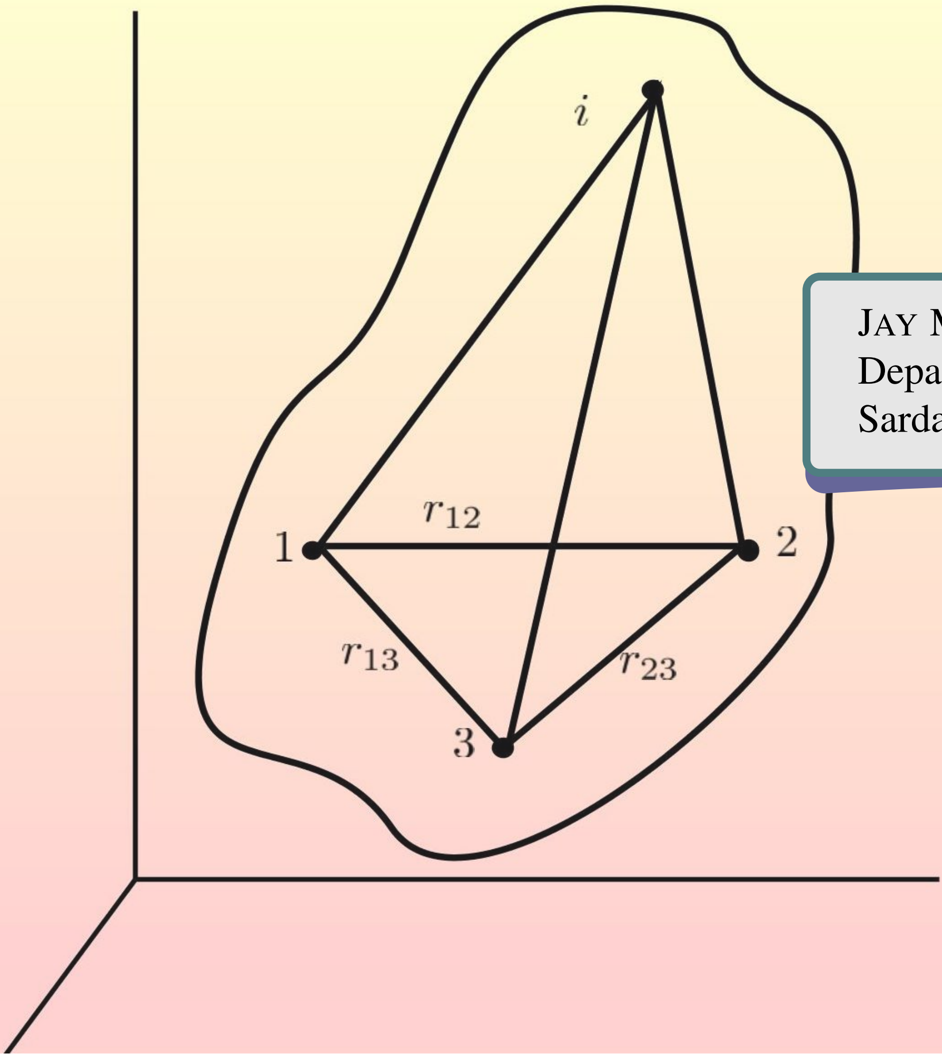
$$r_{ij} = c_{ij}$$

Lecture notes on

Mathematical Classical Mechanics

PS01EMTH22



$$n = 3N - k$$

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$$\delta \bar{r}_i$$

$$\dot{\eta} = J \frac{\partial H}{\partial \eta}$$

Semester - I
2018-19

$$[u, v]_{q,p}$$

Preface and Acknowledgments

These are the lecture notes of the course “Mathematical Classical Mechanics” offered to the M.Sc. (Semester - I/Semester - II) students at Department of Mathematics, Sardar Patel University, 2018-19. These are revision of the lecture notes of past two years of the same course offered by me. These are aimed to provide a reading material to the students in addition to the references mentioned in the university syllabus. These notes are tailored for the Mathematical Classical Mechanics (PS01EMTH22/PS02EMTH22) syllabus of M.Sc. (Semester-I/Semester-II) of the University and do not cover all the topics of Mechanics. Solutions to some of the exercises are not provided and were given as student seminar exercises and discussed in the seminar sessions during the semester.

These handouts are prepared from the recommended reference books as well as lecture notes of past years, and it is not my original work. We mostly followed the book “Classical Mechanics” by H. Goldstein, C. Poole and J. Safko. Many problems and exercises are listed at the end of the chapters. Students are strongly encouraged to refer to the seminar problems and solve these problems in addition to these notes. There may be a few errors/typos in this reading material. The students and interested readers are welcomed to give their valuable suggestions, comments or point out errors, if and whenever, they find any.

Acknowledgment

We are thankful to Prof. A. H. Hasmani for providing his lecture notes including the source files of Unit-1.

Dr. Jay Mehta

Contents

Syllabus

PS01EMTH22: Mathematical Classical Mechanics

Unit I: Constraints and their classification, principle of virtual work, de'Almbert's principle, various forms of Lagrange's equations of motion for holonomic systems, examples.

Unit II: Euler-Lagrange equations in various forms (statements only), Hamilton's variational principle, derivation of Lagrange's equation from Hamilton's variational principle, generalized momentum, cyclic coordinates, general conservation theorem, conservation of linear momentum and angular momentum in Lagrangian formalism and symmetry properties, energy function and conservation of total energy in Lagrangian formalism.

Unit III: Hamilton's canonical equation of motion, relation with Lagrange's equation, cyclic coordinate, Routhian procedure, variational principle approach to Hamilton's equation of motion, examples.

Unit IV: Canonical transformations, generating functions, symplectic condition, infintesimal canonical transformations, examples. Poisson bracket, Lagrange bracket, formal solution of equations of motion in terms of Poisson brackets, examples.

Text Book

1. Goldstein H., Poole C. and Safko J., Classical Mechanics, (Third Edition), Pearson Education, Inc., Indian Low Price Edition, 2002.

   Articles: 1.3, 1.4, 1.5 and 1.6, 4.1 (understanding of constraints and generalized coordinates in a rigid body motion); 2.1, 2.2 (statements only), 2.3, 2.6 and 2.7; 8.1,8.2, 8.3 and 8.5; 9.1,9.2, 9.4, 9.5 and 9.6.

Reference Books

1. Bhatia V. B., Classical Mechanics, Narosa Publishing House, 1997.

2. Sankara Rao K., Classical Mechanics, Prentice-Hall of India, 2005.

CHAPTER 1

Lagrange's Formulation

1.1 Mechanics of a particle

1.1.1 Basic terminology and concepts in Mechanics

There are three parts of Mechanics:

1. **Statics** - Theory of objects at rest.
2. **Kinematics** - Analysis of motions.
3. **Dynamics** - Analysis of the causes of motion.

The following are the components of Mechanics:

1. **Observer:** There exits an observer. By an observer, we mean any object or person capable enough to take observations of any physical process, for example, motion, etc.

2. **Space:** There exits a space in which a physical process takes place including motion. This space is the space around us called the physical space and is denoted by V.

   V is a mathematical space when it is a set with a structure, for example vector space, group, metric space, etc.

   $$\begin{array}{rll} (V,+,\cdot) & V-\text{ set} & +,\cdot-\text{ structure} \\ (X,d) & X-\text{ set} & d-\text{ structure.} \end{array}$$

   Physical space: Passing through any point three mutually perpendicular lines can be drawn. This number three is maximum. For any point $P \in V$, an ordered triple (x,y,z) can be associated with P, called the Cartesian co-ordinates of P, where x,y,z are real numbers.

   $$P = P(x,y,z) \in V \sim \mathbb{R}^3.$$

   Thus, it can be shown that V is a vector space over $\mathbb{R}$ and is isomorphic to $\mathbb{R}^3$. The physical space V is therefore the Euclidean space $\mathbb{R}^3$.

3. **Time:** Time is a real parameter denoted by t. Any physical process occurs in some time interval. Time flows uniformly, i.e. it does not stop. We will assume time to be an absolute parameter, i.e. it does not depend on any physical process or observer.

4. **Space-time:** The space and time together forms an entity called the space-time: $\mathbb{R}^4 \sim \mathbb{R}^3 \times \mathbb{R}$.

$$\text{Event: } (\underbrace{x, y, z,}_{\mathbb{R}^3} \underbrace{t}_{\mathbb{R}}) \in \mathbb{R}^4.$$

   Any event can be described by its place (location) of occurrence and the time of its occurrence. The totality of all the events is called the universe.

   Here the space is assumed to be continuum (i.e. no gap) and homogeneous, i.e. all the points have equal status. Thus, any point can be chose as origin. The space is also assumed to be isotropic, i.e. all directions have equal status. Then any direction can be chose as the fundamental direction.

5. **Motion:** It can be described by the change in position. There are two types of motion:

   (a) *Translation* (with respect to a point) Motion on a straight line.

   (b) *Rotation* (with respect to a straight line) Motion with respect to an axis.

