overline{A} \subset A$ and hence $A = \overline{A}$.

(3) $\Rightarrow$ (1). Assume that inverse image of every closed set is closed. We want to show that f is continuous, i.e. inverse image of every open set is open.

Let V be an open set in Y. Then $B = Y \setminus V$ is a closed subset of Y. Therefore $f^{-1}(B)$ is closed in X. But

$$f^{-1}(B) = f^{-1}(Y \setminus V) = f^{-1}(Y) \setminus f^{-1}(V) = X \setminus f^{-1}(V).$$

Thus, $X \setminus f^{-1}(V)$ is closed in X and so $f^{-1}(V)$ is open in X. Hence, f is continuous. □

Theorem 2.2.7

Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following are equivalent.

(1) f is continuous.
(2) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ and V be a neighborhood of $f(x)$. Since f is continuous, the set $U = f^{-1}(V)$ is a neighborhood of x and

$$f(U) = f(f^{-1}(V)) \subset V.$$

(2) $\Rightarrow$ (1). Let V be an open set in Y. Then we have to show that $f^{-1}(V)$ is open in X. We show this by proving that every point of $f^{-1}(V)$ is its interior point.

Let $x \in f^{-1}(V)$. Then $f(x) \in V$. By the hypothesis, there is a neighborhood U_x of x such that $f(U_x) \subset V$. Then

$$x \in U_x \subset f^{-1}(f(U_x)) \subset f^{-1}(V).$$

Thus, x is an interior point of $f^{-1}(V)$. This complete the proof. □

Exercise 2.8

Let $(X, \mathcal{T})$ be a topological space, $\mathcal{T}'$ be another topology on X and $\mathcal{T}_f$ be the cofinite topology on X.

(a) Show that $i : (X, \mathcal{T}) \to (X, \mathcal{T}')$ defined by $i(x) = x$, $(x \in X)$, is continuous if and only if $\mathcal{T}' \subset \mathcal{T}$.

(b) Show that $(X, \mathcal{T})$ is a T_1-space if and only if $i : (X, \mathcal{T}) \to (X, \mathcal{T}_f)$ defined by $i(x) = x$, $(x \in X)$, is continuous.

Exercise 2.9

Show that a one-one function from a T_1-space to a cofinite topological space is continuous.

Exercise 2.10

Suppose $f : X \to Y$ is continuous. If x is a limit point of $A \subset X$, then prove or disprove: $f(x)$ is a limit point of $f(A)$.

Exercise 2.11

Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at exactly one point.

Exercise 2.12

Determine the continuity of each $f : (\mathbb{R}, \mathcal{T}_1) \to (\mathbb{R}, \mathcal{T}_2)$ defined by the following formula in every pair of topologies $\mathcal{T}_1$ and $\mathcal{T}_2$.

Formula for f:

(a) x
(b) $-x$
(c) $\sin x$
(d) $\cos x$
(e) x^2
(f) x^3

Pairs of topologies:

No.	$\mathcal{T}_1$	$\mathcal{T}_2$
(1)	Lower limit	Lower limit
(2)	Lower limit	Upper limit
(3)	Lower limit	Usual
(4)	Lower limit	Cofinite
(5)	Lower limit	Cocountable
(6)	Upper limit	Lower limit
(7)	Upper limit	Upper limit
(8)	Upper limit	Usual
(9)	Upper limit	Cofinite
(10)	Upper limit	Cocountable
(11)	Usual	Lower limit
(12)	Usual	Upper limit
(13)	Usual	Usual
(14)	Usual	Cofinite
(15)	Usual	Cocountable
(16)	Cofinite	Lower limit
(17)	Cofinite	Upper limit
(18)	Cofinite	Usual
(19)	Cofinite	Cofinite
(20)	Cofinite	Cocountable
(21)	Cocountable	Lower limit
(22)	Cocountable	Upper limit
(23)	Cocountable	Usual
(24)	Cocountable	Cofinite
(25)	Cocountable	Cocountable

2.2.1 Homeomorphism

Definition 2.2.8: Homeomorphism

Let X and Y be topological spaces. A function $f : X \to Y$ is called a *homeomorphism* if it is one-one, onto, and both f and its inverse $f^{-1} : Y \to X$ are continuous.

In other words, $f : X \to Y$ is called homeomorphism if it is bijective and bicontinuous. If f is a homeomorphism, then we say that X is *homeomorphic* to Y or X and Y are *homeomorphic* spaces.

The condition that $f^{-1} : Y \to X$ is continuous means that for every open set U of X, its inverse image under the map f^{-1} is open in Y. The inverse image of U under f^{-1} is same as $f(U)$. Thus, the condition that f^{-1} is continuous is equivalent to saying that $f : X \to Y$ maps open sets of X to open sets of Y. Alternatively defining, a function $f : X \to Y$ is called *homeomorphism* if it is a bijective map such that $f(U)$ is open if and only if U is open.

Thus, a homeomorphism $f : X \to Y$ gives one-one correspondence between the sets X and Y as well as between the open set of X and the open sets of Y. Consequently, any property of X that can be expressed in terms of the topology of X (i.e. the open sets of X) gives a similar property for Y via the correspondence f. Such a property of X is called a *topological property*. In other words, any property that is preserved under a homeomorphism is called a topological property. Just like in algebra any algebraic structure is preserved by an isomorphism, a topological structure is preserved by a homeomorphism.

Exercise 2.13

Let X and Y be topological spaces and $f : X \to Y$ be a homeomorphism. Show that f induces a one-one correspondence between the following.

(a) closed sets of X and closed sets of Y.
(b) limit points of $A \subset X$ and limit points of $f(A) \subset Y$.
(c) boundary of $A \subset X$ and boundary of $f(A) \subset Y$.
(d) interior of $A \subset X$ and interior of $f(A) \subset Y$.
(e) closure of $A \subset X$ and closure of $f(A) \subset Y$.
(f) convergent sequences of X and convergent sequences of Y.

Definition 2.2.9

Suppose X and Y are topolgical spaces and $f : X \to Y$ is an injective continuous map. Let Z be the image of f considered as a subspace of Y. Then the function $f' : X \to Z$ obtained by restricting the range of f is bijective. If f' is a homeomorphism of X with Z, we say that the map $f : X \to Y$ is a *topological imbedding*, or simply *imbedding* of X into Y.

Example 2.2.10. The function $f : \mathbb{R} \to \mathbb{R}$ (with usual topology) defined by $f(x) = 3x+1$ is a homeomorphism.

It is easy to see (**Verify!**) that f is bijective. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(y) = \frac{1}{3}(y-1)$. Then

$$f(g(y)) = f\left(\frac{1}{3}(y-1)\right) = 3\left(\frac{1}{3}(y-1)\right) + 1 = y$$

and

$$g(f(x)) = g(3x+1) = \frac{1}{3}((3x+1)-1) = x.$$

Thus, $g = f^{-1}$.

Let (a,b) be a basis element of $\mathbb{R}$. Then $f^{-1}(a,b) = g(a,b) = \left(\frac{a-1}{3}, \frac{b-1}{3}\right)$ is open in $\mathbb{R}$. This shows that f is continuous.

Conversely, for $(a,b) \subset \mathbb{R}$, $g^{-1}(a,b) = (f^{-1})^{-1}(a,b) = f(a,b) = (3a+1, 3b+1)$ is open in $\mathbb{R}$. Thus, f^{-1} is also continuous and hence f is a homeomorphism. ■

A bijective function $f : X \to Y$ can be continuous without being a homeomorphism, i.e. f^{-1} cannot be continuous. Recall Example 2.2.4, the identity function $g : \mathbb{R}_\ell \to \mathbb{R}$ is bijective and continuous but it is not a homeomorphism as its inverse fails to be continuous. Consider another such example below.

Example 2.2.11. Let S^1 denote the unit circle,

$$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

considered as a subspace of $\mathbb{R}^2$ and let

$$f : [0,1) \to S^1$$

be the map defined by $f(t) = (\cos 2\pi t, \sin 2\pi t)$. From the properties of trigonometric functions, it follows that f is bijective and continuous. However, the function $f^{-1} : S^1 \to [0,1)$ is not continuous since the image of open set $U = \left[0, \frac{1}{4}\right)$ under f is not open in S^1. This is because there is no open set V of $\mathbb{R}^2$ such that the point $p = f(0) \in V \cap S^1 \subset f(U)$.

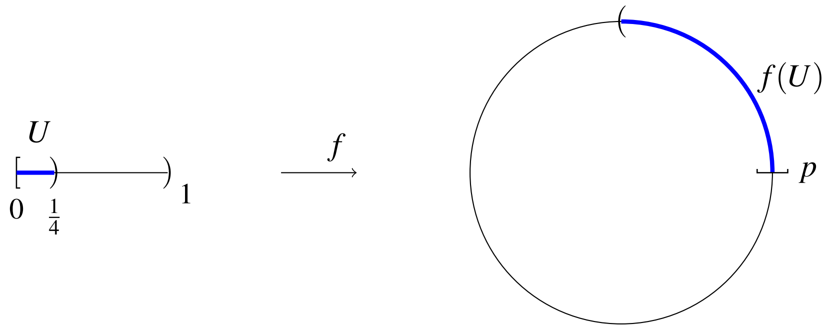


Figure 2.3: Non-open map

■

Example 2.2.12. Consider the function

$$g : [0,1) \to \mathbb{R}^2$$

obtained from the function f in the above example by extending the range from S^1 to $\mathbb{R}^2$. Then from the above example, it follows that the function g is a continuous injective map which is not an imbedding. ■

2.2.2 Constructing Continuous Functions

Theorem 2.2.13: Rules for constructing continuous functions

Let X, Y, and Z be topological spaces.

(a) (Constant function) If $f : X \to Y$ maps all of X into the single point y_0 of Y, then f is continuous.

(b) (Inclusion) If A is a subspace of X, the inclusion function $j : A \to X$ defined by $j(x) = x$ for all $x \in A$ is continuous.

(c) (Composite) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.

(d) (Restricting the domain) If $f : X \to Y$ is continuous, and if A is a subspace of X, then the restricted function $f|_A : A \to Y$ defined by $f|_A(x) = f(x)$ for all $x \in A$ is continuous.

(e) (Restricting or expanding the codomain) Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is continuous.

(f) (Local formulation of continuity) The map $f : X \to Y$ is continuous if X can be written as a union of open sets U_α such that $f|_{U_\alpha}$ is continuous for each α.

Proof. (a) Let $f(x) = y_0$ for all $x \in X$. Let V be open in Y. Then the set

$$f^{-1}(V) = \begin{cases} X & \text{if } y_0 \in V, \\ \emptyset & \text{otherwise.} \end{cases}$$

In either case, $f^{-1}(V)$ is open in X. Hence, $f : X \to Y$ defined by $f(x) = y_0$ is continuous.

(b) If U is open in X, then $j^{-1}(U) = U \cap A$ is open in A by the definition of subspace topology. Hence, the inclusion map $j : A \to X$ is continuous.

(c) Let U be open in Z. Since g is continuous, $g^{-1}(U)$ is open in Y. Since f is continuous, $f^{-1}(g^{-1}(U))$ is open in X. But

$$f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U).$$

Thus, $g \circ f : X \to Z$ is continuous.

(d) The restriction of the function f on A is the composition of the inclusion map $j : A \to X$ and $f : X \to Y$, i.e.

$$f|_A = f \circ j.$$

Hence, $f|_A : A \to Y$ is continuous.

(e) Let $f : X \to Y$ be continuous and $f(X) \subset Z \subset Y$. We show that the function $g : X \to Z$ is continuous. Let B be open in Z. Then $B = U \cap Z$ for some open set U of Y. Since $f(X) \subset Z$, we have $X \subset f^{-1}(f(X)) \subset f^{-1}(Z) = g^{-1}(Z)$ and therefore

$$f^{-1}(U) = g^{-1}(B).$$

Since $f^{-1}(U)$ is open in X, $g^{-1}(B)$ is open in X and hence g is continuous.
Now, if Y is a subspace of Z, then the map $h : X \to Z$ is the composite of the map $f : X \to Y$ and the inclusion map $j : Y \to Z$, i.e. $h = j \circ f$. Hence, h is continuous.

(f) Let, by hypothesis, $X = \bigcup_\alpha U_\alpha$ and $f|_{U_\alpha}$ be continuous for each α. Let V be open in Y. Then

$$f^{-1}(V) \cap U_\alpha = (f|_{U_\alpha})^{-1}(V).$$

Since $f|_{U_\alpha}$ is continuous, the set $f^{-1}(V) \cap U_\alpha$ is open in U_α and hence it is open in X. But

$$f^{-1}(V) = f^{-1}(V) \cap X = f^{-1}(V) \cap \left(\bigcup_\alpha U_\alpha \right) = \bigcup_\alpha (f^{-1}(V) \cap U_\alpha).$$

So $f^{-1}(V)$ is open and hence f is continuous.

□

2.2.3 Pasting Lemma

Theorem 2.2.14: Pasting Lemma

Let X and Y be topological spaces, A and B are closed subsets of X such that $X = A \cup B$. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and

g combine to give a continuous function $h : X \to Y$, defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B. \end{cases}$$

Proof. We want to show that h is continuous. Let C be a closed subset of Y. Now, by the definition of h

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C).$$

Since f is continuous and C is closed, $f^{-1}(C)$ is closed in A and hence it is closed in X. Similarly, $g^{-1}(C)$ is closed in B and therefore closed in X. Since $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X, their union $h^{-1}(C)$ is closed in X and hence h is continuous. □

Observe that the definition of h implies $f = h|_A$ and $g = h|_B$. Then Pasting lemma can be restated as

Pasting lemma

Let X and Y be topological spaces, A and B be closed subsets of X such that $X = A \cup B$, and $f : X \to Y$ be a function. If $f|_A$ and $f|_B$ are continuous, then f is continuous.

Example 2.2.15. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} x & x \leq 0, \\ \frac{x}{2} & x \geq 0. \end{cases}$$

Each of the pieces, i.e. x and $\frac{x}{2}$ of this definition are continuous functions, and they agree on the overlapping domain $\{0\}$. Since domain of each of them is closed, the function h is continuous by Pasting lemma. ■

2.3 The Metric Topology

In this section, we shall see that every metric space is a topological space. Given a metric on a set, we can define topology on it called the *metric topology*.

Definition 2.3.1

A *metric* on a set X is a function

$$d : X \times X \to \mathbb{R}$$

satisfying the following properties.

(1) $d(x,y) \geq 0$ for all $x, y \in X$; $d(x,y) = 0$ if and only if $x = y$.
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$.
(3) (Triangle inequality) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Given a metric d on X, the number $d(x,y)$ is called the *distance* between x and y in the metric d. Given $\varepsilon > 0$, the set

$$B_d(x,\varepsilon) = \{y \in X \mid d(x,y) < \varepsilon\}$$

of all points y whose distance from x is less than ε is called the *ε-ball centered at x*.