   Important assumption about motion

   Any motion is a combination of a single translation and a single rotation.

1.1.2 Dictionary of Mechanics

1. **Particle:** A particle is the smallest unit of matter having inertial property and a definite position. For example, a point (or an object) in $\mathbb{R}^3$ has a definite position and it can be considered as a particle. On the contrary, waves do not have definite position so they cannot be considered as particles.

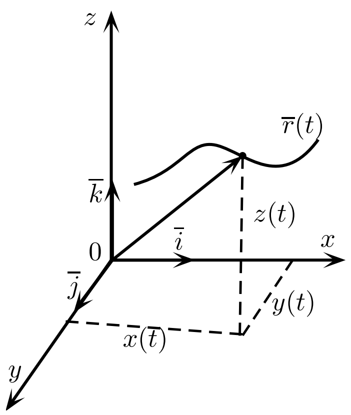
2. **Mass:** The mass of a particle, denote by m, is the quantitative measure of its inertial property. Different particles may have different mass and hence different inertia. The inertia of the particle is measured by its mass.

   The mass of the particle is assumed to be constant during any physical process including motion, i.e. it does not change due to motion.

3. **Position:** A particle has a definite position in the physical space and hence it can be represented by a point in $V = \mathbb{R}^3$. The position vector of the point in $\mathbb{R}^3$ occupied by the particle (at any given time) is the position of the particle. It is denoted by $\bar{r}$.

   Unlike mass, the position of the particle changes with motion or with time. Hence, we write it as a function of time t:

$$\bar{r} = \bar{r}(t)$$
$$\bar{r}(t) = (x(t), y(t), z(t)) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k},$$



where $\bar{i} = (1,0,0), \bar{j} = (0,1,0), \bar{k} = (0,0,1)$. In other words, co-ordinated of a particle are taken as a function of t. A particle is described by its position and its mass.

Different coordinates and relations between them:

$$\begin{aligned} P(x,y,z) &- \text{Cartesian coordinates} \\ P(r,\theta,\phi) &- \text{Spherical coordinates} \\ P(\rho,\phi,z) &- \text{Cylindrical coordinates} \end{aligned}$$

Spherical coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

Cylindrical coordinates

$$\begin{aligned} \rho &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\frac{y}{x} \\ z &= z \end{aligned}$$

4. **Velocity:** The velocity of the particle is denoted by $\bar{v}(t)$ and is given by

$$\bar{v}(t) = \frac{d\bar{r}}{dt} = \dot{\bar{r}} = (\dot{x}(t), \dot{y}(t), \dot{z}(t)).$$

Thus, velocity is the rate of change of position of the particle with respect to time t.

5. **Linear momentum:** Linear momentum is the measure of the linear (translation) motion. It is denote by $\bar{p}$ and is given by

$$\bar{p} = m\bar{v} = m\dot{\bar{r}}.$$

6. **Acceleration:** Acceleration of a particle is denoted by $\bar{a}(t)$ and is given by

$$\bar{a}(t) = \frac{d\bar{v}}{dt} = \frac{d}{dt}\left(\frac{d\bar{r}}{dt}\right) = \frac{d^2\bar{r}}{dt^2} = \ddot{\bar{r}}.$$

Thus, acceleration of a particle is the rate of change of its velocity with respect to time.

7. **Force:** Force on a particle, denoted by $\bar{F}$ is defined as the cause of motion of the particle (i.e. cause of change in its position). Since, a particle has inertial property this definition of force follows from the Newton's first law of motion which states that:
"A particle (or an object) at rest (or stationary) remains at rest and a particle in motion remains in motion (with uniform velocity) unless and until an (external) force is applied to it."

Also, the Newton's second law of motion states that: "force is the rate of change of linear momentum." Therefore, we have

$$\bar{F} = \frac{d\bar{p}}{dt} = \frac{d(m\bar{v})}{dt} = m\frac{d\bar{v}}{dt} = m\bar{a} = m\ddot{\bar{r}}. \tag{1.1}$$

Force in equation (1.1) is a vector quantity and hence it has three components. Thus, (1.1) gives a system of three equations as follows:

$$F_x = m\ddot{x} = m\frac{d^2x}{dt^2}, \qquad F_y = m\ddot{y} = m\frac{d^2y}{dt^2}, \qquad F_z = m\ddot{z} = m\frac{d^2z}{dt^2}. \tag{1.2}$$

These equations are called Newton's equations of motion (NEOM).

Futhermore, equation (1.1) can be used to determine the acceleration of the particle of given mass when the force on the particle is known and vice-versa (i.e. it can be used to determine the force when acceleration of the particle is known).

8. **State of the particle:** The state of a particle at time t is denoted by $\bar{s}(t)$ and is given by

$$\bar{s}(t) = (\bar{r}(t), \bar{v}(t)) = (x(t), y(t), z(t), \dot{x}(t), \dot{y}(t), \dot{z}(t)).$$

9. **Newton's law of indeterminacy:** If the force on a particle is known then the state of the particle at any time t can be determined provided that its initial state is given.

   This is true vice-versa also, i.e. if the state of a particle at some time t is given then the force on the particle can be determined provided that its initial state is given. Let us consider couple of exercises based on this law:

Theorem 1.1.1: Law of conservation of linear momentum

In absence of force, the linear momentum of a particle is conserved.

Proof. We know, by Newton's second law of motion, that force is the rate of change of momentum, i.e.

$$\bar{F} = \frac{d\bar{p}}{dt}.$$

Therefore, in absence of force, we have

$$\frac{d\bar{p}}{dt} = 0$$

and hence the linear momentum $\bar{p}$ is constant with respect to time t. □

10. **Angular velocity:** Rate of change of angular displacement (or angular position) is called angular velocity. It is denote by $\bar{w}$.

11. **Angular momentum:** It is the measure of angular motion. It is denote by $\bar{L}$ and given by

$$\bar{L} = \bar{r} \times \bar{p} = m(\bar{r} \times \bar{v}),$$

   where $\bar{r}$ is the position vector of the particle and $\bar{p}$ is its linear momentum.

12. **Torque (Angular force):** It is the cause of angular (rotational) motion. It is denote by $\bar{N}$ and given by

$$\bar{N} = \bar{r} \times \bar{F}.$$

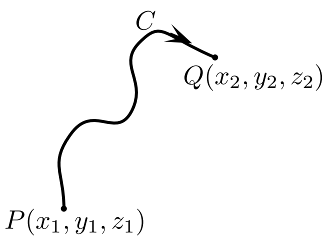
Remark 1.1.2. For vectors $\bar{u} = (u_1, u_2, u_3)$, $\bar{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, the cross product of $\bar{u}$ and $\bar{v}$ is denoted by $\bar{u} \times \bar{v}$ and is defined as

$$\bar{u} \times \bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \bar{i}(u_2v_3 - u_3v_2) + \bar{j}(u_3v_1 - u_1v_3) + \bar{k}(u_1v_2 - u_2v_1).$$

Note that, $\bar{i} \times \bar{i} = \bar{j} \times \bar{j} = \bar{k} \times \bar{k} = 0$ and hence $\bar{u} \times \bar{u} = 0$ for any vector $\bar{u}$.

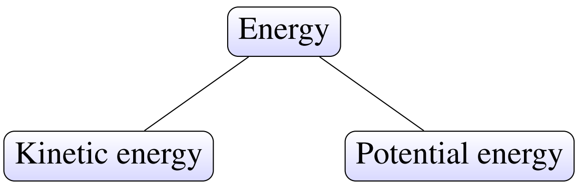
13. **Work:** Work done is a property of the force. It is defined as follows: Suppose a particle is at $P(x_1, y_1, z_1)$ (position 1) and it is brought to $Q(x_2, y_2, z_2)$ (position 2) along a curve C under the force $\bar{F}$. Then the work done by the force $\bar{F}$ is given by

$$W_{12} = \int_{\substack{1 \\ \text{along } C}}^{2} \bar{F}.d\bar{r} \tag{1.3}$$



The integral in equation (1.3) is a line integral. Hence, the work done depends on the path also.

14. **Energy:** Energy is the capacity to do work. There are two types of energy:



   (i) Kinetic Energy: Kinetic energy is the energy of the particle due to motion and it is denoted by T. Kinetic energy of a particle of mass m and moving with speed v is given by $T = \frac{1}{2}mv^2$.

   (ii) Potential Energy: Potential energy is the energy of the particle due to its position and it is denoted by V. It is called internal energy.

1.1.3 Conservative force field

Definition 1.1.3

Any physical quantity (scalar or vector) is said to be a *field quantity* if it can be described as a function of points of the space, i.e. as a function of (x, y, z).

Definition 1.1.4: Conservative force field

A force field $\bar{F} = (x, y, z)$ is said to be *conservative* if work done by $\bar{F}$ does not depend on the path but it depends only on the initial and final positions.

Note: In nature almost all the forces are conservative. For example, gravity (gravitational force) is a conservative force.