Sometimes we avoid writing the metric d in the notation and simply denote the ball as $B(x,\varepsilon)$, where there is no ambiguity.

Definition 2.3.2: Metric topology

If d is a metric on the set X, then the collection of all ε-balls $B_d(x,\varepsilon)$, for $x \in X$ and $\varepsilon > 0$, is a basis for a topology on X, called the *metric topology* induced by d, i.e.,

$$\mathcal{B}_d = \{B_d(x,\varepsilon) \mid x \in X,\ \varepsilon > 0\}$$

is a basis for the metric topology on X.

We verify below that indeed $\mathcal{B}_d$ is a basis for a topology on X.

Proposition 2.3.3

Let (X,d) be a metric space. Then the collection

$$\mathcal{B}_d = \{B_d(x,\varepsilon) \mid x \in X,\ \varepsilon > 0\}$$

is a basis for a topology on X.

Proof. (1) Let $x \in X$. Then $x \in B(x,\varepsilon)$ for any $\varepsilon > 0$.

Before we check the second condition for a basis, we prove the following:

If $y \in B(x,\varepsilon)$, then there is a $\delta > 0$ such that $B(y,\delta) \subset B(x,\varepsilon)$.

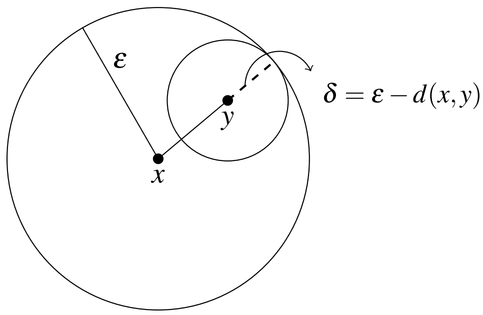


Figure 2.4: Openness of open ball

Since $y \in B(x,\varepsilon)$, we have $d(x,y) < \varepsilon$. Take $\delta = \varepsilon - d(x,y) > 0$. Now let $z \in B(y,\delta)$. Then $d(y,z) < \delta = \varepsilon - d(x,y)$. Therefore, by triangle inequality

$$d(x,z) \leq d(z,y) + d(y,z) < \varepsilon,$$

i.e. $z \in B(x,\varepsilon)$. Therefore, $B(y,\delta) \subset B(x,\varepsilon)$.

Now, we verify the second condition for a basis.

(2) Let $y \in B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$. Then, by the above, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $B(y, \delta_1) \subset B(x_1, \varepsilon_1)$ and $B(y, \delta_2) \subset B(x_2, \varepsilon_2)$. Taking $\delta = \min\{\delta_1, \delta_2\}$, we get $B(y, \delta) \in \mathcal{B}_d$ such that

$$y \in B(y, \delta) \subset B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2).$$

This shows that $\mathcal{B}_d$ is a basis. □

From what we proved above, the open sets in metric topology can be defined as:

> Let (X, d) be a metric space. A set $U \subset X$ is open in the metric topology induced by d if for each $y \in U$ there exists $\delta > 0$ such that $B_d(x, \delta) \subset U$.

Example 2.3.4. Let X be a nonempty set. Define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

It is easy to check that d is a metric which is called the *discrete metric*. The topology induced by d is the *discrete topology* because the basis element $B(x, 1)$ is $\{x\}$. ■

Example 2.3.5. For $x, y \in \mathbb{R}$, define metric d by

$$d(x,y) = |x - y|.$$

Then it is easy to check that d is a metric on $\mathbb{R}$ called the *usual* or *standard*. The topology induced by d is the *usual topology* or the *standard topology* on $\mathbb{R}$.

Any basis element for the metric topology is of the form

$$\begin{aligned} B(x, \varepsilon) &= \{y \in \mathbb{R} \mid B(x,y) < \varepsilon\} \\ &= \{y \in \mathbb{R} \mid |x - y| < \varepsilon\} \\ &= \{y \in \mathbb{R} \mid x - \varepsilon < y < x + \varepsilon\} \\ &= (x - \varepsilon, x + \varepsilon). \end{aligned}$$

Thus, every basis element of a metric is a basis element for the usual topology.

Conversely, let (a, b) be any basis element for the usual topology. Then taking $x = \frac{a+b}{2} \in \mathbb{R}$ and $\varepsilon = \frac{b-a}{2} > 0$, we get

$$(a, b) = B(x, \varepsilon).$$

Thus, the standard metric on $\mathbb{R}$ induces the standard topology on $\mathbb{R}$. ■

Definition 2.3.6: Metrizable space

A topological space X is said to be *metrizable* if there exists a metric d on the set X that induces the topology of X. A metric space is a metrizable space X together with a specific metric d that induces the topology of X.

Exercise 2.14

Show that a metrizable topological space (or a metric space) is Hausdorff.

Definition 2.3.7: Bounded set and Diameter

Let (X,d) be a metric space. A subset A of X is said to be *bounded* if there is some number M such that

$$d(a_1,a_2) \leq M$$

for all $a_1,a_2 \in A$.

If A is nonempty and bounded, then the *diameter* of A is defined to be the number

$$\operatorname{diam} A = \sup\{d(a_1,a_2) \mid a_1,a_2 \in A\}.$$

Boundedness is not a topological property because it depends on a particular metric that is considered on the set X. If X is a metric space with a metric d, then there exists a metric $\bar{d}$ that gives the same topology of X as induced by d, relative to which every subset of X is bounded. One such metric $\bar{d}$ can be defined as follows:

Theorem 2.3.8

Let X be a metric space with metric d. Define $\bar{d} : X \times X \to \mathbb{R}$ by

$$\bar{d}(x,y) = \min\{d(x,y),1\}.$$

Then $\bar{d}$ is a metric that induces the same topology on X as d.

The metric $\bar{d}$ is called *standard bounded metric* corresponding to d.

Proof. First we check that $\bar{d}$ is a metric.

(1) Clearly $\bar{d}(x,y) = \min\{d(x,y),1\} \geq 0$ for all $x,y \in X$. Also,

$$\bar{d}(x,y) = 0 \Leftrightarrow \min\{d(x,y),1\} = 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y.$$

(2) $\bar{d}(x,y) = \min\{d(x,y),1\} = \min\{d(y,x),1\} = \bar{d}(y,x)$ for all $x,y \in X$.

(3) Now we check the triangle inequality

$$\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z). \tag{2.1}$$

If either $d(x,y) \geq 1$ or $d(y,z) \geq 1$, then the right hand side of (2.1) is at least 1. Since (by definition of $\bar{d}$) the left hand side of (2.1) is at most 1, the inequality holds.
Now if both $d(x,y) < 1$ and $d(y,z) < 1$, then we have

$$d(x,z) \leq d(x,y) + d(y,z) = \bar{d}(x,y) + \bar{d}(y,z).$$

Since $\bar{d}(x,z) \leq d(x,z)$ by the definition of $\bar{d}$, the triangle inequality holds for $\bar{d}$.

In a metric space, every basis element containing x contains an ε-ball centered at x with $\varepsilon < 1$. It can be verified that, the collection of all ε-balls with $\varepsilon < 1$ forms a basis for the metric topology for any metric. Then d and $\bar{d}$ induced the same topology on X, for any metric d, such a collection will form the basis for the topology induced by $\bar{d}$ also. □

Definition 2.3.9

Given $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we define the *norm* of $\mathbf{x}$ by

$$\|\mathbf{x}\| = (|x_1|^2 + \cdots + |x_n|^2)^{\frac{1}{2}}.$$

We define the *Euclidean metric* d on $\mathbb{R}^n$ by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \left[(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2\right]^{\frac{1}{2}}.$$

We define the *square metric* or the *sup metric* ρ by the equation

$$\rho(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

The verification that d and ρ are metrics on $\mathbb{R}^n$ is left as an exercise.

Note that for $n = 1$, i.e. on $\mathbb{R}$, both the metrics coincide with the standard metric for $\mathbb{R}$ (as seen in Example 2.3.5). In the plane $\mathbb{R}^2$, the basis elements under the metric d can be viewed as circular regions while the basis elements under the metric ρ can be viewed as square regions with sides parallel to the axes (similar to what we have seen in Example 1.2.4). Our next goal is to show that both the metrics induce the usual topology on $\mathbb{R}^n$ for which we first prove the following lemma which is similar to Lemma 1.2.9.

Lemma 2.3.10

Let d and d' be two metrics on the set X, and let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies induced by them respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each $x \in X$ and each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$B_{d'}(x, \delta) \subset B_d(x, \varepsilon).$$

Proof. Suppose that $\mathcal{T}'$ is finer than $\mathcal{T}$, i.e. $\mathcal{T} \subset \mathcal{T}'$. Then given a basis element $B_d(x, \varepsilon)$ for $\mathcal{T}$, by Lemma 1.2.9, there is a basis element B' for $\mathcal{T}'$ such that

$$x \in B' \subset B_d(x, \varepsilon).$$

So, we can find a $\delta > 0$ such that $x \in B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$.

Conversely, suppose that the ε-δ condition holds. We show that $\mathcal{T} \subset \mathcal{T}'$. Let B be a basis element for $\mathcal{T}$ containing. Then we can find an ε-ball $B_d(x, \varepsilon)$ centered at x contained in B. By given condition, there is δ such that

$$x \in B_{d'}(x, \delta) \subset B_d(x, \varepsilon) \subset B.$$

It follows (by Lemma 1.2.9) that $\mathcal{T}'$ is finer than $\mathcal{T}$. □

Theorem 2.3.11

The topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$

Proof. Let $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. First we compare the metrics d and ρ to show that

$$\rho(\mathbf{x},\mathbf{y}) \leq d(\mathbf{x},\mathbf{y}) \leq \sqrt{n}\,\rho(\mathbf{x},\mathbf{y}). \tag{2.2}$$

$$\begin{aligned}
& |x_i - y_i|^2 \leq (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 && \forall\, i = 1,2,\ldots,n \\
\Rightarrow\ & |x_i - y_i| \leq \left[(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2\right]^{\frac{1}{2}} && \forall\, i = 1,2,\ldots,n \\
\Rightarrow\ & \max_{1\leq i\leq n} \{|x_i - y_i|\} \leq d(\mathbf{x},\mathbf{y}) \\
\Rightarrow\ & \rho(\mathbf{x},\mathbf{y}) \leq d(\mathbf{x},\mathbf{y}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& (x_i - y_i)^2 \leq \left(\max_{1\leq i\leq n} \{|x_i - y_i|\}\right)^2 && \forall\, i = 1,2,\ldots,n \\
\Rightarrow\ & (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 \leq n \left(\max_{1\leq i\leq n} \{|x_i - y_i|\}\right)^2 \\
\Rightarrow\ & \left[(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2\right]^{\frac{1}{2}} \leq \sqrt{n} \max_{1\leq i\leq n} \{|x_i - y_i|\} \\
\Rightarrow\ & d(\mathbf{x},\mathbf{y}) \leq \sqrt{n}\,\rho(\mathbf{x},\mathbf{y}).
\end{aligned}$$

Now, for all $\mathbf{x}$ and ε if $\mathbf{y} \in B_d(\mathbf{x},\varepsilon)$, then $d(\mathbf{x},\mathbf{y}) < \varepsilon$. But by (2.2), $\rho(\mathbf{x},\mathbf{y}) < \varepsilon$ and so $\mathbf{y} \in B_\rho(\mathbf{x},\varepsilon)$. Therefore,

$$\boxed{B_d(\mathbf{x},\varepsilon) \subset B_\rho(\mathbf{x},\varepsilon)}.$$

Similarly, the second inequality in (2.2) gives

$$\boxed{B_\rho\left(\mathbf{x}, \frac{\varepsilon}{\sqrt{n}}\right) \subset B_d(\mathbf{x},\varepsilon).}$$

By previous lemma, it follows that the two metric topologies are the same.

Now, we show that the product topology is the same as that induced by the metric ρ. Let

$$B = (a_1, b_1) \times \cdots \times (a_n, b_n)$$

be a basis element for the product topology on $\mathbb{R}^n$ containing $\mathbf{x} = (x_1, \ldots, x_n)$. Then for each i, there is an ε_i such that

$$(x_i - \varepsilon_i, x_i + \varepsilon_i) \subset (a_i, b_i).$$

Choosing $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$, we get $B_\rho(\mathbf{x},\varepsilon) \subset B$. Thus, the topology induced by ρ is finer than the product topology.

Conversely, let $B_\rho(\mathbf{x},\varepsilon)$ be a basis element for the ρ-topology and $\mathbf{y} \in B_\rho(\mathbf{x},\varepsilon)$. We have to find a basis element B for the product topology such that

$$\mathbf{y} \in B \subset B_\rho(\mathbf{x},\varepsilon).$$

This follows trivially by taking $B = B_\rho(\mathbf{x},\varepsilon)$, because

$$B_\rho(\mathbf{x},\varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon)$$

is itself a basis element for the product topology. This completes the proof. □

2.3.1 Continuity and Sequences in Metrizable Spaces

We are familiar with the ε-δ definition of continuity of a function between metric spaces. In topological space, we have seen that a function is continuous if inverse image of any open set under that function is open. In metrizable spaces, both these definitions are equivalent which is precisely proved in the theorem given below.

Theorem 2.3.12

Let X and Y be metrizable topological spaces with metrics d_X and d_Y respectively and $f : X \to Y$ be a function. Then the continuity of f is equivalent to the condition that given $\varepsilon > 0$ and $x \in X$, there exists $\delta > 0$ such that

$$d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon.$$

Proof. Suppose f is continuous. Then inverse image of every basis element under f is open. Therefore, given $x \in X$ and $\varepsilon > 0$ the set

$$f^{-1}(B(f(x), \varepsilon))$$

is open in X and contains x. So it contains some δ-ball $B(x, \delta)$ centered at x. Now, if $y \in B(x, \delta)$, then

$$f(y) \in f(B(x, \delta)) \subset f(f^{-1}(B(f(x), \varepsilon)) \subset B(f(x), \varepsilon).$$

Hence, $d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

Conversely, assume that the ε-δ condition holds. Let V be open in Y. We want to show that $f^{-1}(V)$ is open in X. Let $x \in f^{-1}(V)$. Since $f(x) \in V$ and V is open, there is an ε-ball $B(f(x), \varepsilon)$ centered at $f(x)$ such that

$$f(x) \in B(f(x), \varepsilon) \subset V.$$

By the hypothesis, there exists $\delta > 0$ such that $d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$. Let $y \in B(x, \delta)$. Then $f(y) \in B(f(x), \varepsilon) \subset V$. So, $y \in f^{-1}(V)$. Therefore,

$$x \in B(x, \delta) \subset f^{-1}(V)$$

and hence $f^{-1}(V)$ is open in X. □

In analysis we know that if x lies in the closure of some set A, then there is a sequence of points of A converging to x. This is not true in general in topological spaces. However, the result holds for metrizable spaces.

Lemma 2.3.13: The sequence lemma

Let X be a topological space and $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$. The converse holds if X is metrizable.

Proof. Suppose $\{x_n\}$ is a sequence in A such that $x_n \to x$. Then by the definition of a convergent sequence, every neighborhood of x contains a point of A. Hence, $x \in \overline{A}$.

Conversely, assume that X is a metrizable space and $x \in \overline{A}$. Let d be a metric for the topology of X. Then by definition of closure, for each $n \in \mathbb{N}$, we have $B_d(x, \frac{1}{n}) \cap A \neq \emptyset$. Choose $x_n \in B_d(x, \frac{1}{n}) \cap A$ for each n. Then $\{x_n\}$ is a sequence in A.

Claim. $x_n \to x$.

Let U be an open set containing x. Then there exits $\varepsilon > 0$ such that

$$x \in B(x, \varepsilon) \subset U.$$

Choosing $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$, we get

$$x_n \in B(x, \varepsilon) \subset U, \quad \forall\, n \geq N.$$

Hence the claim. □

Theorem 2.3.14

Let X and Y be topological spaces and $f : X \to Y$ be a function. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.

Proof. Suppose that f is continuous and $x_n \to x$. We want to prove that $f(x_n) \to f(x)$. Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is neighborhood of x, and so there exits $N \in \mathbb{N}$ such that $x_n \in f^{-1}(V)$ for all $n \geq N$. Then $f(x_n) \in f(f^{-1}(V)) \subset V$ for all $n \geq N$. Hence, $f(x_n) \to f(x)$.

Conversely, assume that X is a metrizable space and $f(x_n) \to f(x)$ whenever $x_n \to x$. We want to prove that f is continuous. We prove this by showing that $f(\overline{A}) \subset \overline{f(A)}$ (by Theorem 2.2.6). Let $x \in \overline{A}$. Then, by the sequence lemma, there is a sequence $x_n \in A$ such that $x_n \to x$. By assumption $f(x_n) \to f(x)$. Since $f(x_n) \in f(A)$, again by the sequence lemma, $f(x) \in \overline{A}$. Hence, $f(\overline{A}) \subset \overline{f(A)}$ and so f is continuous. □

Note that in both the results above, we did not completely use the fact that X is metrizable. We used that there is a countable collection of balls $B_d(x, \frac{1}{n})$ centered at x. This leads us to a new definition below.

Definition 2.3.15: First Countability Axiom

A space X is said to have a *countable basis at the point* x if there is a countable collection $\{U_n\}_{n \in \mathbb{N}}$ of neighborhoods of x such that any neighborhood U of x contains at least one of the sets U_n.

A space X that has a countable basis at each of its points is said to satisfy the *first countability axiom*.

Note that every metrizable space satisfies the first countability axiom, but the converse is not true.

Exercise 2.15

Show that

1. A discrete topological space is metrizable.

2. An infinite set with cofinite topology is not metrizable.

Exercise 2.16

Let (X,d) be a metric space, $x \in X$, and $A \subset X$. Define distance of x from A as

$$d(x,A) = \inf\{d(x,y) \mid y \in A\}.$$

Show that

$$x \in \overline{A} \Leftrightarrow d(x,A) = 0.$$

CHAPTER 3

Connectedness and Compactness

In this chapter, we shall discuss mainly two topological properties, called connectedness and compactness on which important theorems in calculus like intermediate value theorem, maximum value theorem, and uniform continuity theorem depend.

3.1 Connected Spaces

Definition 3.1.1: Separation and Connected space

Let X be a topological space. A *separation* of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be *connected* if there does not exist a separation of X.

In other words, a topological space X is said to be *disconnected* if there exist two subsets $U, V \subset X$ such that

- U, V are open
- $U \neq \emptyset, V \neq \emptyset$
- $U \cup V = X$
- $U \cap V = \emptyset$

The pair (U, V) is called a *separation* of X and X is said to be *connected* if it is not disconnected.

Connectedness is clearly a topological property since it is defined in terms of the open sets of X. Therefore, if X is a connected space then any space which is homeomorphic to X is also connected.

Observe that the sets U, V in the separation of X are complements of each other. Hence U (and V) is a non-empty proper subset of X which is both open and closed. In this regard, we have the following result.

Proposition 3.1.2

A topological space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.

Proof. Assume first that X is connected. Suppose, if possible, A is a nonempty proper subset of X that is both open and closed in X. Then taking $U = A$ and $V = X \setminus A$, we get a separation of X which is contradiction to our assumption that X is connected. Thus, only subsets of X that are both open and closed in X are $\emptyset$ and X itself.

Conversely, assume that only subsets of X that are both open and closed in X are $\emptyset$ and X. Suppose U and V give a separation of X. Then U and V are nonempty open subsets of X such that $U \cup V = X$ and $U \cap V = \emptyset$. But then $U = X \setminus V$. Therefore U is also closed. Since V is nonempty, U is a proper subset of X. Thus, U is a nonempty proper subset of X which is both open and closed in X. This is contradiction to our assumption. Hence, X must be connected. □

Another criterion which can be considered equivalent to the definition of connected space is given by the following exercise.

Exercise 3.1

Let X be a topological space. Then X is disconnected if and only if there is a continuous function f from X onto $\{0,1\}$, where $\{0,1\}$ carries the discrete topology.

Now suppose X is a topological space and Y is a subspace of X. When can we say that Y is connected? The following is a useful way of defining connectedness of a subspace Y of a space X.

Lemma 3.1.3

Let X be a topological space and Y be a subspace of X. A separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other.

The space Y is connected if there exists no separation of Y.

Proof. Suppose that A and B form a separation of Y. Then A is both open and closed in Y. The closure of A in Y is the set $\overline{A} \cap Y$, where $\overline{A}$ is the closure of A in X. Since A is closed in Y, $\overline{A} \cap Y = A$. But $A \cap B = \emptyset$. Therefore, $\overline{A} \cap B = \emptyset$. Since $\overline{A}$ is the union of A and its limit points, B contains no limit points of A. Similarly, A does not contain any limit points of B.

Conversely, suppose that A and B are disjoint nonempty sets whose union is Y, neither of which contains a limit point of the other. Since $A \cap B = \emptyset$ and $A' \cap B = \emptyset$, it follows that $\overline{A} \cap B = \emptyset$. Similarly $A \cap \overline{B} = \emptyset$. Since $A \cup B = Y$, we conclude that $\overline{A} \cap Y = A$ and $\overline{B} \cap Y = B$. Thus, both A and B are closed in Y. Since $A = Y \setminus B$ and $B = Y \setminus A$, they both are also open in Y. Hence, they give a separation of Y. □

Example 3.1.4. Any singleton set is connected as there is no separation possible.

Any set X with indiscrete topology is connected as the only open sets are X and $\emptyset$ and hence there is no separation of X.

Any discrete space X with more than one point is not connected. If x is an element of X, then the sets $U = \{x\}$ and $V = X \setminus \{x\}$ are both nonempty open disjoint sets whose union is X. Hence, U, V give a separation of X. ■

Example 3.1.5. Let $Y = [-1,0) \cup (0,1]$ be the subspace of $\mathbb{R}$. Each of the sets $[-1,0)$ and $(0,1]$ is nonempty and open in Y. Therefore they form a separation of Y. Observe that none of them contains a limit point of the other. ■

Example 3.1.6. Consider the subspace $Y = [-1,1]$ of $\mathbb{R}$. The sets $[-1,0]$ and $(0,1]$ are both nonempty and disjoint, but they do not form a separation of Y as the set $[-1,0]$ is not open in Y. Alternatively, 0 is the limit point of the second set $(0,1]$ which is contained in the first set $[-1,0]$. Hence, they do not form a separation of Y. ■

We cannot simply say that a set is connected if one or few pairs fail to give a separation. There may be a separation possible or otherwise we have to prove that the given set is connected. The question here is whether the set $X = [-1,1]$ is connected or not. Is it possible to give some separation of X? The answer will be disclosed at the end of this section.

Example 3.1.7. The set of rationals $\mathbb{Q}$ as a subspace of $\mathbb{R}$ is not connected.

In fact the only connected subsets of $\mathbb{Q}$ are one-point sets (i.e. singletons). Let Y be a subset of $\mathbb{Q}$ containing more than one point. Let $p, q \in Y$. Then we can choose an irrational number a such that $p < a < q$. Then the separation of Y is given by the sets

$$(-\infty, a) \cap \mathbb{Q} \quad \text{and} \quad (a, \infty) \cap \mathbb{Q}.$$

Such a space is called totally disconnected. Hence, $\mathbb{Q}$ is called totally disconnected. ■

From the above example, we have the following definition and a partial answer to the succeeding exercise.

Definition 3.1.8: Totally disconnected

A topological space is said to be *totally disconnected* if its only connected subspaces are one-point sets.

Exercise 3.2

Show that if X (with more than one point) has the discrete topology, then X is totally disconnected. Does the converse hold?

Exercise 3.3

Show that $\mathbb{R}_\ell$ is totally disconnected.

Exercise 3.4

1. Is $\mathbb{R}$ or $\mathbb{Q}$ with cofinite topology $\mathcal{T}_f$ connected? Justify?
2. When is a set X with cofinite topology $\mathcal{T}_f$ connected, disconnected or totally disconnected?

Example 3.1.9. Consider the subset $X = \{(x,y) \in \mathbb{R}^2 \mid xy = 1\}$ of the plane $\mathbb{R}^2$. Then X is not connected as the two sets

$$\{(x,y) \in \mathbb{R}^2 \mid x > 0,\ y = \tfrac{1}{x}\} \text{ and } \{(x,y) \in \mathbb{R}^2 \mid x < 0,\ y = \tfrac{1}{x}\}$$

give a separation of X as shown in the figure.

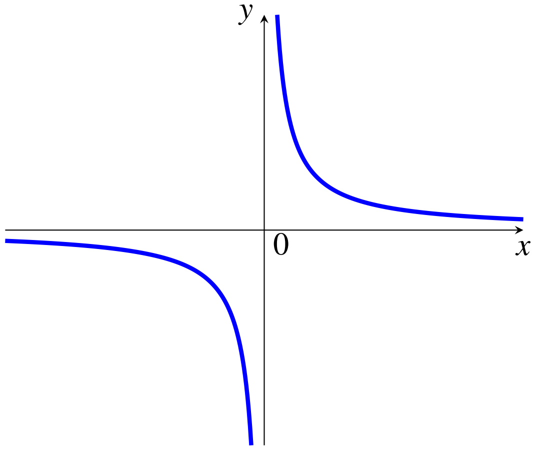


Figure 3.1: A disconnected subset of $\mathbb{R}^2$

■

Exercise 3.5

Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X such that $\mathcal{T}' \supset \mathcal{T}$. Which of the following holds?

1. $(X,\mathcal{T})$ is connected implies $(X,\mathcal{T}')$ is connected.
2. $(X,\mathcal{T}')$ is connected implies $(X,\mathcal{T})$ is connected.

Lemma 3.1.10

If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within either C or D.

Proof. Since C and D are both open in X, the sets $C \cap Y$ and $D \cap Y$ are open in Y. They are disjoint and their union is Y. If $C \cap Y \neq \emptyset$ and $D \cap Y \neq \emptyset$, then they form a separation of Y. Since Y is connected, one of $C \cap Y$ and $D \cap Y$ is empty. Hence, Y must lie entirely either in C or in D. □

Theorem 3.1.11

The union of a collection of connected subspaces of X that have a point in common is connected.

Proof. $\{A_\alpha\}$ be a collection of connected subspaces of a topological space X and $p \in \bigcap_\alpha A_\alpha$. We prove that $Y = \bigcup A_\alpha$ is connected.

Suppose that $Y = C \cup D$ is a separation of Y. The point p is one of the sets C or D. Suppose $p \in C$. Since A_α is connected, it must lie entirely in C or D. But $p \in A_\alpha$ and $p \in C$ so it cannot lie in D. Hence, $A_\alpha \subset C$ for every α, and so $\bigcup A_\alpha \subset C$. This is contradiction to the assumption that $D \neq \emptyset$. Hence, $Y = \bigcup A_\alpha$ must be connected. □

In particular, if A and B are connected sets and $A \cap B \neq \emptyset$, then $A \cup B$ is connected.

The above lemma says that if $\{A_\alpha\}$ is an arbitrary collection of connected subspaces of X with nonempty intersection, then their union is connected. If we consider a countable collection, then we get the same result with a weaker hypothesis as given in the exercise below.

Exercise 3.6

Let $\{A_n\}_{n\in\mathbb{N}}$ be a collection of connected subsets of X such that $A_k \cap A_{k+1} \neq \emptyset$ for all $k \in \mathbb{N}$. Show that $\bigcup_{n=1}^{\infty} A_n$ is connected.

Another variation of the above result where all A_α's do not have nonempty intersection but they intersect a connected space A. This means $\cup A_\alpha$ is connected. See the exercise below.

Exercise 3.7

Let $\{A_\alpha\}$ be a sequence of connected subsets of a topological space X and $A \subset X$ be connected such that $A \cap A_\alpha \neq \emptyset$ for all α. Show that $A \cup (\cup_\alpha A_\alpha)$ is connected.

Theorem 3.1.12

Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is also connected.

In other words, If B is formed by adding some or all the limit points of a connected subspace A, then B is connected.

Proof. Let A be connected and $A \subset B \subset \overline{A}$. Suppose $B = C \cup D$ is a separation of B. Since $A \subset B$ is connected, by above lemma, the set A must lie entirely in C or in D. Without the loss of generality, suppose $A \subset C$. Then $\overline{A} \subset \overline{C}$. Since $B \subset \overline{A}$ and $\overline{C} \cap D = \emptyset$, we have $B \cap D = \emptyset$, i.e. $B \subset C$. This is contradiction to the fact that D is a nonempty subset of B.

Hence, B must be connected. □

Exercise 3.8

Let A and B be two connected subsets of a topological space X such that $A \cap \bar{B} \neq \emptyset$. Then show that $A \cup B$ is connected.

The above theorem says that closure of a connected set is connected. However, the converse is not true. We have seen that $\mathbb{Q}$ is (totally) disconnected. We shall see at the end of this section that its closure is connected, i.e. $\overline{\mathbb{Q}} = \mathbb{R}$ is connected.

The following theorem says that continuous image of a connected set is connected.

Theorem 3.1.13

The image of a connected space under a continuous map is connected.