Theorem 1.1.5

For a conservative force field $\bar{F}$, the following statements are equivalent:

1. Force $\bar{F}$ is conservative.
2. $\oint_C \bar{F}.d\bar{r} = 0$, where C is a closed curve, i.e. work done along a closed path is zero.
3. The curl of $\bar{F}$ is the zero vector, i.e. $\nabla \times \bar{F} = 0$, where $\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$.
4. The force can be written as the negative gradient of a potential, i.e. $\bar{F} = -\nabla V$ for some scalar field $V(x,y,z)$. Equivalently
$$(F_x, F_y, F_z) = -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right).$$

Example 1.1.6. Is the force $\bar{F} = (2xy-1, x^2+z, y)$ conservative? If yes then determine the corresponding potential V such that $\bar{F} = -\nabla V$.

Solution. We have

$$\begin{aligned}
\nabla \times \bar{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy-1 & x^2+z & y \end{vmatrix} \\
&= \hat{i}\left(\frac{\partial}{\partial y}(y) - \frac{\partial}{\partial z}(x^2+z)\right) - \hat{j}\left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial z}(2xy-1)\right) + \hat{k}\left(\frac{\partial}{\partial x}(x^2+z) - \frac{\partial}{\partial y}(2xy-1)\right) \\
&= \hat{i}(1-(0+1)) - \hat{j}(0-0) + \hat{k}(2x-2x) \\
&= \hat{i}(0) - \hat{j}(0) + \hat{k}(0) = 0.
\end{aligned}$$

Thus, the given force $\bar{F}$ is conservative.

Now, we find the corresponding potential V. $\bar{F} = -\nabla V$ implies

$$(2xy-1, x^2+z, y) = -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right).$$

Therefore,

$$\frac{\partial V}{\partial x} = -2xy+1, \qquad \frac{\partial V}{\partial y} = -x^2 - z, \qquad \frac{\partial V}{\partial z} = -y.$$

From the first equation above, we have

$$V_x = -\int\limits_{y \text{ constant}} (2xy-1)dx = -x^2y + x + f_1(y,z).$$

Differentiating above partially with respect to y, we get

$$\frac{\partial V}{\partial y} = -x^2 + \frac{\partial f_1}{\partial y}$$

$$-x^2 - z = -x^2 + \frac{\partial f_1}{\partial y}$$
$$\Rightarrow \frac{\partial f_1}{\partial y} = -z.$$

Integrating with respect to y and keeping z constant, we get $f_1(y,z) = -yz + f_2(z)$. Then it has the form

$$V = -x^2y + x - yz + f_2(z).$$

Now, differentiating partially with respect to z, we get

$$\frac{\partial V}{\partial z} = -y + \frac{\partial f_2}{\partial z}$$
$$-y = -y + \frac{\partial f_2}{\partial z}$$
$$\Rightarrow \frac{\partial f_2}{\partial z} = 0 \Rightarrow f_2 = c.$$

Therefore, $\boxed{V = -x^2y + x - yz + c}$. □

Potential energy of a particle situated in a conservative force field $\bar{F}$ is denoted by V and is given by $\bar{F} = -\nabla V$. Potential energy of a particle at point $P(x,y,z)$ corresponding to a conservative force is given by

$$V(P) = V(x,y,z) = -\int_{\infty}^{P(x,y,z)} \bar{F}.d\bar{r}, \tag{1.4}$$

where ∞ is the point where potential energy is zero.

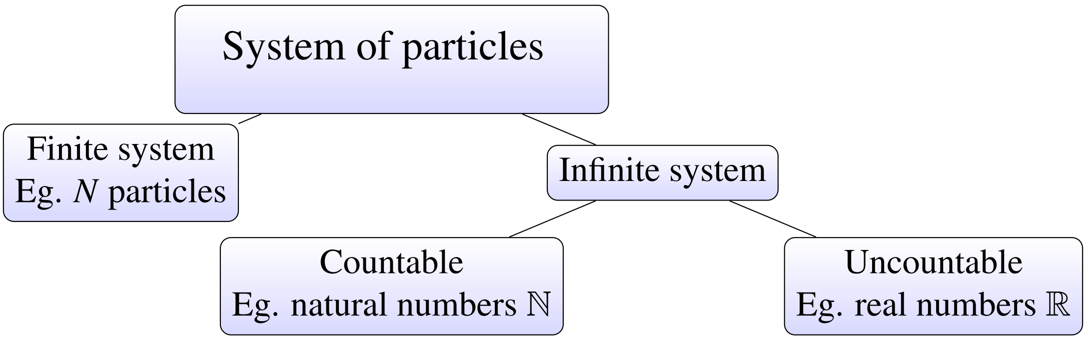
Remark 1.1.7. 1. Potential energy of a particle is *not unique*. In fact, if $V(x,y,z)$ is potential energy of the particle then $V_1 = V + \lambda$, where λ is constant is also potential energy. In other words, potential energy is unique up to addition of a constant.

2. If a particle of mass m placed at a height h in the gravitational force field of the earth then its potential energy is given by $V = mgh$, where g is gravitational acceleration.

Theorem 1.1.8: Law of conservation of energy

In a conservative force field the sum of kinetic energy and potential energy of a particle is conserved (i.e. it does not change with respect to time). This implies $E = T + V$ is constant (E is called the total energy).

1.2 Mechanics of a system of particles



1.2.1 Finite system of particles

Here we consider a system of N number of particles.

- ⋆ **Position of the system:** Each particle in the system has a definite position. Let $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N$ be the positions of the particles.

  For the system as a whole the position is given by the N position vectors of the particles in the system. Since each position vector of the particle is in $\mathbb{R}^3$, the position of the system is a vector in $\mathbb{R}^{3N}$ given by a $3N$-tuple $(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_N, y_N, z_N)$, where $\bar{r}_i = (x_i, y_i, z_i)$, $1 \le i \le N$.

- ⋆ **Masses:** Each particle in the system has a definite mass. Let $m_1, m_2, \ldots, m_N$ be the masses of the N particles in the system. Then the (total) mass of the system is given by

$$M = \sum_{i=1}^{N} m_i = m_1 + m_2 + \cdots + m_N.$$

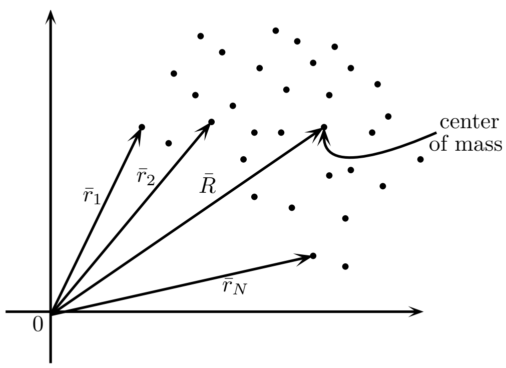
- ⋆ **Linear momentum of the system:** Total linear momentum of the system is denoted by $\bar{P}$ and is given by the sum of the linear momenta of the particles in the system, i.e.

$$\bar{P} = \sum_{i=1}^{N} \bar{p}_i = \sum_{i=1}^{N} m_i \dot{\bar{r}}_i.$$

- ⋆ **Center of Mass:**

  The center of mass of a system of N-particles can be described as a point having position vector $\bar{R}$ given by

$$\bar{R} = \frac{\sum_{i=1}^{N} m_i \bar{r}_i}{\sum_{i=1}^{N} m_i} = \frac{m_1\bar{r}_1 + m_2\bar{r}_2 + \cdots + m_N\bar{r}_N}{m_1 + m_2 + \cdots + m_N}.$$



Remark 1.2.1. Note that the (total) linear momentum of the system of particles is given as the sum of the linear momenta of the particles in the system. Since, the center of mass of the system has position vector $\bar{R}$ one would also like to define the linear momentum of the system as

$$\bar{P} = M\dot{\bar{R}} = \left(\sum_{i=1}^{N} m_i\right)\dot{\bar{R}}.$$

In what follows, we show that both these definitions coincide.

⋆ **Linear momentum and center of mass:**

$$\begin{aligned}\bar{P} = \sum_{i=1}^{N} \bar{p}_i = \sum_{i=1}^{N} m_i\dot{\bar{r}}_i & \\ &= m_1\dot{\bar{r}}_1 + m_2\dot{\bar{r}}_2 + \cdots + m_N\dot{\bar{r}}_N \\ &= m_1\frac{d\bar{r}_1}{dt} + m_2\frac{d\bar{r}_2}{dt} + \cdots + m_N\frac{d\bar{r}_N}{dt} \\ &= \frac{M\frac{d}{dt}\left(m_1\bar{r}_1 + m_2\bar{r}_2 + \cdots + m_N\bar{r}_N\right)}{M} \\ &= M\frac{\frac{d}{dt}\left(\sum_{i=1}^{N} m_i\bar{r}_i\right)}{M} = M\dot{\bar{R}}\end{aligned}$$

Thus, the linear momentum of a system of particles can be given as

$$\boxed{\bar{P} = M\dot{\bar{R}}}. \tag{1.5}$$

The expression in (1.5) can be interpreted as follows: the linear momentum of the system is equal to the velocity of the center of mass multiplied by the total mass of the system.

We may thus regard center of mass as a particle having position vector $\bar{R}$ and mass equal to the total mass of the system. In other words, we say that the total mass of the system is concentrated at the center of mass.

⋆ **Conservation of total linear momentum:** We know that $\bar{P} = \sum_{i=1}^{N} \bar{p}_i$. Therefore,

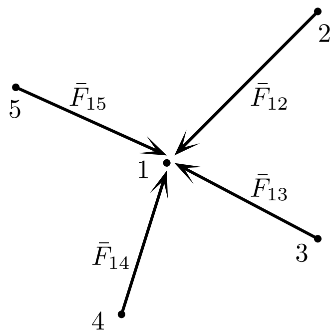
$$\dot{\bar{P}} = \sum_{i=1}^{N} \dot{\bar{p}}_i = \sum_{i=1}^{N} \bar{F}_i\ , \tag{1.6}$$

where $\bar{F}_i$ is the force on the i-th particle. Now, force on the i-th particle comprises of two parts: *internal force* and *external force*.

Internal force:

The internal force on a particle is the force exerted by (i.e. force due to) the other particles of the system. The internal force on the i-th particle is given by

$$\bar{F}_i^{(\text{int})} = \sum_{\substack{j=1\\ j\neq i}}^{N} \bar{F}_{ij}. \tag{1.7}$$



External force: External force on a particle is the force exerted from outside the system. The total force on i-th particle is the sum of the external and internal forces, i.e.

$$\bar{F}_i = \bar{F}_i^{(\text{int})} + \bar{F}_i^{(\text{ext})} = \bar{F}_i^{(\text{ext})} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \bar{F}_{ij}. \tag{1.8}$$

Thus, the total force on the system is given by

$$\bar{F} = \sum_{i=1}^{N} \bar{F}_i = \sum_{i=1}^{N} \bar{F}_i^{(\text{ext})} + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \bar{F}_{ij}.$$

Thus, from equations (1.6) and (1.8), we have

$$\dot{\bar{P}} = \sum_{i=1}^{N} \bar{F}_i^{(\text{ext})} + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \bar{F}_{ij}. \tag{1.9}$$

The second sum in the RHS of the above equation is the sum of pairs $\bar{F}_{ij}$ and $\bar{F}_{ji}$.

$$\dot{\bar{P}} = \sum_{i=1}^{N} \bar{F}_i^{(\text{ext})} + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} (\bar{F}_{ij} + \bar{F}_{ji}).$$

By Newton's third law of motion (principle of action and reaction), $\bar{F}_{ij} = -\bar{F}_{ji}$ and hence $\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{N} (\bar{F}_{ij} + \bar{F}_{ji}) = 0$ which implies $\dot{\bar{P}} = \sum\limits_{i=1}^{N} \bar{F}_i^{(\text{ext})} = \bar{F}^{(\text{ext})}$.

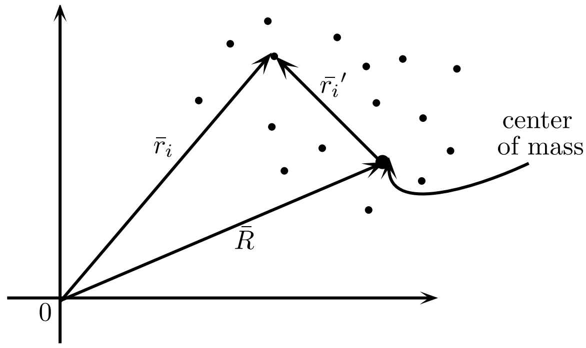
⋆ **Law of conservation of (total) linear momentum:** Total linear momentum of the system is conserved if the total external force is zero provided that the internal forces are Newtonian.

⋆ **Angular momentum of a system:** The angular momentum of a system of particles is the sum of the angular momenta of the particles in the system, i.e.

$$\bar{L} = \sum_{i=1}^{N} \bar{L}_i = \sum_{i=1}^{N} \bar{r}_i \times \bar{p}_i.$$

⋆ **Angular momentum and center of mass:**

As shown in figure, if $\bar{r}_i$ and $\bar{r}_i{}'$ denotes position of the i-th particle with respect to origin and center of mass respectively, then



$$\bar{r}_i = \bar{R} + \bar{r}_i{}' \quad \text{or} \quad \bar{r}_i{}' = \bar{r}_i - \bar{R}.$$

Therefore, $\dot{\bar{r}}_i = \dot{\bar{R}} + \dot{\bar{r}}_i'$ and so

$$\begin{aligned}
\bar{L} &= \sum_{i=1}^{N} (\bar{R} + \bar{r}_i{}') \times m_i(\dot{\bar{R}} + \dot{\bar{r}}_i') \\
&= \sum_{i=1}^{N} m_i(\bar{R} \times \dot{\bar{R}} + \bar{r}_i{}' \times \dot{\bar{R}} + \bar{R} \times \dot{\bar{r}}_i' + \bar{r}_i{}' \times \dot{\bar{r}}_i') \\
&= \sum_{i=1}^{N} m_i(\bar{R} \times \dot{\bar{R}}) + \sum_{i=1}^{N} m_i(\bar{r}_i{}' \times \dot{\bar{R}}) + \sum_{i=1}^{N} m_i(\bar{R} \times \dot{\bar{r}}_i') + \sum_{i=1}^{N} m_i(\bar{r}_i{}' \times \dot{\bar{r}}_i')
\end{aligned}$$

Now, clearly the last two terms in the above expression vanishes and

$$\sum_{i=1}^{N} m_i(\bar{R} \times \dot{\bar{R}}) = \left(\bar{R} \times \sum_{i=1}^{N} m_i \dot{\bar{R}}\right) = \bar{R} \times M\dot{\bar{R}} = \bar{L}_{\text{cm}}.$$

Thus, the total angular momentum about a point O is the angular momentum of motion concentrated at the center of mass, plus the angular momentum about the center of mass.

1.3 Constraints and their classification

1.3.1 Constraints

In many real life situations the moving objects are restricted or constrained to move such that its coordinates and/or velocity components must satisfy some given relation at any instant of time. It is possible to express such a restriction as an equation or inequality involving coordinates. Mathematically this means that for a restricted motion the coordinates involved are not all independent. We define below the constraint in this context.

Definition 1.3.1: Constraint

Constraint is defined as a restriction on motion.

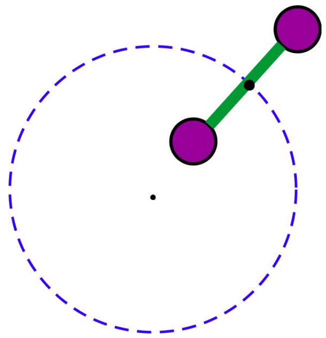
A constraint is given as a part of the problem. The forces which are responsible for restricting the motion of the object are called *constraint forces*. They are as such unknown forces. Also they are very strong that they barely allow the body under consideration to deviate even slightly. The effect of the constraint forces is to keep the constraint relations satisfied.

Example 1.3.2. Motion of a particle on a sphere.

It is required for the particle to be on the sphere, that way a restriction on the motion is imposed. This restriction can be expressed mathematically as an equation satisfied by the coordinates of the particle.

Example 1.3.3. Motion of two particles connected by an extensible weightless rod.

In this example two particles are connected by the rod and hence they have to move in such a way that they remain connected (with fixed distance between them) throughout the motion.



Example 1.3.4. In the Example 1.3.3 we put a further restriction on the motion by insisting that the center of the rod moves on a circle.

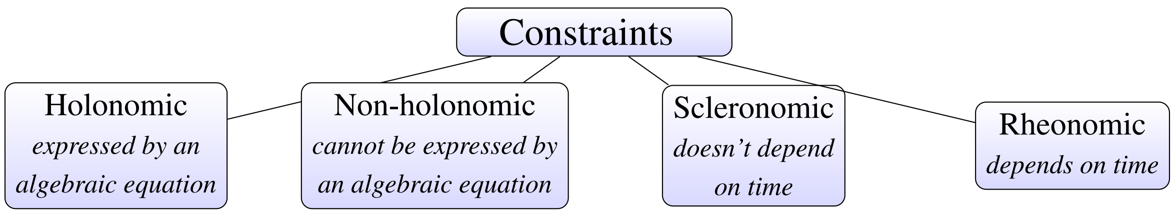
Remark 1.3.5. From the above two examples we understand the constraints as restriction on motion which is mentioned in the description of the mechanical system. We note that a constraint is due to some unknown force. This force is not an external force or internal force hence we call it *constraint force*.

Example 1.3.6. Motion of a particle on XY-plane. In general, motion of particle on any plane.

Example 1.3.7. Motion of a particle in a rectangular box.

1.3.2 Classification of constraints

The constraints are classified according to their nature. There are mainly four types of constraints which we shall discuss in this chapter. They are as follows:



Definition 1.3.8: Holonomic Constraint

A constraint is said to be holonomic if it can be mathematically described as an algebraic equation between coordinates of particles in the system.