Proof. Let X and Y be topological spaces. Let $f : X \to Y$ be a continuous map, and X be connected. We want to prove that the image space $Z = f(X)$ is connected. Since the map obtained from f by restricting its codomain to the space Z is also continuous, it suffices to prove the result for a continuous surjective map

$$g : X \to Z.$$

Suppose $Z = A \cup B$ is a separation of Z, where A and B are two nonempty disjoint sets open in Z. Since g is continuous, $g^{-1}(A)$ and $g^{-1}(B)$ are open in X. Since g is surjective, they are nonempty subsets of X. Since $A \cap B = \emptyset$ and $A \cup B = Z$, we have

$$g^{-1}(A) \cap g^{-1}(B) = g^{-1}(A \cap B) = g^{-1}(\emptyset) = \emptyset$$
$$g^{-1}(A) \cup g^{-1}(B) = g^{-1}(A \cup B) = g^{-1}(Z) = X.$$

Thus, $g^{-1}(A)$ and $g^{-1}(B)$ give a separation of X which is a contradiction to our assumption that X is connected. This completes the proof. □

As an immediate consequence of the above theorem, we can say that if Y is homeomorphic to a connected space X, then Y must be connected too. This shows that connectedness is a topological property.

If X and Y are connected topological spaces, then what can be said about the connectedness of the product space $X \times Y$? The following theorem addresses this question.

Theorem 3.1.14

A finite Cartesian product of connected spaces is connected.

Proof. First we prove the result for the product of two connected spaces X and Y.

Choose a base point (a,b) in the product space $X \times Y$. Observe that the "horizontal slice" $X \times \{b\}$ is homeomorphic to X, under the correspondence $(x,b) \mapsto x$, and hence it is connected. Similarly for $x \in X$, the "vertical slice" $\{x\} \times Y$, being homeomorphic to Y, is connected. Consequently, the "T-shaped" space

$$T_x = (X \times \{b\}) \cup (\{x\} \times Y)$$

is connected for each x because it is the union of the two connected spaces having the point (x,b) in common (as shown in the figure below). Therefore, the union $\bigcup_{x \in X} T_x$ of all these T-shaped spaces is connected because they have the point (a,b) (in fact, whole $X \times \{b\}$) in common. Observe that this union is equal to $X \times Y$ and hence $X \times Y$ is connected.

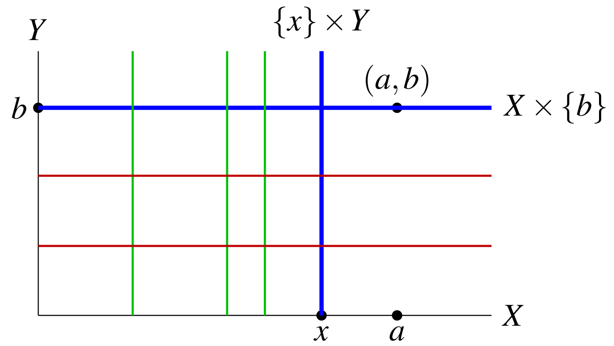


Figure 3.2: The "T-shaped" space

The proof for finite product of connected spaces follows by induction (**Verify!**) and using the fact that

$$X_1 \times \cdots \times X_n \text{ is homeomorphic to } (X_1 \times \cdots \times X_{n-1}) \times X_n.$$

□

The following are some natural exercises to practice about the connectedness and the product of two spaces which would benefit for better understanding of the concept.

Exercise 3.9

Prove or disprove: For topological spaces X and Y

1. If $X \times Y$ is connected, then X and Y both must be connected.
2. If $X \times Y$ is connected, then either X or Y must be connected.
3. If $X \times Y$ is disconnected, then X and Y both must be disconnected.
4. If $X \times Y$ is disconnected, then either X or Y must be disconnected.

The immediate and natural question one would like to ask here is what about the connectedness of arbitrary (i.e. infinite) product of connected spaces. Is it connected? It depends on the topology of the product space. As far as our syllabus is concerned, we are not going into depth but here is where the box and product topology on arbitrary product of $\mathbb{R}$ with itself differ. We end this discussion by stating the following relevant example.

Example 3.1.15. Consider the Cartesian product $\mathbb{R}^\omega$. Assuming that $\mathbb{R}$ is connected, the space $\mathbb{R}^\omega$ is not connected in the box topology but it is connected in the product topology. ▪

Exercise 3.10

Let X and Y be topological spaces. Let A be a proper subset of X and B be a proper subset of Y. If X and Y are connected, show that

$$(X \times Y) \smallsetminus (A \times B)$$

is connected. Hence, (we deduce that) $\mathbb{R}^2 \smallsetminus \mathbb{Q}^2$, i.e. $(\mathbb{R} \times \mathbb{R}) \smallsetminus (\mathbb{Q} \times \mathbb{Q})$ is connected.

3.1.1 Connected Subspaces of the Real Line

Our goal in this subsection is to prove that connected subsets of $\mathbb{R}$ (with usual topology) are intervals and $\mathbb{R}$ itself. For any $a \in \mathbb{R}$, we consider $\{a\}$ as an interval, by convention. The proof does not depend on any algebraic property of $\mathbb{R}$ but it depends only on the ordered structure of $\mathbb{R}$. The proof given in the book by Munkres is for more general set called a linear continuum, having the same order properties as that of $\mathbb{R}$.

Here we present two proofs of the fact that an interval in $\mathbb{R}$ is connected. The first proof presented below is given in the book by Simmons and the second proof is exclusively from the lecture notes of my teacher Dr. D. J. Karia. Students may learn/refer to any proof (even other than these two proofs) they prefer.

Theorem 3.1.16

A subspace of the real line $\mathbb{R}$ is connected if and only if it is an interval. In particular, $\mathbb{R}$ is connected.

Proof. Let X be a subspace of $\mathbb{R}$. First we prove that if X is connected, then it is an interval.

Suppose, if possible, X is not an interval. Then there exist real numbers x, y, and z such that $x < y < z$, where $x, z \in X$ but $y \notin X$. It is easy to see that the sets

$$(-\infty, y) \cap X \quad \text{and} \quad (y, \infty) \cap X$$

give a separation of X and hence X is not connected. Thus, if X is connected, then it is an interval.

Conversely, we want to show that if X is an interval, then it is connected. Suppose, if possible, X is not connected. Let $X = A \cup B$ be a separation of X. Since A and B are nonemtpy, we can choose points $x, z \in X$ such that $x \in A$ and $z \in B$. Since A and B are disjoint, we have $x \neq z$. Without the loss of generality, assume that $x < z$. Since X is an interval, the interval $[x, z] \subset X$, and each point in $[x, z]$ is either in A or in B. Define

$$y = \sup([x, z] \cap A).$$

Clearly, $x \le y \le z$ and so $y \in X$. We know that if $A \subset \mathbb{R}$ is bounded and closed, then $\sup A \in A$. Since A is closed, $y \in A$. Therefore, we can conclude that $y < z$. Again by the definition of supremum, $y + \varepsilon \in B$ for every $\varepsilon > 0$ such that $y + \varepsilon \le z$. Since B is closed, this implies that $y \in B$. Thus, $y \in A \cap B = \emptyset$, which is a contradiction. Hence, X must be connected. ☐

Another proof for the converse of the above result, i.e. any interval of $\mathbb{R}$ is connected, is presented below.

Theorem 3.1.17

The interval (a, b) is a connected subset of $\mathbb{R}$.

Proof. We set $I = (a, b)$. Suppose I is disconnected. So, there exists a continuous function f from I onto $\{0, 1\}$. Fix $c, d \in I$ such that $f(c) = 0, f(d) = 1$. Without loss of generality, we assume that $c < d$, so that $[c, d] \subset I$. Let

$$E = \{x \in I : x > c, f(x) = 1\}.$$

Clearly, $d \in E$. So, $E \neq \emptyset$. Also, c is a lower bound of E. Hence E has the infimum in $[c, d]$; and hence in I. Let

$$r = \inf E.$$

Claim: $f(r) = 1$.

If $r = d$, then $f(r) = 1$. So, assume that $r < d$. Consider an integer $n_0 > \frac{1}{d-r}$. Then $\frac{1}{n_0} < d - r$. So $r + \frac{1}{n} \in (r, d)$ for all $n \ge n_0$. Now for each such n, $r + \frac{1}{n}$ is not a lower bound of E. Hence there exists $a_n \in E$ such that $a_n < r + \frac{1}{n}$. Hence $f(a_n) = 1$. But $r = \inf E$. Hence $a_n \ge r$. So, $a_n \in [r, r + \frac{1}{n})$ for every $n \ge n_0$. But then $a_n \to r$. Hence $f(a_n) \to f(r)$. Hence

$$f(r) = 1. \tag{3.1}$$

Now $f(c) = 0$ and $f(r) = 1$. So, $c \neq r$. Let $x \in (c, r)$. If $f(x) = 1$, then $x > c \Rightarrow x \in E$, contradicting to $r = \inf E$. Thus $f(x) = 0$. Let $n_1 > \frac{1}{r-c}$. Then for every $n > n_1$, $r - \frac{1}{n} > r - \frac{1}{n_1} > c$. Hence $f(r - \frac{1}{n}) = 0$. But $r - \frac{1}{n} \to r$ gives

$$f(r) = 0. \tag{3.2}$$

This is a contradiction. Thus I must be connected. ☐

Corollary 3.1.18

The intervals of the form

$$[a,b), \ -\infty < a < b \leq \infty$$
$$(a,b], \ -\infty \leq a < b < \infty$$
$$[a,b], \ -\infty \leq a < b \leq \infty$$

are connected.

Proof. Let J be interval of any form mentioned above. Then $(a,b) \subset J \subset \overline{(a,b)}$. Since (a,b) is connected, J is connected. □

The above corollary implies that $\mathbb{R}$ is connected.

Exercise 3.11

Show that

1. The unit circle in $\mathbb{R}^2$ given by $S^1 = \{(x,y) \mid x^2+y^2 = 1\}$ is connected.
2. The open unit disc in $\mathbb{R}^2$ given by $\mathbb{D} = \{(x,y) \mid x^2+y^2 < 1\}$ is connected.

Exercise 3.12

Prove or disprove:

1. There exists connected sets A and B such that $A \cap B$ and $A \cup B$ are both disconnected.
2. There exists disconnected sets A and B such that $A \cap B$ and $A \cup B$ are both connected.

Exercise 3.13

Prove or disprove:
There exists connected sets A and B such that $A \cap B$ is disconnected.

Exercise 3.14

Which of the following sets is/are connected? Justify.

$$X = \{z \in \mathbb{C} : |z| < 1\} \cup \{z \in \mathbb{C} : |z-2| < 1\},$$
$$Y = \{z \in \mathbb{C} : |z| \leq 1\} \cup \{z \in \mathbb{C} : |z-2| < 1\},$$
$$Z = \{z \in \mathbb{C} : |z| \leq 1\} \cup \{z \in \mathbb{C} : |z-2| \leq 1\}.$$

Exercise 3.15

Let $f : S^1 \to \mathbb{R}$ be a continuous map. Show that there exists a point x of S^1 such that $f(x) = f(-x)$.

Exercise 3.16

Let $f : X \to X$ be continuous. Show that if $X = [0,1]$, then there is a point x such that $f(x) = x$. Such a point x is called a *fixed point* of f. What happens of X is $[0,1)$ or $(0,1)$?

Exercise 3.17

If A is a connected subspace of X, does it follow that A° and $\text{Bd}(A)$ are connected? Does the converse hold? Justify.

Exercise 3.18

Let X be a topological space with VIP topology and $|X| \geq 2$. Let A, B be two connected non-singleton subsets of X. Show that $A \cap B$ is connected.

3.1.2 Components

Definition 3.1.19: Components

Let X be a topological space. For $x, y \in X$ define $x \sim y$ if there is a connected subspace of X containing both x and y. Then $\sim$ is an equivalence relation on X and the equivalence classes are called the *components* of the *connected components* of X.

Exercise 3.19

Show that the relation $\sim$ defined above (for $x, y \in X$, $x \sim y$ if there is a connected subspace of X containing both x and y) is an equivalence relation on X

The components ofa topological space have the following property.

Theorem 3.1.20

The components of X are connected disjoint subspaces of X whose union is XS, such that each nonempty connected subspace of X intersects only one of them.

Proof. The components of X, being equivalence classes, are disjoint and their union is X.

Let A be a connected subspace of X. If A intersects the components C_1 and C_2 of X in points x_1 and x_2 (say) respectively, then $x_1 \sim x_2$. By definition of equivalence relation, this is possible only if $C_1 = C_2$. Thus, a connected subspace of X intersects only one of the component of X.

Now we show that a component C of X is connected. Choose a point x_0 of C. For each point x of C, we know that $x_0 \sim x$. So there is a connected subspace A_x of X containing x_0 and x. By above argument, $A_x \subset C$ (since each connected subspace of X is contained in exactly one of the components of X).

Since $x_0 \in \bigcap_x A_{x \in C}$, the intersection $\bigcap_x A_{x \in C}$ is nonempty. Since each A_x is connected, by Theorem 3.1.11, their union is connected. Hence,

$$C = \bigcup_{x \in C} A_x$$

is connected. □

3.2 Compact Spaces

Definition 3.2.1: Covering and Open Cover

Let X be a topological space. A collection $\mathscr{A}$ of subsets of X is said to *cover* or a *covering* of X if the union of elements of $\mathscr{A}$ is equal to X.

A cover $\mathscr{A}$ is said to be an *open cover* or *open covering* if the elements of $\mathscr{A}$ are open subsets of X.

Definition 3.2.2: Compact Space

A topological space X is said to be *compact* if every open covering $\mathscr{A}$ of X contains a finite subcollection that also covers X. Such a collection is called a *finite subcover* of $\mathscr{A}$.

In other words, X is said to be compact if every open cover of X has a finite subcover.

Compactness is also a topological property since it is defined in terms of the open sets. Thus, if X is a compact space then any topological space which homeomorphic to X is also compact.

Example 3.2.3. Any indiscrete space X is compact, for the only open sets are X and $\emptyset$. So every open cover of X or any subset of X comprises of X itself or $\{\emptyset, X\}$ both of which are finite.

In general any space $(X, \mathcal{T})$, where $\mathcal{T}$ itself comprises of finitely many elements, is compact. In particular, any finite set X is compact because $\mathcal{T}$ is finite. ∎

Example 3.2.4. The real line $\mathbb{R}$ is not compact because the covering of $\mathbb{R}$ by open intervals

$$\mathscr{A} = \{(n, n+2) \mid n \in \mathbb{Z}\}$$

does not have a finite subcollection that covers $\mathbb{R}$ (**Verify!**). ∎

One can find many other open covers of $\mathbb{R}$ which do not have a finite subcover. To show that a set is not compact, only one example of such an open cover is sufficient.