Example 1.3.9. In the Example 1.3.2 above if (x,y,z) are coordinates of the particle and if the radius of the sphere is R then this constraint can be described as

$$x^2 + y^2 + z^2 = R^2.$$

This relation is an algebraic equation and hence the constraint is a holonomic constraint.

Example 1.3.10. Motion of a particle on a circle. A circle is in a plane, say XY-plane. Let the radius of the circle be R, then there are two constraints in the system and both are holonomic. They are $x^2+y^2=R^2$ and $z=0$.

Example 1.3.11. Motion of a particle in a plane. Here the constraint is that the coordinates of the particle, say (x,y,z) satisfy the equation of the plane $ax+by+cz+d=0$ for some constants a,b,c,d. This is the only constraint of motion.

In particular, if the particle moves in XY-plane, then the constraint is $z=0$ which is holonomic.

Example 1.3.12. Motion of a particle on a line or axis (linear motion). Suppose the particle movies in X-axis. Then the system has two holonomic constraints which are $y=0$ and $z=0$.

Example 1.3.13. If we take $x=0$, $y=0$ and $z=0$ or x,y then the system has 3 constraints and the particle is at rest at origin, i.e. motion is not possible in this case. If $x=a, y=b$ and $z=c$ are constants then the particle rests at point (a,b,c). In this case also all the three constraints are holonomic.

Definition 1.3.14: Non-holonomic Constraint

If a constraint cannot be described as an algebraic equation then it is called a non-holonomic constraint.

We note here that the relation describing a constraint may be an inequality, a transcendental equation or a differential equation. In these situations the constraint is no more a holonomic constraint.

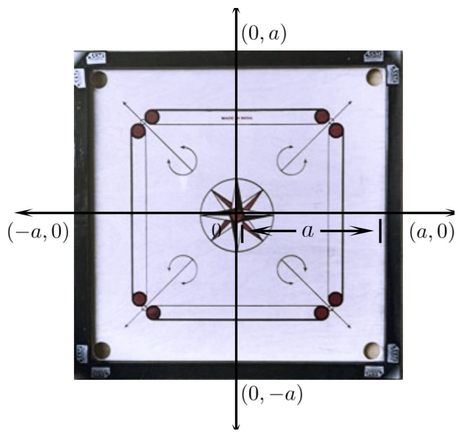
Example 1.3.15. In Example 1.3.7, i.e. motion of a particle inside a rectangular box, the constraints can be described with the help of inequalities and hence they are non-holonomic. If the length of sides of the rectangular box are a, b and c, then the constraints can be expressed as three inequalities of the form

$$|x| \le a,\ |y| \le b,\ |z| \le c.$$

Example 1.3.16. Motion on the carom-board can be considered as another example of a non-holonomic constraints. If the center of the carom board is considered as origin then the constraints can be given by

$$-a \le x,y \le a \quad \text{and} \quad z=0. \tag{1.10}$$

Thus, there are total three constraints here (i.e. $|x| \le a, |y| \le a, z=0$) among which the two constraints on x and y are non-holonomic while the constraint $z=0$ is a holonomic constraint.



Example 1.3.17. Consider a system of two particles joined by a mass less rod of fixed length. Suppose for simplicity, the system is confined to the horizontal plane. Suppose further that

the system is so constrained that the centre of the rod cannot have a velocity component perpendicular to the rod.

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the coordinates of the particles in the system. In this example there are following restrictions

(i) the motion is confined to the plane $z = 0$, i.e. for the particles $z_1 = 0$ and $z_2 = 0$,

(ii) the particles are connected by a rod of fixed length say l, this can be described by $(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2$ and

(iii) the center of the rod cannot have a velocity component perpendicular to the rod.

It is clear that the three constraints in the first two points above are holonomic constraints. Now we analyze the last constraint. It is clear that this constraint can be described as

$$(\dot{x}_1 + \dot{x}_2)\sin\theta = (\dot{y}_1 + \dot{y}_2)\cos\theta \tag{1.11}$$

This is a differential equation which can not be integrated and from this we can not get an algebraic equation connecting the coordinates. Consequently, it is non-holonomic.

Remark 1.3.18. Observe that in Examples 1.3.13, 1.3.16 and 1.3.15, the number of constraints is 3. Thus, all of them have same number of constraints. However, the difference is that in Example 1.3.13 the particle is at rest, while in case of Example 1.3.16 motion in possible and planner and in Example 1.3.15 the motion is possible in all the 3-dimensions.

Thus, the state or the motion of a particle cannot be determined merely from the number of constraints.

Definition 1.3.19: Rheonomic Constraint

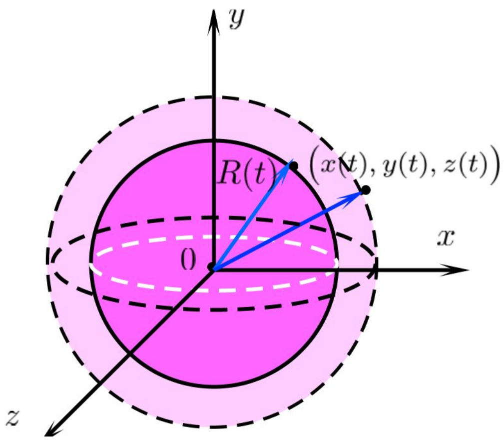
A constraint is said to be rheonomic if the constraint relation depend explicitly on time.

Example 1.3.20. Motion of a particle on an expanding sphere. Here the relation describing the constraint is

$$x^2 + y^2 + z^2 = R(t)^2,$$

where $R(t)$ is an increasing function of time. This relation depends on time and hence the constraint is rheonomic.

Similarly, motion of a particle on a contracting or shrinking sphere is also a system with a rheonomic constraint. In this case the radius $R(t)$ is a decreasing function of time t.



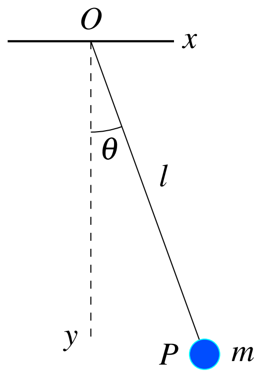
Definition 1.3.21: Scleronomic Constraint

A constraint is said to be scleronomic if the constraint relation does not depend explicitly on time.

Most of the constraints discussed above are scleronomic. All the examples (except Example 1.3.20) considered above are examples of scleronomic constraints.

Example 1.3.22. Describe simple pendulum and state all its constraints. Also mention the types of constraints.

Solution.
A particle (of mass m) is suspended at a point (say) P from a fixed point (say) O by an in-extendable string (as shown in the figure). The motion is fixed in a vertical plane under gravity. As shown in the figure if the length of the string OP be l. Then the two constraints are:



1. $z = 0$ (holonomic and scleronomic).
2. $x^2 + y^2 = l^2$ (holonomic and scleronomic), and x, y can be given by $x = l\cos\theta$, $y = l\sin\theta$, where θ is the angle made with the y-axis as shown in the figure.

□

Note that motion of a particle moving on a circle (in a plane) also has the same constraints as that of a simple pendulum. Hence the knowledge of constraints is insufficient to determine the motion or nature of a mechanical system.

Remark 1.3.23. There are other types of constraints also. They are based on the conservation of energy and also on on the possibility of forward and backward motion. They are beyond the scope of our syllabus and hence we will not discuss them.

1.3.3 Difficulties Imposed by Constraints

Consider a system of N-particles. Let $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N$ be the position vectors and $m_1, m_2, \ldots, m_N$ be the masses of these particles respectively. Suppose there are some constraints on the motion of this system. Due to constraints following difficulties arise:

1. Coordinates $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N$ are not independent.

   Constraints are relations between coordinates of the particles in the system. As a result, we have different equations which are not all independent. Hence the system becomes difficult to solve.

2. Constraints could be interpreted as effect of some forces. We know that constraints are due to unknown forces (i.e. constraint forces are unknown). Thus, these forces are not specified in the problem, they are unknowns in the problem.

   Equations of motion for such a system have the form,

   $$m\frac{d^2\bar{r}}{dt^2} = \bar{F} = \bar{F}^{(a)} + \bar{F}^{(c)}, \tag{1.12}$$

   where $\bar{F}^{(a)}$ and $\bar{F}^{(c)}$ are applied forces and constraint forces respectively. Since constraint forces are not known in above equation, left hand side as well as right hand side contain unknowns. This makes the problem unsolvable. Also, $\bar{r}$'s are not independent because of this system of equations given in (1.12) is a coupled system.

In the next section remedy to above mentioned difficulties in some situations is discussed.

1.3.4 Generalized coordinates and Degrees of Freedom

Suppose for a given system of N-particles the number of constraints is k. These constraints can be expressed mathematically as k-equations,

$$\begin{aligned} f_1(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N, t) &= 0, \\ f_2(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N, t) &= 0, \\ \vdots \qquad & \vdots \\ f_k(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N, t) &= 0. \end{aligned} \tag{1.13}$$

The coordinates are $3N$ in number namely, $x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_N, y_N, z_N$. They are unknowns in the equations of motion. These $3N$-parameters satisfy k-equations given in (1.13), hence number of independent parameters in the problem is $3N - k$. This leads to the following definition:

Definition 1.3.24: Degrees of Freedom

The minimum number of independent parameters required for the mathematical description of the given system is called degrees of freedom of the given system.