Thus, the real $\mathbb{R}$ with usual topology is not compact. However, it may be compact with other topologies, for example, as seen in the first example, $\mathbb{R}$ with the indiscrete topology is compact. Another such example is the following exercise.

Exercise 3.20

Show that $\mathbb{R}$ with cofinite topology is compact.

Example 3.2.5. The subspace

$$X = \{0\} \cup \{\tfrac{1}{n} \mid n \in \mathbb{N}\}$$

of $\mathbb{R}$ is compact.

Given an open cover $\mathscr{A}$ of X, there is an element U of $\mathscr{A}$ containing 0. Then the set contains all but finitely many points $\frac{1}{n}$. For each point of X not in U, choose an element of $\mathscr{A}$ containing it. The collection of these finitely many elements of $\mathscr{A}$ together with the set U forms a finite subcover of $\mathscr{A}$.

Since $\mathscr{A}$ was arbitrary open cover, we say that every open cover of X has a finite subcover and hence X is compact. ∎

If X is a compact set and $Y \subset X$, what can be said about the compactness of Y? It is not necessarily true that a subset of a compact set is compact. Consider the following exercise in which we consider a subset of the compact set X seen in the above example.

Exercise 3.21

Note that $X = \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ is a compact subspace of $\mathbb{R}$. Let $Y = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$.Show that Y is not compact.

Example 3.2.6. The interval $(0,1]$ of $\mathbb{R}$ is not compact. The open covering

$$\mathscr{A} = \left\{\left(\tfrac{1}{n},1\right] \mid n \in \mathbb{N}\right\}$$

contains no finite subcollection which covers $(0,1]$. Similarly, the interval $(0,1)$ is not compact.

However, the interval $[0,1]$ is compact, we shall see the proof of which later in the next section. ∎

Exercise 3.22

Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X such that $\mathcal{T}' \supset \mathcal{T}$. Which of the following holds?

1. $(X,\mathcal{T})$ is compact implies $(X,\mathcal{T}')$ is compact.
2. $(X,\mathcal{T}')$ is compact implies $(X,\mathcal{T})$ is compact.

Exercise 3.23

Let X be a non-empty set and $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X. If X is compact and T_2 in both the topologies, then show that either both are equal or uncomparable.

Lemma 3.2.7

Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection which covers Y.

Proof. Suppose Y is compact and $\mathscr{A} = \{A_\alpha\}_{\alpha\in\Lambda}$ is an open cover of Y by sets open in X, i.e. here $Y \subset \cup A_\alpha$. Then the collection

$$\{A_\alpha \cap Y \mid \alpha \in \Lambda\}$$

is a cover of Y by sets open in Y. Since Y is compact, a finite subcollection

$$\{A_{\alpha_1} \cap Y, \dots, A_{\alpha_n} \cap Y\}$$

covers Y. Then $\{A_{\alpha_1}, \dots, A_{\alpha_n}\}$ is a subcollection of $\mathscr{A}$ that covers Y.

Conversely, suppose the given condition holds. We want to show that Y is compact. Let $\mathscr{A}' = \{A'_\alpha\}$ be a cover of Y by sets open in Y. Since A'_α is open in Y, for each α, there exists set A_α open in X such that

$$A'_\alpha = A_\alpha \cap Y.$$

Then the collection $\mathscr{A} = \{A_\alpha\}$ is a covering of Y by sets open in X. By the hypothesis, some finite subcollection of $\mathscr{A}$, say $\{A_{\alpha_1}, \dots, A_{\alpha_n}\}$ covers Y. Then $\{A'_{\alpha_1}, \dots, A'_{\alpha_n}\}$ is a subcollection of $\mathscr{A}'$ that covers Y. □

If a set X has the discrete topology, then only compact subsets of X are finite subsets. What about the converse? This question is addressed by the following exercise.

Exercise 3.24

Prove or disprove: Let $(X, \mathcal{T})$ be a topological space. If the only compact subsets of X are finite sets, then $(X, \mathcal{T})$ is a discrete topological space.

Theorem 3.2.8

Every closed subspace of a compact space is compact.

Proof. Let X be a compact topological space and let Y be a closed subspace of X. We want to show that Y is compact.

Let $\mathscr{A}$ be a cover of Y by sets open in X. Since Y is closed in X, $X \setminus Y$ is open in X. Let

$$\mathscr{B} = \mathscr{A} \cup \{X \setminus Y\}.$$

Then $\mathscr{B}$ is an open cover of X. Since X is compact, some finite subcollection of $\mathscr{B}$ covers X. If this subcollection contains the set $X \setminus Y$, then we discard $X \setminus Y$, otherwise we consider it as it is. Clearly, the resulting subcollection is a finite subcollection of $\mathscr{A}$ that covers Y.

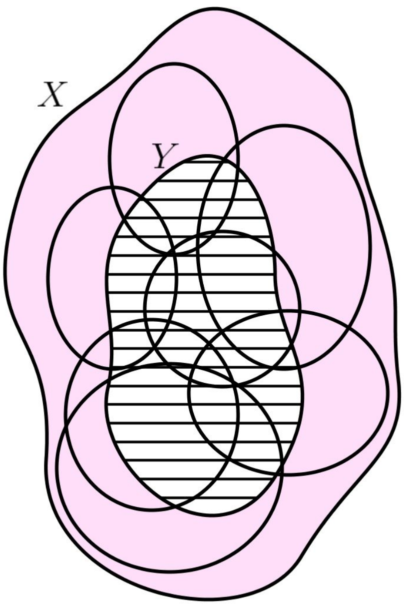


Figure 3.3: Closed subset of a compact set is compact

□

Theorem 3.2.9

Every compact subspace of a Hausdorff space is closed.

Proof. Let X be a Hausdorff topological space and let Y be a compact subspace of X. We show that Y is closed by proving that $X \setminus Y$ is open.

Let $x_0 \in X \setminus Y$. We show that there is a neighborhood of x_0 that is disjoint from Y. Since X is Hausdorff, for each point y of $Y (\subset X)$, we get neighborhoods U_y and V_y such that

$$x_0 \in U_y, \quad y \in V_y, \quad U_y \cap V_y = \emptyset.$$

Observe that the collection $\{V_y \mid y \in Y\}$ is a cover of Y by sets open in X. Since Y is compact, a finite subcollection $\{V_{y_1}, \ldots, V_{y_n}\}$ covers Y. Let

$$V = V_{y_1} \cup \cdots \cup V_{y_n}$$

and

$$U = U_{y_1} \cap \cdots \cap U_{y_n}.$$

Then U and V are both open and $Y \subset V$.

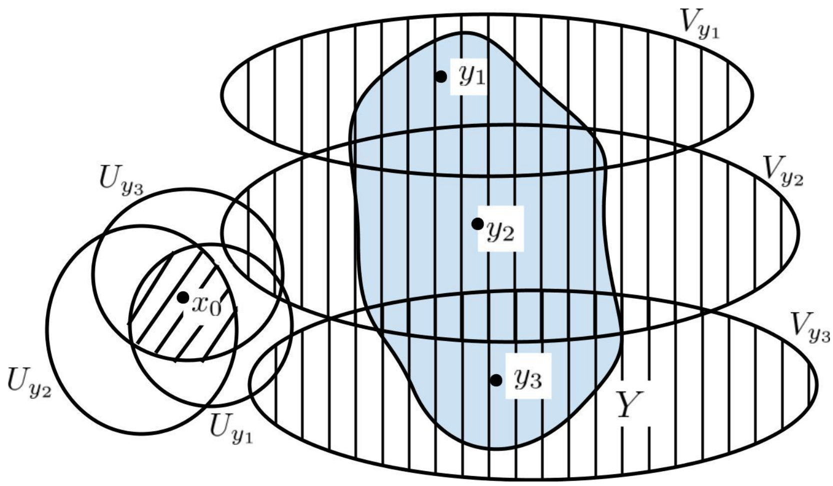


Figure 3.4: Compact subset of a T_2 space is closed

Claim: $U \cap V = \emptyset$.

If $z \in U \cap V$, then $z \in U_{y_i}$ for all i and $z \in V_{y_i}$ for some i. Therefore, $z \in U_{y_i} \cap V_{y_i} = \emptyset$ which is contradiction. Hence, $U \cap V = \emptyset$. Since, $Y \subset V$, U is disjoint from Y. Thus,

$$x_0 \in U \subset X \setminus Y.$$

So, $X \setminus Y$ is open and hence Y is closed. □

Since every metric space is Hausdorff (see Exercise 2.14), the above result is true for every metric space also.

The result we obtained from the proof of the previous theorem which will be useful later. We state it below separately as a lemma.

Lemma 3.2.10

Let X be a Hausdorff space, Y be a compact subspace of X, and $x_0 \notin Y$. Then there exists disjoint open sets U and V of X containing x_0 and Y respectively.

Remark 3.2.11. If we show that interval $[a,b]$ is compact, then by Theorem 3.2.8, every closed subspace of $[a,b]$ is compact. On the other hand, from Theorem 3.2.9, the intervals (a,b) and $(a,b]$ cannot be compact as they are not closed subsets of Hausdorff space $\mathbb{R}$.

Note that the Hausdorff condition of Theorem 3.2.9 is necessary. For example, every subset of $\mathbb{R}$ is compact with the cofinite topology as seen Exercise 3.20.

Theorem 3.2.12

The image of a compact space under a continuous map is compact.

Proof. Let X be a compact space and $f: X \to Y$ be a continuous map. We want to show that $f(X)$ is compact. Let $\mathscr{A}$ be a cover of $f(X)$ by sets open in Y. Then the collection

$$\{f^{-1}(A) \mid A \in \mathscr{A}\}$$

is a collection of sets covering X. Since f is continuous, these sets are open in X. Since X is compact, finitely many of them, say

$$f^{-1}(A_1), \dots, f^{-1}(A_n)$$

cover X, i.e.

$$X = f^{-1}(A_1) \cup \dots \cup f^{-1}(A_n).$$

Therefore

$$f(X) = f(f^{-1}(A_1)) \cup \dots \cup f(f^{-1}(A_n)) \subset A_1 \cup \dots \cup A_n.$$

Thus, the sets $A_1, \dots, A_n$ cover $f(X)$ and hence $f(X)$ is compact. □

The above theorem says that continuous image of compact set is compact. As a consequence, we can conclude that if a space Y is homeomorphic to a compact space X, then Y must be compact. Thus, compactness is also a topological property.

One of the important result to verify whether a given map is a homeomorphism or not is given by the following theorem.

Theorem 3.2.13

Let $f: X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. We have to show that $f^{-1}: Y \to X$ is continuous. For this we show that inverse image of a set, closed in X, under f^{-1} is closed in Y. If C is a closed subset of X. Then we have to show that $(f^{-1})^{-1}(C) = f(C)$ is closed in Y. In other words, we have to show that $f: X \to Y$ maps closed sets to closed sets. So, the proof goes as follows:

- Let A be closed in X.
- We know that closed subset of compact space is compact. Therefore A is compact.
- Since f is continuous and onto, and since continuous image of compact set is compact, the set $f(A)$ is compact in Y.
- We know that compact subset of a Hausdorff space is closed. Therefore, $f(A)$ is closed.

This completes the proof. □

Exercise 3.25

Does there exist spaces X and Y such that $f : X \to Y$ is a continuous bijection but not a homeomorphism and $g : Y \to X$ is a continuous bijection but not a homeomorphism?

As seen in the previous section, a finite product of connected space is connected, while the connectedness of an arbitrary product of connected spaces depends on the topology of the product space. The similar question one would like to ask here is what can be said about the compactness of the product space (finite and arbitrary product) of compact spaces? The answer is true. That is, finite product of compact spaces is compact. In fact, the result is, product of infinitely many compact spaces is compact, which is known as *Tychonoff theorem*. We skip this topic as it is beyond the scope of our syllabus.

Exercise 3.26

Which of the following statements are true?

1. There exists compact sets A, B such that $A \cup B$ and $A \cap B$ are not compact.
2. There exists non-compact sets A, B such that $A \cup B$ and $A \cap B$ are compact.

Exercise 3.27

If a set A is compact, is it true that ∂A, i.e. boundary of A is compact?

Next we study another criterion for a space to be compact in terms of closed sets rather than the open sets.

Definition 3.2.14: FIP

A collection $\mathcal{C}$ of subsets of X is said to have the *finite intersection property* if for every finite subcollection

$$\{C_1, \dots, C_n\}$$

of $\mathcal{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty.

Theorem 3.2.15

Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets of X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty.

Proof. Suppose X is compact. Let $\mathcal{C}$ be any collection of closed sets of X having finite intersection property. We want to show that $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

Suppose, if possible, $\bigcap_{C \in \mathcal{C}} C = \emptyset$. Let $\mathscr{A} = \{X \setminus C \mid C \in \mathcal{C}\}$. Then $\mathscr{A}$ is a collection of open subsets of X. Now,

$$\bigcap_{C \in \mathcal{C}} C = \emptyset \Rightarrow X \setminus \bigcap_{C \in \mathcal{C}} C = X$$

$$\Rightarrow \bigcup_{C\in\mathscr{C}} (X \setminus C) = X$$
$$\Rightarrow \bigcup_{A\in\mathscr{A}} A = X.$$

Therefore, $\mathscr{A}$ is an open cover of X and since X is compact, $X = A_1 \cup \cdots \cup A_n$. Let $C_i = X \setminus A_i$. Then $C_1 \cap \cdots \cap C_n = \emptyset$ which is a contradiction to our assumption that $\mathscr{C}$ has finite intersection property. Therefore $\bigcap_{C\in\mathscr{C}} C$ must be nonempty.

Conversely, assume that for every collection $\mathscr{C}$ of closed sets of X having the finite intersection property, $\bigcap_{C\in\mathscr{C}} C \neq \emptyset$. We want to show that X is compact.

Let $\mathscr{A}$ be an open covering of X. Let $\mathscr{C} = \{X \setminus A \mid A \in \mathscr{A}\}$. Then $\mathscr{C}$ is a collection of closed subsets of X. Now,

$$\bigcup_{A\in\mathscr{A}} A = X \Rightarrow X \setminus \bigcup_{A\in\mathscr{A}} A = \emptyset$$
$$\Rightarrow \bigcap_{A\in\mathscr{A}} (X \setminus A) = \emptyset$$
$$\Rightarrow \bigcap_{C\in\mathscr{C}} C = \emptyset.$$

By our assumption $\mathscr{C}$ cannot have finite intersection property. Therefore, there is a finite subcollection $C_1, \ldots, C_n$ of $\mathscr{C}$ such that $C_1 \cap \cdots \cap C_n = \emptyset$. Let $A_i = X \setminus C_i$. Then $A_1 \cup \cdots \cup A_n = X$. Thus, $\mathscr{A}$ has a finite subcover and hence X is compact. □

3.2.1 Compact Subspaces of the Real Line

In the remaining part of this section, like in case of connectedness, we study the compact subsets of the real line $\mathbb{R}$ and a very famous theorem called Heine-Borel Theorem which states that a closed and bounded subset of $\mathbb{R}$ (with usual topology) is compact. The core idea of the proof is in proving that a closed interval $[a,b]$ of $\mathbb{R}$ is compact. There are many proofs for this. The one given in the book by Simmons uses Finite Intersection Property and later it is remarked by the author that the same can be proved independently. Here we present an independent proof of $[a,b]$ being compact but the students are free to write/learn any proof which they prefer.