It is clear that for a system of N-particles with k-constraints the degrees of freedom is $3N - k$. Degrees of freedom of a system is denoted by n, i.e.

$$n = 3N - k.$$

Definition 1.3.25: Generalized Coordinates

Consider a system of N particles and k constraints having n degrees of freedom. Then

$$n = 3N - k.$$

Thus, the number of independent variables or parameters required to describe the motion or the position of the system is n. Let $q_1, q_2, \ldots, q_n$ be chosen to describe the motion of the system. These parameters are called *generalized coordinates* for the given system.

Example 1.3.26. Determine degrees of freedom in case of motion of a particle on a sphere and assign generalized coordinates.

Solution. The number of particles is 1, i.e. $N = 1$. As seen in Example 1.3.2, we have only constraint in this case which is $x^2 + y^2 + z^2 = R^2$. Therefore, $k = 1$. Hence, degrees of freedom is

$$n = 3N - k = 3 - 1 = 2.$$

Since the degrees of freedom is 2, we will have 2 generalized coordinates in this case. Expressing it in usual spherical coordinates, we have

$$x = R \sin\theta \sin\phi$$

$$y = R\sin\theta\cos\phi$$
$$z = R\cos\theta$$

We assign the two generalized coordinates $q_1 = \theta$ and $q_2 = \phi$. Clearly, R can be obtained from θ and ϕ from the above three relations among them. Note that two generalized coordinates θ and ϕ are independent. □

Example 1.3.27. Determine degrees of freedom for motion of a simple pendulum (or motion of particle on a circle) and assign generalized coordinates.

Solution. The number of particle(s) in this case is again 1, i.e. $N = 1$. As seen in Exercise 1.3.22, we have two constraints in this case which are $z = 0$ and $x^2 + y^2 = l^2$, where l is the length of the string at which the particle is tied. Therefore, $k = 2$. Hence, degrees of freedom is

$$n = 3N - k = 3 - 2 = 1.$$

Since the degrees of freedom is 1 in this case, we have only one generalized coordinate. Writing in terms of plane polar coordinates, $x = l\cos\theta$ and $y = \ \sin\theta$, we see that either θ or l can be assigned as a generalized coordinate. The other coordinate can be obtained from the assigned generalized coordinate by the relations $l = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. So in this case there is only one generalized coordinate. □

Example 1.3.28. Determine the degrees of freedom and assign generalized coordinates for motion of a particle in any plane.

Solution. Let $ax + by + cz + d = 0$ be the equation of the plane. Here the number of particle is $N = 1$ and the constraint is also 1 (given by the equation of plane). Therefore, $k = 1$ and hence degrees of freedom is $n = 3N - k = 2$. Choosing $q_1 = x$ and $q_2 = y$ as generalized coordinates, from above equations, we have $x = q_1, \quad y = q_2, \quad z = -\frac{a}{c}q_1 - \frac{b}{c}q_2 - \frac{d}{c}$. □

1.3.5 Constraints and generalized coordinates in a rigid body

Definition 1.3.29: Rigid body

A rigid body can be defined as a system of particle in which distance between any two particles remains constant (during the motion) and does not vary with time.

Constraints in a rigid body

If r_i and r_j denote the coordinates of the i^{th} and the j^{th} particle in a rigid body respectively and r_{ij} denotes the distance between them, then the constraints in a rigid body are given by

$$r_{ij} = c_{ij} \quad \text{or} \quad (r_i - r_j)^2 - c_{ij}^2 = 0,$$

where c_{ij}'s are constants. All the constraints in a rigid body motion are expressed by an algebraic equation and hence they all are holonomic.

The question is how many such constraints are there? We determine the number of constraints in a rigid body with N particles. Since the distance of first particle of other $N-1$ particles remains constant, we have the following $N-1$ constraints:

$$r_{12} = c_{12},\ r_{13} = c_{13}, \dots, r_{1N} = c_{1N},$$

where c_{ij}'s are constants. Now, consider second particle. Since its distance from all other particles is constant and $r_{12} = r_{21} = c_{12}$ is already considered (counted) once, we have the following $N-2$ new constraints:

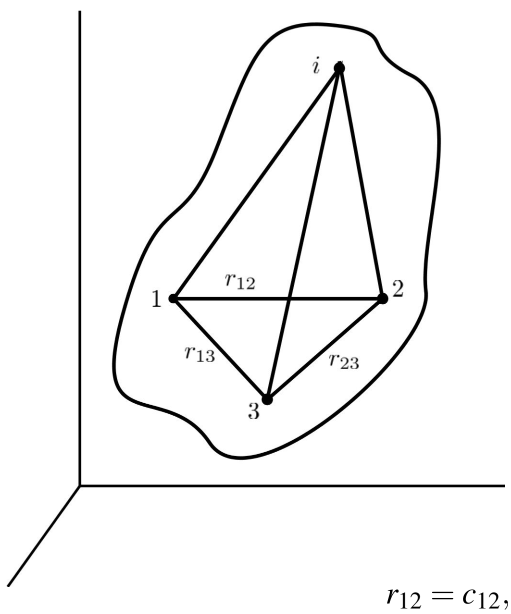
$$r_{23} = c_{23},\ r_{24} = c_{24}, \dots, r_{2N} = c_{2N}.$$

Thus, in this manner, a rigid body with N particles has $\boxed{\frac{1}{2}N(N-1)}$ constraints expressed as equations of the form $r_{ij} = c_{ij}$.

Degrees of freedom in a rigid body

We now determine how many independent coordinates are required to specify the configuration of a rigid body. We shall show that it should be done by just 6 independent coordinates, i.e. degrees of freedom of a rigid body (with at least three particles) is 6, irrespective of the number of particles (≥ 3) in the rigid body.

Suppose there are N particle in the rigid body. Then we have $3N$ coordinates. Since the distance between every pair of particles is fixed, as seen above, we have $\frac{1}{2}N(N-1)$ constraint equations.



Note that the degrees of freedom in this case cannot be merely computed by subtracting $\frac{1}{2}N(N-1)$ from $3N$ as

$$3N - \frac{1}{2}N(N-1) \leq 0, \quad \forall\, N \geq 7.$$

To fix a point in a rigid body, it suffices to specify its distances from any three non-collinear points. Thus, once the position of the three of the (non-collinear) particles of the rigid body are determined, the positions of all the remaining particles are fixed by the constraints. The number of degrees of freedom therefore must be at most nine. Furthermore, the three reference points are not independent. They are related by the three constraints given by

$$r_{12} = c_{12},\ r_{23} = c_{23},\ r_{13} = c_{13}.$$

This reduces the degrees of freedom of the system to *six*.

1.3.6 Transformation equations

During the course of motion the configuration of the system keeps on changing with time and hence usual coordinates as well as generalized coordinates are functions of time t, i.e.

$$q_i \equiv q_i(t), \qquad i = 1, 2, \dots, n$$

or

$$\bar{r}_i \equiv \bar{r}_i(t), \qquad i = 1,2,\ldots,n.$$

Let $\bar{r}_1$,$\bar{r}_2$,...,$\bar{r}_N$ be the position vectors (i.e. usual coordinates) of the particles in the system. Then they can be expressed in terms of generalized coordinates $q_1, q_2, \ldots, q_n$ as follows:

$$\begin{aligned}
\bar{r}_1 &\equiv \bar{r}_1(q_1, q_2, \ldots, q_n, t),\\
\bar{r}_2 &\equiv \bar{r}_2(q_1, q_2, \ldots, q_n, t),\\
\vdots & \qquad \vdots\\
\bar{r}_N &\equiv \bar{r}_N(q_1, q_2, \ldots, q_n, t).
\end{aligned} \tag{1.15}$$

In principle, these relations are invertible and hence we can write

$$\begin{aligned}
q_1 &\equiv q_1(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N, t),\\
q_2 &\equiv q_2(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N, t),\\
\vdots & \qquad \vdots\\
q_n &\equiv q_n(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N, t).
\end{aligned} \tag{1.16}$$

The system of equations given in (1.15) and (1.16) are called transformations equations as they transform usual coordinates into generalized coordinates and vice-versa. If the equations of motions are expressed in terms of generalized coordinates then they will be independent. Thus, the introduction of generalized coordinates resolves the difficulty of equations of motion (i.e. the first difficulty due to constraints is resolved).

In the case of holonomic constraints the constraint equations are algebraic equations and hence they are used to determine generalized coordinates. In other cases it is difficult to determine generalized coordinates. We will discuss only holonomic systems.