Theorem 3.2.16: Heine-Borel Theorem

A closed and bounded subset of $\mathbb{R}$ is compact.

Proof. A closed and bounded subset of $\mathbb{R}$ is a closed subset of the closed interval $[a,b]$ for some $a,b \in \mathbb{R}$. Since closed subset of a compact space is compact, it suffices to prove that $[a,b]$ is compact. If $a = b$, then $[a,b] = \{a\}$ which is clearly compact. Hence, we prove that $[a,b]$, where $a,b \in \mathbb{R}$, and $a < b$ is compact.

The proof is given below. □

The interval $[a,b]$ being homeomorphic to $[0,1]$ and compactness being a topological property, we prove that $[0,1]$ is compact to complete the proof of Heine-Borel theorem. The following proof of $[0,1]$ being compact is not yet found in the literature. This interesting and simple proof is taken from the lecture notes of my teacher Dr. D. J. Karia, for which we are extremely grateful to him. The proof is not yet found in the literature.

Proposition 3.2.17

$[0,1]$ is a compact subset of $\mathbb{R}$.

Proof. Let $\mathscr{A}$ be an open cover of $[0,1]$ in $\mathbb{R}$. Let

$$A = \{s \in (0,1] : [0,s] \text{ can be covered by finitely many elements of } \mathscr{A}\}.$$

We show that $1 \in A$. We shall show this in the following three steps.

• First observe that $A \neq \emptyset$. For that, let $G \in \mathscr{A}$ such that $0 \in G$. But G is open. So, there exists an open interval (a,b) such that $0 \in (a,b) \subset G$. If $1 < b$, then $[0,1] \subset (a,b) \subset G$, and so, $1 \in A$. If $b \leq 1$, then $[0, \frac{b}{2}] \subset (a,b) \subset G$. Thus $\frac{b}{2} \in A$. Hence, $A \neq \emptyset$.

• Now let $r = \sup A$. Clearly, $0 < r \leq 1$. We claim that $r \in A$. Suppose $r \notin A$. Since $\mathscr{A}$ covers $[0,1]$, there exists $G \in \mathscr{A}$ such that $r \in G$. So there exists an open interval (a,b) such that $r \in (a,b) \subset G$. If $a < 0$, we replace a by 0 and if $b > 1$, then we replace b by 1 and in that way we assume that $r \in (a,b) \subset [0,1]$. Choose $t \in (a,r)$. Since $t < r$, t is not an upper bound of A. Hence we get $s \in A$ such that $t < s$. Since $s \in A$, $[0,s]$ can be covered by finitely many elements of $\mathscr{A}$. But then one more element G is required to cover $[s,r]$. Thus $[0,r]$ can be covered by finitely many elements of $\mathscr{A}$. Thus $r \in A$.

• Suppose, if possible, that $r < 1$. Then we choose $t \in (r,b)$ and see that $[r,t]$ is covered by G. Thus $[0,t]$ is covered by finitely many elements of $\mathscr{A}$. Thus $t \in A$. Since $t > r$, r is not the supremum of A. Thus $r = 1$. Hence $1 \in A$. This completes the proof. □

Corollary 3.2.18

For $a, b \in \mathbb{R}$, $a < b$, the closed interval $[a,b]$ is compact.

Proof. Define $f : [0,1] \to [a,b]$ by $f(t) = a + (b-a)t$, $(t \in [0,1])$. Then f is one-one, onto, continuous function. Since $[a,b]$ is a continuous image of a compact space, it is compact. □

This completes the proof of Heine-Borel Theorem. What about its converse? Is every compact subset of $\mathbb{R}$ necessarily closed as well as bounded. The converse is true.

Since a compact subset of a Hausdorff space is closed, and $\mathbb{R}$ is a T_2-space, we conclude that a compact subset of $\mathbb{R}$ is closed. Also, a compact subset of a metric space (see Exercise below) is bounded. The real line $\mathbb{R}$ being a metric space, compact subset of $\mathbb{R}$ is also bounded.

Exercise 3.28

Let (X,d) be a metric space. If $Y \subset X$ is compact, then Y is bounded. That is, in other words, a compact subset of a metric space is bounded.

Further, find a metric space in which not every closed and bounded subspace is compact.

Combining this with the Heine-Borel theorem, we have

A subset of $\mathbb{R}$ (with usual topology) is compact if and only if it is closed and bounded.

Exercise 3.29

Check whether $[0,1]$ is compact with respect to the following topologies on $\mathbb{R}$.

Usual topology, lower limit $\mathcal{T}_\ell$ (or upper limit $\mathcal{T}_u$), cofinite topology $\mathcal{T}_f$, cocountable topology $\mathcal{T}_c$, and $\mathbb{R}_k$, i.e. $(\mathbb{R}, \mathcal{T}_K)$.

Exercise 3.30

Show that the unit circle in $\mathbb{R}^2$ given by $S^1 = \{(x,y) \mid x^2+y^2 = 1\}$ is compact.

Exercise 3.31

Check whether the following spaces are homeomorphic or not. Assume usual or subspace topology whenever not specified.

1. $(0,1)$ and (a,b), $a,b \in \mathbb{R}$.
2. (a,b) and (c,d), $a,b,c,d \in \mathbb{R}$.
3. $[a,b)$ and $(c,d]$, $a,b,c,d \in \mathbb{R}$.
4. (a,b) and $\mathbb{R}$, $a,b \in \mathbb{R}$.
5. (a,b) and $[c,d)$, $a,b,c,d \in \mathbb{R}$.
6. S^1 and (a,b).
7. S^1 and $[a,b]$.
8. S^1 and $[a,b)$.
9. S^1 and $\{(x,y) \in \mathbb{R}^2 \mid \max\{|x|,|y|\} = 1\}$.
10. S^1 and $\mathbb{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$.
11. $\mathbb{D}$ and $\overline{\mathbb{D}} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$.
12. $\mathbb{D}$ and $\{(x,y) \in \mathbb{R}^2 \mid |x| < 1,\ |y| < 1\}$.
13. $\mathbb{R}$ and $\mathbb{R}^2$ (in general $\mathbb{R}^n$ for $n \geq 2$).
14. The figures represented by the letters X and Y as subspaces of $\mathbb{R}^2$.
15. $\mathbb{R}^n$ and $\mathbb{R}^m$ for $n,m \geq 2$, $n \neq m$.

Exercise 3.32

Let $f : X \to Y$ be a map, where Y is compact Hausdorff space. Show that f is continuous if and only if the *graph of f*,

$$G_f = \{(x, f(x)) \mid x \in X\}$$

is closed in $X \times Y$.

3.3 The Countability Axioms

Let us recall the notion of first countability axiom before we define what is called second countability axiom.

Definition 3.3.1: First-countable

A space X is said to have a *countable basis at x* if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of its points is said to satisfy the *first countability axiom*, or to be *first-countable*.

3.3.1 Second-Countable

Definition 3.3.2: Second-countable

If a space X has a countable basis for its topology, then X is said to satisfy the *second countability axiom*, or to be *second-countable*.

Clearly, the second axiom implies the first. If $\mathcal{B}$ is a countable basis for the topology of X, then the subcollection of $\mathcal{B}$ consisting of those elements of $\mathcal{B}$ which contain the point x forms a countable basis at x.

Example 3.3.3. The real line $\mathbb{R}$ (with the usual topology) has a countable basis

$$\mathcal{B} = \{(a,b) \mid a < b,\ a,b \in \mathbb{Q}\}.$$

Similarly, $\mathbb{R}^n$ has a countable basis of the collection of all products of intervals having rational end points. ■

We prove the following result for second countability axiom. The proof for the first countability axiom is similar.

Theorem 3.3.4

A subspace of a second-countable space is second-countable, and a countable product of second-countable spaces is second-countable.

Proof. Let X be a second-countable space and $A \subset X$. If $\mathcal{B}$ is a countable basis for X, then $\{B \cap A \mid B \in \mathcal{B}\}$ is a countable basis for the subspace A of X. Hence, a subspace of a second-countable space is second-countable.

Let X_i be second-countable topological spaces. If $\mathcal{B}_i$ is a countable basis for the space X_i, then the collection $\prod U_i$, where $U_i \in \mathcal{B}_i$ for finitely many values of i and $U_i = X_i$ for all other values of i, is a countable basis for $\prod X_i$. □

Definition 3.3.5: Separable space

A subset A of a space X is said to be *dense* in X if $\overline{A} = X$.

A space having a countable dense set is said to be *separable*.

Example 3.3.6.
1. A finite set is separable with any topology on it.
2. $\mathbb{R}$ is separable because $\mathbb{Q}$ is dense in $\mathbb{R}$
3. $\mathbb{R}$ with the cocountable topology is not separable as every countable subset of $\mathbb{R}$ is closed and hence it is not dense in $\mathbb{R}$.

■

The next theorem states that every second-countable space is separable.

Theorem 3.3.7

Suppose that X has a countable basis. Then

(1) Every open covering of X contains a countable subcollection covering X.

(2) There exists a countable subset of X that is dense in X.

Proof. Let $\{B_n\}$ be a countable basis for X.

(1) Let $\mathscr{A}$ be an open covering of X. For each $n \in \mathbb{N}$, for which it is possible, choose an element $A_n \in \mathscr{A}$ such that $B_n \subset A_n$. Then clearly the collection $\mathscr{A}'$ of these sets A_n is a countable subcollection of $\mathscr{A}$, since it is indexed by a subset J of $\mathbb{N}$.
Claim: $\mathscr{A}'$ covers X.
Let $x \in X$. Since $\mathscr{A}$ is an open cover, there exists $A \in \mathscr{A}$ such that $x \in A$. Since A is open, there is a basis element B_n such that $x \in B_n \subset A$. Since $B_n \subset A$, the element A_n of $\mathscr{A}'$ containing B_n is defined. Since $x \in B_n$ and $B_n \subset A_n$, we have $x \in A_n$. Hence, $\mathscr{A}'$ covers X.

(2) From each $n \in \mathbb{N}$, choose $x_n \in B_n$. Let $D = \{x_n \mid n \in \mathbb{N}\}$.
Claim: D is dense in X, i.e. $\overline{D} = X$.
Let $x \in X$. By the definition of basis, $x \in B_n$ for some n. Then $x_n \in B_n \cap D$. Thus, every basis element containing x intersects D. So $x \in \overline{D}$. Hence, D is dense in X.

□

CHAPTER 4

Regular Spaces, Normal Spaces, and Complete Metric Spaces

4.1 The Separation Axioms (revisited)

In this section, we recall a separation axiom we have already seen called the Hausdorff axiom or the T_2-axiom, and we see another stronger separation axiom and its relationship with the other axioms.

These axioms are called separation axioms because they define separation of two points, two closed sets, a point and a set in terms of (disjoint) open sets. This separation is different from the separation (disconnectedness) we saw in the section on connected spaces.

4.1.1 Regular Spaces

Definition 4.1.1: Regular space

Let X be a topological space. Suppose that the one point sets are closed in X. Then X is said to be *regular* if for each pair of a closed set B and a point $x \notin B$, there exist disjoint open sets containing x and B respectively.

The condition that if x is a point and B is a closed set not containing x, then there exist open sets U and V such that $x \in U$, $B \subset V$, and $U \cap V = \emptyset$ is called the T_3-*axiom*. Thus, a regular space is a space which is T_1 and T_3.

It is evident that a regular space is Hausdorff. This can also be seen in the schematic representation of T_3-axiom given below. However, the converse is not true which we shall see later in this section by means of an example.

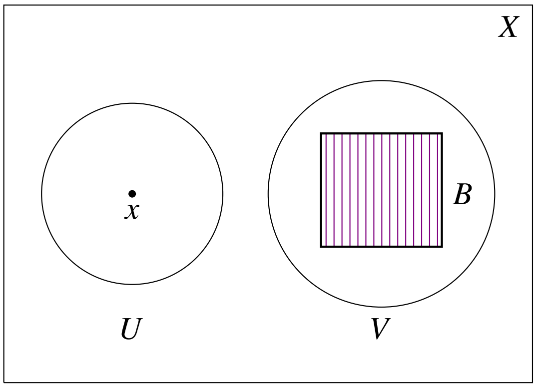


Figure 4.1: Schematic T_3-spaces

Lemma 4.1.2

Let X be a topological space and let one-point sets be closed in X. Then X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subset U$.

Proof. Suppose that X is regular. Let x be a given point and U be a neighborhood of x. Let $B = X \setminus U$. Then B is a closed set and $x \notin B$. Since X is regular, there exist open sets V and W such that $x \in V$, $B \subset W$, and $V \cap W = \emptyset$.

Claim: $\overline{V} \cap B = \emptyset$.
Let $y \in \overline{V} \cap B$. Since $y \in \overline{V}$, every neighborhood of y must intersect V. Also, $y \in B \subset W$. But $V \cap W = \emptyset$. Thus, W is a neighborhood of y which does not intersect V. This is contradiction to $y \in \overline{V}$. Hence, $\overline{V} \cap B = \emptyset$.

Therefore $\overline{V} \subset X \setminus B$, i.e. $\overline{V} \subset U$.

Conversely, suppose x is a point of X and B is a closed set not containing x. Let $U = X \setminus B$. Then U is a neighborhood of x. By hypothesis, there is a neighborhood V of x such that $\overline{V} \subset U$. Then V and $X \setminus \overline{V}$ are disjoint neighborhoods of x and B respectively. Since every singletons are closed in X, the space X is regular. □

Theorem 4.1.3

(1) A subspace of a Hausdorff space is Hausdorff.
(2) A subspace of a regular space is regular.

Proof. (1) Let X be a Hausdorff topological space and Y be a subspace of X. Let x and y be two points of Y. Since X is Hausdorff, there exist disjoint neighborhoods U and V of x and y respectively in X. Then $U \cap Y$ and $V \cap Y$ are disjoint neighborhoods of x and y respectively in Y.

(2) Let X be a regular space and Y be a subspace of X. Then one-point sets are closed in Y. Let x be a point of Y and B be a closed subset of Y such that $x \notin B$. Since B is closed in Y, we have $\overline{B} \cap Y = B$, where $\overline{B}$ is the closure of B in X.

Therefore $x \notin \overline{B}$. Thus, $x \in X$ and $\overline{B}$ is a closed set in X not containing x. By regularity of X, we get disjoint open sets U and V containing x and $\overline{B}$ respectively. Then $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y containing x and B respectively.

□

A similar result is true for arbitrary product of spaces, i.e. a product of Hausdorff spaces is Hausdorff and a product of regular spaces is regular. The results concerning arbitrary products are not in our syllabus and so we skip the proof here but we state the following exercise for a finite product. It is sufficient to prove for the product of two spaces and then it can be extended by induction to a finite product.

Exercise 4.1

Let X and Y be two topological spaces. Show that

1. If X and Y are Hausdorff, then $X \times Y$ is Hausdorff.
2. If X and Y are regular, then $X \times Y$ is regular.

As remarked earlier that a regular space is Hausdorff but the converse is not true. The following is the example of a Hausdorff space which is not regular.

Example 4.1.4. The space $\mathbb{R}_K$ is Hausdorff but not regular.

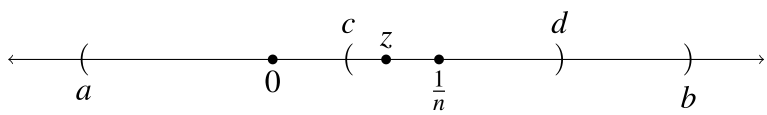
Recall that $\mathbb{R}_K$ denotes $\mathbb{R}$ with the topology generated by basis consisting of all open intervals (a,b) and sets of the form $(a,b) \setminus K$, where $K = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. It is Hausdorff since any two distinct points have disjoint open intervals containing them. We show that it is not regular.

Note that the set K is closed (since its complement is open) in $\mathbb{R}_K$ and $0 \notin K$. Suppose that there exist disjoint open sets U and V containing 0 and K respectively. Since U is open and $0 \in U$ it contains a basis element containing 0. It must be of the form $(a,b) \setminus K$, since each basis element of the form (a,b) containing 0 intersects K.

Choose n sufficiently large such that $\frac{1}{n} \in (a,b)$. Now choose a basis element about $\frac{1}{n}$ contained in V. Clearly, it must be of the form (c,d). Finally choose z such that

$$\max\left\{c, \frac{1}{n+1}\right\} < z < \frac{1}{n}.$$

Then, as shown in the figure below, $z \in U \cap V = \emptyset$ which is a contradiction. Hence, $\mathbb{R}_K$ is not regular.



■

Exercise 4.2

Show that a metric space is regular.

Exercise 4.3

Show that a compact Hausdorff space is regular.

Exercise 4.4

Show that if X is a regular space, then every pair of points of X have neighborhoods whose closures are disjoint.

Exercise 4.5

Let $f, g : X \to Y$ be continuous functions and Y be a Hausdorff space. Show that

$$\{x \in X \mid f(x) = g(x)\}$$

is closed in X.

4.1.2 Normal Spaces

Definition 4.1.5: Normal space

Let X be a topological spaces such that one-point sets are closed in X. Then X is said to be *normal* if for each pair of disjoints closed sets A and B, there exist disjoint open sets containing A and B respectively.

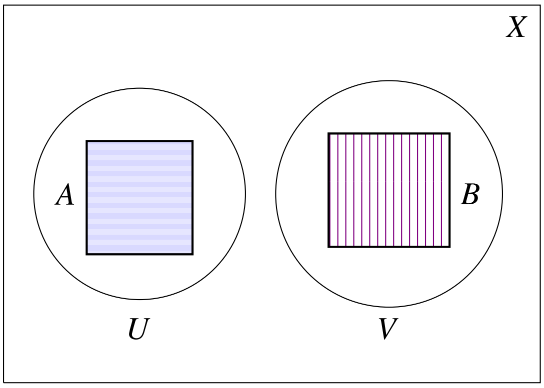


Figure 4.2: Schematic T_4-spaces

The condition that for every pair of disjoint closed sets A and B, there exists open sets U and V such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$ is called the T_4-*axiom*. Thus, a T_4 space which is also T_1 is called a *normal space*.

Clearly a normal space is regular, since singleton sets are closed. Thus, normality is stronger than regularity. However, the converse is not true. The above figure shows a schematic representation of the T_4-axiom.

Lemma 4.1.6

Let X be a topological space and let one-point sets be closed in X. Then X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$.

Proof. Similar to the proof of Lemma 4.1.2 (replace the point x by the set A in that proof). □

Example 4.1.7. The space $\mathbb{R}_\ell$ is normal.

Since $\mathbb{R}_\ell$ is finer than $\mathbb{R}$ (with usual topology), it follows that every singleton sets are closed in $\mathbb{R}_\ell$.

Now let A and B be two disjoint closed sets in $\mathbb{R}_\ell$. For each point of A choose a basis element $[a, x_a)$ not intersecting B, and for each point b of B choose a basis element $[b, x_b)$ not intersecting A. Then the sets

$$U = \bigcup_{a \in A} [a, x_a) \quad \text{and} \quad V = \bigcup_{b \in B} [b, x_b)$$

are disjoint **Verify!** open sets containing A and B respectively.

Hence, $\mathbb{R}_\ell$ is normal. ■

Example 4.1.8. The Sorgenfrey plane $\mathbb{R}_\ell^2$ is regular but not normal.

By the above example, we know that $\mathbb{R}_\ell$ is normal and hence regular. Also by the Exercise 4.1 (2), we know that product of regular spaces is regular. Therefore, we conclude that $\mathbb{R}_\ell^2$ is a regular space. We do not give the proof that it is not normal as it is beyond the scope of our syllabus. ■

Now we see three results which ensure normality of a space under certain hypotheses.

Theorem 4.1.9

Every regular space with a countable basis is normal.

Proof. Let X be a regular space with a countable basis $\mathcal{B}$. Let A and B be two disjoint closed subsets of X. Since X is regular, each point x of A has a neighborhood U which does not intersect B. Again by regularity (Lemma 4.1.2), we get a neighborhood V of x such that $\overline{V} \subset U$. Choose an element of basis $\mathcal{B}$ containing x and contained in V.

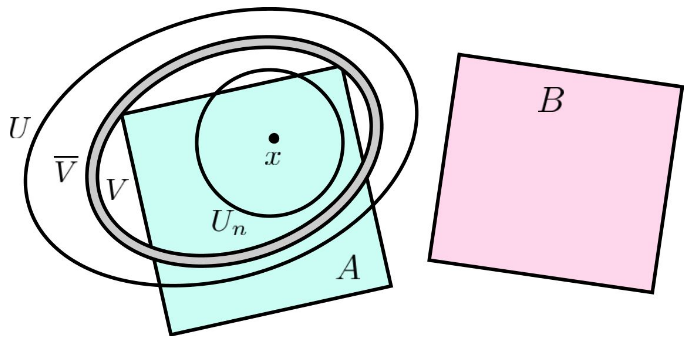


Figure 4.3

By choosing such a basis element for each $x \in A$, we get a covering and hence a countable covering of A by open sets whose closures do not intersect B. We denote this countable covering of A by $\{U_n\}$. Similarly, choose a countable open covering $\{V_n\}$ of B such that $\overline{V}_n$ is disjoint from A. Then the sets $U = \bigcup U_n$ and $V = \bigcup V_n$ are open sets containing A and B respectively. But they may not be disjoint.

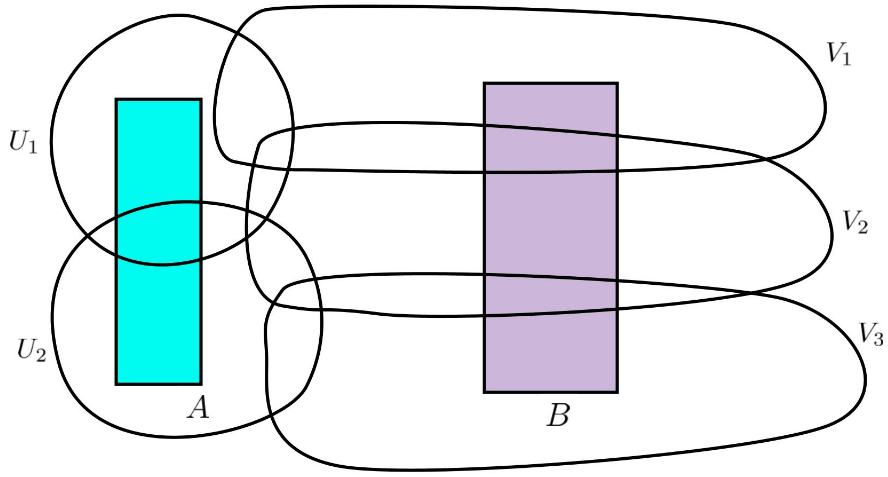


Figure 4.4

Now, we construct two disjoint open sets U' and V' containing A and B respectively. For each $n \in \mathbb{N}$, define

$$U'_n = U_n \setminus \bigcup_{i=1}^{n} \overline{V}_i \quad \text{and} \quad V'_n = V_n \setminus \bigcup_{i=1}^{n} \overline{U}_i.$$

Note that each set U'_n is the difference of an open set U_n and a closed set $\bigcup\limits_{i=1}^{n} \overline{V}_i$. Hence, U'_n is open for all n. Similarly, V'_n is open for all n. Also, the collection $\{U'_n\}$ covers A since each $x \in A$ belongs to U_n for some n but does not belong to $\overline{V}_i$ for all i. Similarly, the collection $\{V'_n\}$ covers B.

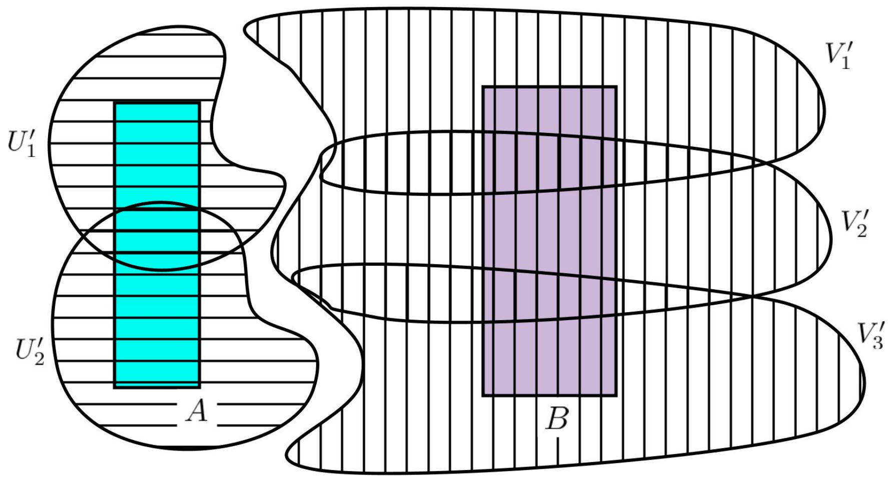


Figure 4.5

Define the open sets

$$U' = \bigcup_{n\in\mathbb{N}} U'_n \quad \text{and} \quad V' = \bigcup_{n\in\mathbb{N}} V'_n.$$

Claim: $U' \cap V' = \emptyset$.

Let $x \in U' \cap V'$. Then $x \in U'_j \cap V'_k$ for some j and k. Suppose $j \leq k$. Then from the definition of U'_j, we get $x \in U_j$. Since $j \leq k$, it follows from the definition of V'_k that $x \notin \overline{U}_j$ which is a contradiction. Similarly, we get contradiction if $j \geq k$.

Hence, X is normal. □

Theorem 4.1.10

Every metrizable space is normal.

Proof. Let X be a metrizable topological space with metric d which induces the topology on X. Let A and B be disjoint closed subsets of X. For each $a \in A$, choose ε_a such that the ball $B(a, \varepsilon_a)$ does not intersect B. Similarly, for each $b \in B$, choose ε_b such that the ball $B(b, \varepsilon_b)$ does not intersect A. Take

$$U = \bigcup_{a \in A} B\left(a, \tfrac{\varepsilon_a}{2}\right) \quad \text{and} \quad V = \bigcup_{b \in B} B\left(b, \tfrac{\varepsilon_b}{2}\right).$$

Then U and V are open sets containing A and B respectively.
Claim: $U \cap V = \emptyset$.

Let $z \in U \cap V$. Then

$$z \in B\left(a, \tfrac{\varepsilon_a}{2}\right) \cap B\left(b, \tfrac{\varepsilon_b}{2}\right)$$

for some $a \in A$ and $b \in B$. By triangle inequality,

$$d(a,b) \leq d(a,z) + d(z,b) < \tfrac{\varepsilon_a + \varepsilon_b}{2}.$$

If $\varepsilon_a \leq \varepsilon_b$, then $d(a,b) < \varepsilon_b$ which means that the ball $B(b, \varepsilon_b)$ contains a. This is not possible. On the other hand, if $\varepsilon_b \leq \varepsilon_a$, then $d(a,b) < \varepsilon_a$ which means that $B(a, \varepsilon_a)$ contains b. This is also not possible and hence $U \cap V = \emptyset$. □

Theorem 4.1.11

Every compact Hausdorff space is normal.

Proof. Let X be a compact Hausdorff space. First we show that X is regular. Let $x \in X$ and B be a closed subset of X such that $x \notin B$. Then by Lemma 3.2.10, there exists disjoint open sets containing x and B respectively. Hence, X is regular.

Now, we show that X is normal. Let A and B be disjoint closed sets of X. For each $a \in A$, by regularity of X, choose disjoint open sets U_a and V_a containing a and B respectively. Then the collection $\{U_a\}_{a \in A}$ is a cover of A. Since A is compact, A is covered by finitely many sets $U_{a_1}, \ldots, U_{a_n}$. Then

$$U = U_{a_1} \cup \cdots \cup U_{a_n} \quad \text{and} \quad V = V_{a_1} \cap \cdots \cap V_{a_n}$$

are disjoint open sets containing A and B respectively. Hence, X is normal. □

The following diagram gives a pictorial presentation of the separation axioms we studied so far.