Now, we demonstrate how the transformation equations are obtained by means of the following example:

Example 1.3.30. Consider motion of a particle on a sphere.

$$x^2 + y^2 + z^2 = R^2.$$

As seen in Example 1.3.26, we know that, the usual coordinates are $\bar{r} = (x, y, z)$ and the assigned generalized coordinates $q_1 = \theta$ and $q_2 = \phi$. Then the transformation relation $\bar{r} \equiv \bar{r}(q_1, q_2)$ is given as:

$$\begin{aligned}
x &= R\sin\theta\sin\phi\\
y &= R\sin\theta\cos\phi\\
z &= R\cos\theta
\end{aligned}$$

and the transformation relations $q_i \equiv q_i(\bar{r}) \equiv q_i(x, y, z)$ for $i = 1, 2$ are given as follows:

$$\begin{aligned}
q_1 &= \tan^{-1}\left(\frac{y}{x}\right)\\
q_2 &= \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)
\end{aligned}$$

1.3.7 Generalized Velocities

In analogy with velocity defined as time derivative of position vector, we define time derivative of a generalized coordinate q_j as *generalized velocity* it is denoted by $\dot{q}_j$. It is easy to see from equation (1.16) that

$$\dot{\bar{r}}_i = \sum_{j=1}^{N} \frac{\partial \bar{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \bar{r}_i}{\partial t} \qquad i = 1,2,\ldots,N. \tag{1.17}$$

Above equations provide relation between usual velocities and generalized velocities. If the constraints are scleronomic then above equations reduce to

$$\dot{\bar{r}}_i = \sum_{j=1}^{N} \frac{\partial \bar{r}_i}{\partial q_j}\dot{q}_j \qquad i = 1,2,\ldots,N. \tag{1.18}$$

Theorem 1.3.31: Cancellation of dots

If constraints are scleronomic then

$$\frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_k} = \frac{\partial \bar{r}_i}{\partial q_k}, \qquad i = 1,2,\ldots,N,\ k = 1,2,\ldots,n.$$

Proof. Equation (1.18) is differentiated partially with respect to $\dot{q}_k$ to get

$$\frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_k} = \sum_{j=1}^{N} \frac{\partial \bar{r}_i}{\partial q_j}\frac{\partial \dot{q}_j}{\partial \dot{q}_k},$$

The second term inside the summation on the right hand side of above equation vanishes except for $j = k$ and hence the proof. □

This theorem is referred as the law of cancellation of dots.

1.4 Principle of Virtual Work

1.4.1 Virtual Displacement and Virtual Work

Definition 1.4.1: Virtual Displacement

Consider a system of N-particles. Let $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N$ be the position vectors of the particles of the system. The *virtual displacement* of the ith particle in the system is the displacement occurring without the change in time (instantaneous), i.e. virtual displacement is the change $\delta\bar{r}_i$ which is infinitesimal and instantaneous. It is consistent with the forces and constraints of the system.

Definition 1.4.2: Virtual Work

The work done (by the forces in system) due to virtual displacement is called *virtual work*. It is denoted by δW and is given by

$$\delta W = \sum_{i=1}^{N} \bar{F}_i \cdot \delta \bar{r}_i.$$

1.4.2 Principle of Virtual Work

A system of N-particles is said to be in *equilibrium* if the net force (i.e. total force) on each particle is zero. In other words, if $\bar{F}_i$ denotes the total force on the ith particle of the system then the system is said to be in equilibrium if $\bar{F}_i = 0$ for all $i = 1, 2, \ldots, N$.

We know that the virtual work is the word done due to virtual displacement. The principle of virtual work states that:

Theorem 1.4.3

Total virtual work done on a system of particles in equilibrium vanishes.

Proof. Consider a system of N-particles in equilibrium, i.e. the total force on each particles is zero. Let $\bar{F}_i$ denote the total force on the i^{th} particle for $i = 1, 2, \ldots, N$. Let $\delta \bar{r}_i$ denote the virtual displacement of the i^{th} particle.

Since the system is in equilibrium, $\bar{F}_i = 0$ for all $i = 1, 2, \ldots, N$. Therefore, $\bar{F}_i \cdot \delta \bar{r}_i = 0$ for all $i = 1, 2, \ldots, N$ and hence

$$\sum_{i=1}^{N} \bar{F}_i \cdot \delta \bar{r}_i = 0 \Rightarrow \delta W = 0.$$

□

1.4.3 Refined version of Principle of Virtual Work

As before, consider a system of N-particles in equilibrium and let $\bar{F}_i$ be the total force on the i^{th} particle. Then $\bar{F}_i$ is the sum of applied forces denoted by $\bar{F}_i^{(a)}$ and constraint forces denoted by $\bar{F}_i^{(c)}$, i.e. for $i = 1, 2, \ldots, N$,

$$\bar{F}_i = \bar{F}_i^{(a)} + \bar{F}_i^{(c)}.$$

Since the system is in equilibrium, by principle of virtual work, the virtual work $\delta W = 0$, i.e.

$$\delta W = \sum_{i=1}^{N} \bar{F}_i \cdot \delta \bar{r}_i = \sum_{i=1}^{N} \bar{F}_i^{(a)} \cdot \delta \bar{r}_i + \sum_{i=1}^{N} \bar{F}_i^{(c)} \cdot \delta \bar{r}_i = 0. \tag{1.19}$$

It is observed that if the motion is on a surface then the constraint force $\bar{F}_i^{(c)}$ is the deviation normal to the surface and displacement $\delta \bar{r}_i$ is along the tangential direction. Since the tangent and normal are perpendicular to each other, their dot product is zero, i.e. $\bar{F}_i^{(c)} \cdot \delta \bar{r}_i$ vanishes for all $i = 1, 2, \ldots, N$. Hence, the work done by constraint forces is zero or in other words, constraint forces are workless. In most of the cases, constraint forces are workless, i.e.

$$\sum_{i=1}^{N} \bar{F}_i^{(c)} \cdot \delta \bar{r}_i = 0.$$

Using this in equation (1.19), we get

$$\sum_{i=1}^{N} \bar{F}_i^{(a)} \cdot \delta \bar{r}_i = 0. \tag{1.20}$$

Thus, the principle of virtual work can be rewritten as

"If a system is in equilibrium and the constraint forces are workless then the total virtual work done by applied forces is zero."

1.5 D'Alemberts Principle and Lagrange's Equations

1.5.1 D'Alemberts Principle

As stated earlier constraints are due to unknown forces and hence in the equations of motion these forces appear as unknowns. This difficulty can be resolved by an alternate statement of principle of virtual work. Principle of virtual work states that, "the virtual work done by forces in equilibrium is zero". D'Alembert's principle is derived as below.

Jean le Rond d'Alembert (1717-1783)

Consider a system of N-particles. Let $\bar{F}_i$ be force on i^{th} particle and $\delta \bar{r}_i$ denote virtual displacement of that particle. Let $\bar{F}_i^{(a)}$ and $\bar{F}_i^{(c)}$ be applied forces and constraint forces on i^{th} particle respectively. By Newton's equations of motion, we write

$$\bar{F}_i = \dot{\bar{p}}_i \qquad \forall\, i = 1, 2, \ldots, N.$$

or

$$\bar{F}_i - \dot{\bar{p}}_i = 0, \qquad \forall\, i = 1, 2, \ldots, N.$$

Therefore,

$$\sum_{i=1}^{N} (\bar{F}_i - \dot{\bar{p}}_i) = 0.$$

Thus the system is in equilibrium under effective force $\sum_{i=1}^{N} (\bar{F}_i - \dot{\bar{p}}_i)$ and hence by principle of virtual work, we get

$$\sum_{i=1}^{N} (\bar{F}_i - \dot{\bar{p}}_i) \cdot \delta \bar{r}_i = 0. \tag{1.21}$$

Further the forces $\bar{F}_i$ are sum of applied forces $\bar{F}_i^{(a)}$ and constraint forces $\bar{F}_i^{(c)}$ and hence (1.21) can be rewritten as

$$\sum_{i=1}^{N} \left(\bar{F}_i^{(a)} - \dot{\bar{p}}_i \right) \cdot \delta \bar{r}_i + \sum_{i=1}^{N} \bar{F}_i^{(c)} \cdot \delta \bar{r}_i = 0. \tag{1.22}$$

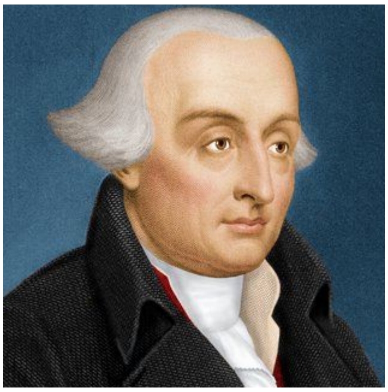
If the constraints are workless then the second term on the left hand side of (1.22) vanishes and hence it reduces to

$$\sum_{i=1}^{N}\left(\bar{F}_i^{(a)}-\dot{\bar{p}}_i\right)\cdot\delta\bar{r}_i=0 \tag{1.23}$$

This is called D'Alembert's principle. From (1.23) it can be seen that the constraint forces are removed from the equation. It is known that equations of motion can be derived from principle of virtual work. We will derive them from D'Alembert's principle which is an alternate form of the principle of virtual work.