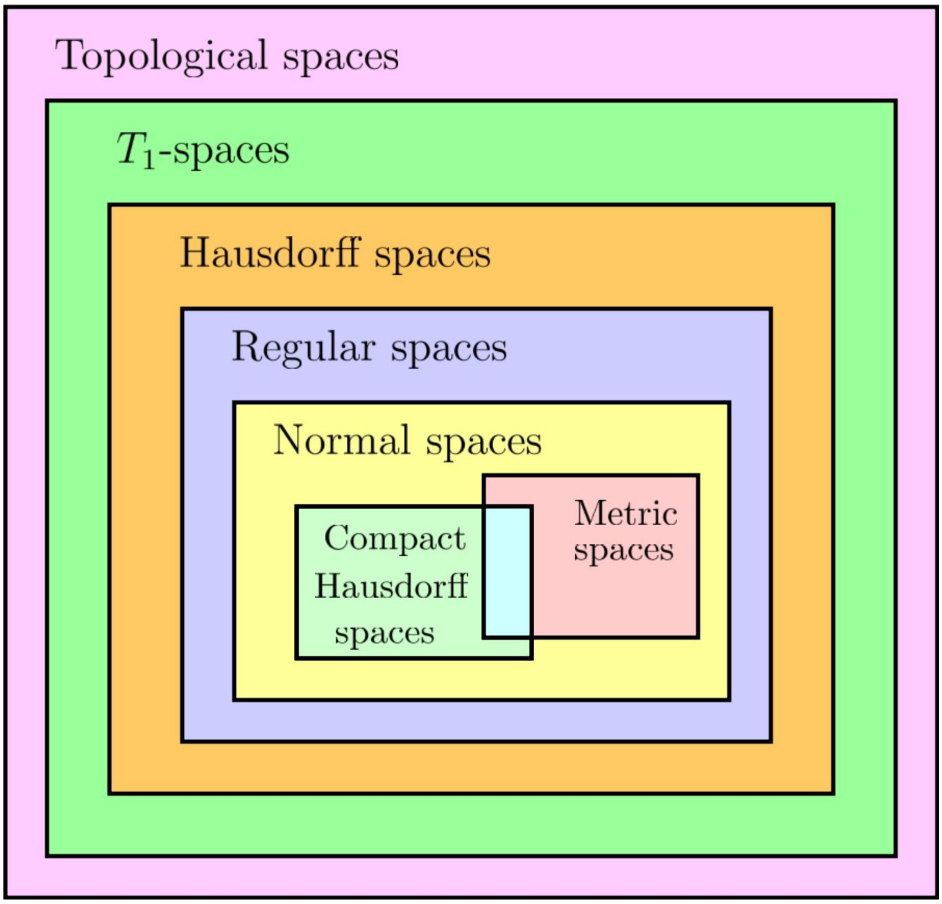


Figure 4.6: The separation properties

Exercise 4.6

Show that a closed subspace of a normal space is normal.

Exercise 4.7

Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a set X. If one of the spaces $(X,\mathcal{T})$ and $(X,\mathcal{T}')$ is Hausdorff (or regular, or normal), then what can be said about the other?

4.2 Urysohn Lemma and Tietze Extension Theorem

Theorem 4.2.1: Urysohn lemma

Let X be a normal space. Let A and B be disjoint closed subsets of X. Let $[a,b]$ be a closed interval in the real line. Then there exists a continuous map

$$f : X \to [a,b]$$

such that

$$f(x) = \begin{cases} a & \forall\, x \in A, \\ b & \forall\, x \in B. \end{cases}$$

Theorem 4.2.2: Tietze extension theorem

Let X be a normal space and A be a closed subspace of X.

(a) Any continuous map of A into the closed interval $[a,b]$ of $\mathbb{R}$ may be extended to a continuous map of all X into $[a,b]$.

(b) Any continuous map of A into $\mathbb{R}$ may be extended to a continuous map of all X into $\mathbb{R}$.

4.3 Complete Metric Spaces

Let (X,d) be a metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is *convergent* if there exits a point x in X such that for each $\varepsilon > 0$, there exists a positive integer n_0 such that $d(x_n,x) < \varepsilon$ for all $n \geq n_0$. Equivalently, for each open ball $B(x,\varepsilon)$ centered at x, there exists a positive integer n_0 such that $x_n \in B(x,\varepsilon)$ for all $n \geq n_0$. Note that sometimes the open ball $B(x,\varepsilon)$ is denoted as $S_\varepsilon(x)$ and called as "*open sphere*" centered at x.

The point x is called *limit* of the sequence x_n. We denote it by $x_n \to x$ or $\lim x_n = x$ and verbally say that "x_n approaches x", or "x_n converges to x". Since every metric space is Hausdorff and a sequence in a Hausdorff space has at most one limit, it follows that if $x_n \to x$, then x is unique.

Definition 4.3.1: Cauchy sequence

Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is said to be *Cauchy* if for every $\varepsilon > 0$, there exists a positive integer n_0 such that $d(x_n,x_m) < \varepsilon$ for all $n,m \geq n_0$.

Every convergent sequence is Cauchy. To see this, let $x_n \to x$. Then for each $\varepsilon > 0$, there exists an integer n_0 such that $d(x_n,x) < \frac{\varepsilon}{2}$ for all $n \geq n_0$. Therefore, by the triangle inequality

$$d(x_n,x_n) \leq d(x_n,x) + d(x,x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, every convergent sequence is Cauchy. However, the converse is not true. For example, consider the subspace $X = (0,1]$ of $\mathbb{R}$. Clearly the sequence $x_n = \frac{1}{n}$ is Cauchy but it is not convergent in X because the point 0 to which it tends to is not in the space X.

Thus, not every Cauchy sequence is convergent. The notion of convergence does not merely depend on the sequence itself but it also depends on the space because convergent sequence must converge to a point in that space.

Definition 4.3.2: Complete metric space

A metric space in which every Cauchy sequence is convergent is called a *complete metric space*.

The real line $\mathbb{R}$ and the complex plane $\mathbb{C}$ are complete metric spaces. The completeness of $\mathbb{C}$ depends on the completeness of the real line. The space $(0,1]$ mentioned above is not complete, but it can be made complete by adjoining the point 0 to it to form a larger space $[0,1]$. In fact, any metric space which is not complete can be made so by suitably adjoining additional points to it.

Observe that the terms *limit* and *limit point* are different. A sequence may have a limit but cannot have a limit point; whereas a set of points of a sequence may have a limit point but cannot have a limit. For instance, the constant real sequence $\{1,1,\ldots,1,\ldots\}$ is convergent to limit 1, but the set of points of the sequence is the singleton set $\{1\}$ and it does not have any limit point. The following result relates these two concepts to each other.

Theorem 4.3.3

If a convergent sequence in a metric space has infinitely many distinct points, then its limit is a limit point of the set of points of the sequence.

Proof. Let X be a metric space and $\{x_n\}$ be a convergent sequence in X with limit x. Suppose, if possible, x is not a limit point of the set of points (i.e. the range) of the sequence $\{x_n\}$. Then there exists an open ball $B(x,\varepsilon)$ centered at x which does not contain any point of the sequence different from x. However, since $x_n \to x$, there is an integer n_0 such that $x_n \in B(x,\varepsilon)$ for all $n \geq n_0$. Then $x_n = x$ for all $n \geq n_0$. From this we conclude that the sequence $\{x_n\}$ has only finitely many distinct points which is a contradiction. □

Suppose X is a complete metric space and Y is a subspace of X. What condition ensures the completeness of the subspace Y? The following theorem guarantees the completeness of subspaces of a complete metric spaces.

Theorem 4.3.4

Let X be a complete metric space and Y be a subspace of X. Then Y is complete if and only if it is closed.

Proof. Assume that Y is complete as a subspace of X. We show that it is closed. Let y be a limit point of Y. Then for each positive integer n, the open ball $B\left(y,\frac{1}{n}\right)$ contains a point y_n in Y. Clearly, the sequence $\{y_n\}$ converges to y in X and hence it is a Cauchy sequence. Since Y is complete, $y \in Y$. Therefore, Y is closed.

Conversely, assume that Y is a closed subspace of X. We show that it is complete. Let $\{y_n\}$ be a Cauchy sequence in Y. Then it is also a Cauchy sequence in X, and since X is complete, y_n converges to a point x in X. We want to show that $x \in Y$.

- If $\{y_n\}$ has only finitely many distinct points and $y_n \to x$, then x is a point of the sequence which is repeated infinitely many times. Since x is a point of the sequence $\{y_n\}$ in Y, it must be in Y.
- On the other hand, if $\{y_n\}$ has infinitely many distinct points and $y_n \to x$, then (by previous theorem) x must be the limit point of set of points of the sequence $\{y_n\}$. Therefore, x is also a limit point of Y and since Y is closed, $x \in Y$.

□

4.3.1 Cantor's Intersection Theorem

Let (X,d) be a metric space and $A \subset X$. Recall that the *diameter* of the set A is denoted by $d(A)$ or $\operatorname{diam}(A)$ and is defined as

$$\operatorname{diam}(A) = \sup\{d(x,y) \mid x,y \in A\}.$$

A sequence $\{A_n\}$ of subsets of a metric space is called a *decreasing* sequence if

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots .$$

The following result gives conditions under which the intersection of such a sequence is nonempty.

Theorem 4.3.5: Cantor's Intersection Theorem

Let X be a complete metric space, and let $\{F_n\}$ be a decreasing sequence of nonempty closed subsets of X such that $\operatorname{diam}(F_n) \to 0$. Then $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Proof. Suppose $x,y \in F$. Since

$$d(x,y) \le \operatorname{diam}(F) \le \operatorname{diam}(F_n) \to 0,$$

it follows that $d(x,y) = 0$, i.e. $x = y$. Thus, F cannot have more than one point. Hence, it suffices to show that F is nonempty.

For each $n \in \mathbb{N}$, let x_n be a point in F_n. Let $\varepsilon > 0$ be given. Since $\operatorname{diam}(F_n) \to 0$, there exists $N \in \mathbb{N}$ such that $\operatorname{diam}(F_n) < \varepsilon$ for all $n \ge N$. Let $n,m > N$ and without loss of generality, assume $n < m$. Then $x_m \in F_m \subset F_n$. Hence,

$$d(x_n,x_m) \le \operatorname{diam}(F_n) < \varepsilon \qquad \text{for all } n,m \ge N,$$

i.e. $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $\{x_n\}$ has a limit, say $x \in X$.

We now show that $x \in F$. For this it suffices to show that $x \in F_{n_0}$ for a fixed but arbitrary n_0.

- If $\{x_n\}$ has finitely many distinct points, then since $x_n \to x$, the point x is repeated infinitely many times in the sequence, and therefore $x \in F_{n_0}$.
- If $\{x_n\}$ has infinitely many distinct points, then x is the limit point of the set of points of the sequence. Therefore, it is a limit point of the subset $\{x_n \mid n \ge n_0\}$ of the set of points of the sequence, and hence it is a limit point of F_{n_0}. Since F_{n_0} is closed, $x \in F_{n_0}$.

□

4.3.2 Baire's Theorem

Definition 4.3.6: Nowhere dense set

A subset A of a metric space is said to be *nowhere dense* if its closure has empty interior.

It is easy to see that

A is nowhere dense $\Leftrightarrow \overline{A}$ does not contain any nonempty open set
$\Leftrightarrow$ each nonempty open set has a nonempty open subset disjoint from $\overline{A}$
$\Leftrightarrow$ each nonempty open set has a nonempty open subset disjoint from A
$\Leftrightarrow$ each nonempty open set contains an open ball disjoint from A.

Definition 4.3.7

Let (X,d) be a metric space $a \in X$ and $r > 0$, then by the *closed sphere with radius r centered at a* we mean the set

$$B[a,r] = S_r[a] = \{x \in X : d(x,a) \leq r\}.$$

The next result states that a complete metric space cannot be covered by a sequence of nowhere dense sets.

Theorem 4.3.8: Baire's Theorem

If $\{A_n\}$ is a sequence of nowhere dense sets in a complete metric space X, then there exists a point in X which is not in any of the A_n's.

In other words, a complete metric space is of *second category*. That is, if $\{A_n\}$ is a sequence of nowhere dense sets in a complete metric space (X,d), then $\bigcup\limits_{n=1}^{\infty} A_n \neq X$. That is, there exists a point x outside the set $\bigcup\limits_{n=1}^{\infty} A_n$.

Proof. Let (X,d) be a complete metric space and $\{A_n\}$ be a sequence of nowhere dense sets in X. Since $(\overline{A_1})^\circ = \emptyset$, there is an open set $U_0 \subset X$ such that $A_1 \bigcap U_0 = \emptyset$.[1] Let $B_1 = B(x_1,r_1)$ be an open ball of radius $r_1 < 1$ such that $B_1 \subset U_0$.[2] That is $B_1 \bigcap A_1 = \emptyset$. Let $F_1 = B[x_1, \frac{r_1}{2}]$. Now $(\overline{A_2})^\circ = \emptyset$. So, choose an open ball $B_2 = B(x_2,r_2) \subset F_1^\circ$ of radius $r_2 < \frac{1}{2}$ such that $B_2 \bigcap A_2 = \emptyset$. Define $F_2 = B[x_2, \frac{r_2}{2}]$. Continuing in this way, for every $n \in \mathbb{N}$, we get the open balls $B_n = B(x_n,r_n) \subset F_{n-1}^\circ$ with $r_n < \frac{1}{n}$ and $B_n \bigcap A_n = \emptyset$. Define $F_n = B[x_n, \frac{r_n}{2}]$. Thus for every $n \in \mathbb{N}$, F_n is a closed set, $F_n \subset F_{n-1}$ and $\operatorname{diam}(F_n) \leq 2\frac{r_n}{2} = r_n < \frac{1}{n}$. Hence by the Cantor's Intersection Theorem, $\bigcap\limits_{n=1}^{\infty} F_n = \{x\}$, some singleton. Clearly, for any $i \in \mathbb{N}$, $x \in \bigcap\limits_{n=1}^{\infty} F_n \subset F_i$ and $F_i \bigcap A_i = \emptyset$. So, $x \notin A_i$. Thus $x \notin \bigcup\limits_{n=1}^{\infty} A_n$. Thus $\bigcup\limits_{n=1}^{\infty} A_n \neq X$. Hence X is of second category. $\square$

The following is an equivalent form of the Baire's theorem stated above.

Theorem 4.3.9: Baire's theorem

[1] For example $U_0 = X \setminus A_1$

[2] Take any point in $a \in U_0$, then there is $r > 0$ such that $B(a,r) \subset U_0$. If $r < 1$, take $B_1 = B(a,r)$. Otherwise take $B_1 = B(a, 0.8)$. $B(a,0.8) \subset B(a,r) \subset U_0$.

> If a complete metric space is the union of a sequence of its subsets, then the closure of at least one set in the sequence must have nonempty interior.

Exercise 4.8

Let X be a metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $x_n \to x$ and $y_n \to y$, then show that $d(x_n, y_n) \to d(x, y)$.

Exercise 4.9

Show that a Cauchy sequence is convergent if and only if it has a convergent subsequence.

Exercise 4.10

Show that a closed set is nowhere dense if and only if its complement is everywhere dense.

Index

Symbols

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