1.5.2 Lagrange's Equations of Motion

In this section, we shall derive equations of motion from D'Alembert's principle called the Lagrange's equation of motion (LEOM).



Joseph-Louis Lagrange (1736-1813)

Consider a system of N-particles having masses $m_1, m_2, .., m_N$ and position vectors $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N$ respectively. Let the degrees of freedom for this system be n and $q_1, q_2, \ldots, q_n$ be chosen as generalized coordinates. We shall also assume that the constraints are workless and scleronomic, so that D'Alembert's principle holds.

The transformation relations between the usual coordinates and the generalized coordinates are given as follows:

$$\bar{r}_i\equiv\bar{r}_i(q_1,q_2,\ldots,q_n),\qquad i=1,2,\ldots,N. \tag{1.24}$$

Also, we know that these relations are invertible and so

$$q_j\equiv q_j(\bar{r}_1,\bar{r}_2,\ldots,\bar{r}_N),\qquad j=1,2,\ldots,n. \tag{1.25}$$

D'Alembert's principle can be written as

$$\sum_{i=1}^{N}\bar{F}_i^{(a)}\cdot\delta\bar{r}_i=\sum_{i=1}^{N}\dot{\bar{p}}_i\cdot\delta\bar{r}_i, \tag{1.26}$$

where $\bar{F}_i^{(a)}$ are the applied forces. Our aim is to derive equations of motion by transforming above equation in terms of generalized coordinates. From (1.24), the change $\delta\bar{r}_i$ in $\bar{r}_i$ is given by,

$$\delta\bar{r}_i=\sum_{j=1}^{n}\frac{\partial\bar{r}_i}{\partial q_j}\delta q_j\qquad i=1,2,\ldots,N. \tag{1.27}$$

Using this, the LHS of (1.26) can be written as

$$\begin{aligned}\sum_{i=1}^{N}\bar{F}_i^{(a)}\cdot\delta\bar{r}_i&=\sum_{i=1}^{N}\sum_{j=1}^{n}\left(\bar{F}_i^{(a)}\cdot\frac{\partial\bar{r}_i}{\partial q_j}\delta q_j\right)\\&=\sum_{j=1}^{n}\left(\sum_{i=1}^{N}\bar{F}_i^{(a)}\cdot\frac{\partial\bar{r}_i}{\partial q_j}\right)\delta q_j\end{aligned}$$

$$= \sum_{j=1}^{n} Q_j \, \delta q_j, \tag{1.28}$$

where we define

$$Q_j = \sum_{i=1}^{N} \bar{F}_i^{(a)} \cdot \frac{\partial \bar{r}_i}{\partial q_j} \qquad j = 1, 2, \ldots, n. \tag{1.29}$$

These quantities Q_j are called *generalized forces* or components of generalized force. Now, we simplify the RHS of (1.26). For this, consider

$$\frac{d}{dt}\left(\dot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\right) = \ddot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j} + \dot{\bar{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right). \tag{1.30}$$

Also from (1.24), for $i = 1, 2, \ldots, N$, we have

$$\dot{\bar{r}}_i = \sum_{k=1}^{n} \frac{\partial \bar{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \bar{r}_i}{\partial t}.$$

Differentiating this further partially with respect to q_j, we get

$$\begin{aligned}
\frac{\partial \dot{\bar{r}}_i}{\partial q_j} &= \frac{\partial}{\partial q_j}\left(\sum_{n=1}^{n} \frac{\partial \bar{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \bar{r}_i}{\partial t}\right) \\
&= \sum_{n=1}^{n} \frac{\partial}{\partial q_j}\left(\frac{\partial \bar{r}_i}{\partial q_k} \dot{q}_k\right) + \frac{\partial}{\partial q_j}\left(\frac{\partial \bar{r}_i}{\partial t}\right) \\
&= \sum_{k=1}^{n} \frac{\partial^2 \bar{r}_i}{\partial q_k \partial q_j} \dot{q}_k + \frac{\partial^2 \bar{r}_i}{\partial t \partial q_j}
\end{aligned} \tag{1.31}$$

On the other hand from (1.24), we also have

$$\frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right) = \sum_{k=1}^{n} \frac{\partial^2 \bar{r}_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 \bar{r}_i}{\partial t \partial q_j} \tag{1.32}$$

Comparing the RHS of the above two equations (1.31) and (1.32), we get

$$\frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\bar{r}}_i}{\partial q_j} \tag{1.33}$$

Using this in (1.30)

$$\frac{d}{dt}\left(\dot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\right) = \ddot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_i} + \dot{\bar{r}}_i \cdot \left(\frac{\partial \dot{\bar{r}}_i}{\partial q_j}\right) \tag{1.34}$$

Now, since the constraints are scleronomic, by Theorem 1.3.7, we have $\dfrac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_j} = \dfrac{\partial \bar{r}_i}{\partial q_j}$.

Using this in LHS of above equation (1.34), we get

$$\frac{d}{dt}\left(\dot{\bar{r}}_i \cdot \frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_j}\right) = \ddot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j} + \dot{\bar{r}}_i \cdot \left(\frac{\partial \dot{\bar{r}}_i}{\partial q_j}\right)$$

or

$$\ddot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j} = \frac{d}{dt}\left(\dot{\bar{r}}_i \cdot \frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_j}\right) - \dot{\bar{r}}_i \cdot \left(\frac{\partial \dot{\bar{r}}_i}{\partial q_j}\right)$$

$$= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\dot{\bar{r}}_i \cdot \dot{\bar{r}}_i\right)\right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2}\dot{\bar{r}}_i \cdot \dot{\bar{r}}_i\right)$$
$$= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\bar{v}_i \cdot \bar{v}_i\right)\right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2}\bar{v}_i \cdot \bar{v}_i\right)$$
$$= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}v_i^2\right)\right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2}v_i^2\right) \tag{1.35}$$

Then the RHS of (1.26) becomes

$$\sum_{i=1}^{N} \dot{\bar{p}}_i \cdot \delta\dot{\bar{r}}_i = \sum_{i=1}^{N}\sum_{j=1}^{n} m_i \ddot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\delta q_j$$

Using equation (1.35) in above expression, we get

$$\sum_{i=1}^{N} \dot{\bar{p}}_i \cdot \delta\dot{\bar{r}}_i = \sum_{i=1}^{N}\sum_{j=1}^{n} m_i\left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}v_i^2\right)\right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2}v_i^2\right)\right]\delta q_j$$
$$= \sum_{i=1}^{N}\sum_{j=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}m_i v_i^2\right)\right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2}m_i v_i^2\right)\right]\delta q_j$$
$$= \sum_{j=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\sum_{i=1}^{N}\left(\frac{1}{2}m_i v_i^2\right)\right) - \frac{\partial}{\partial q_j}\sum_{i=1}^{N}\left(\frac{1}{2}m_i v_i^2\right)\right]\delta q_j$$
$$= \sum_{j=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j, \tag{1.36}$$

where T $\left(= \sum_{i=1}^{N}\frac{1}{2}m_i v_i^2\right)$ is the total kinetic energy of the system. Finally substituting equations (1.28) and (1.36) in (1.26), we get

$$\sum_{j=1}^{n} Q_j\,\delta q_j = \sum_{j=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j$$

or

$$\sum_{j=1}^{n}\left[Q_j - \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right)\right]\delta q_j = 0 \tag{1.37}$$

Further, if constraints are holonomic then the generalized coordinates q_j are independent and hence (1.37) holds if individual term in the summation vanishes, i.e.,

$$Q_j - \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right) = 0, \qquad j = 1,2,\ldots,n$$

or

Lagrange's equations of motion (General Form)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j, \quad j = 1,2,\ldots,n. \tag{1.38}$$

Equations (1.38) are called *Lagrange's equations of motion in general for*m. These equations form a simultaneous system of second order ordinary differential equations. Solution of this system will determine generalized coordinates $q_1, q_2, \ldots, q_n$ as functions of time.

In the discussion of constraints it was seen that due to constraints there were two types of difficulties arise namely, (i) coordinates are not independent and (ii) constraint forces occur in the equations of motion as unknowns. It can be noted from Lagrange's equations of motion that, these equations of motion are written only for the applied forces, the constraint forces no more appear in the equations and also they are written for independent coordinates q_j's, thus both the difficulties imposed by constraints are overcome.

Lagrange's equations of motion in general form can be used for various forms of forces in the nature. In the next section we derive special cases of Lagrange's equations of motion.

Exercise 1.5.1: LEOM for particle moving in space

Obtain Lagrange's equations of motion for a particle moving in space under a force $\bar{F}$ using Cartesian coordinates.

Solution. We know that Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j, \qquad j = 1, 2, \ldots, n. \tag{1.39}$$

In this case, we have only one particle and no constraints, i.e. $N = 1$ and $k = 0$. So, we have $n = 3N - k = 3$ generalized coordinates, say

$$q_1 = x, \quad q_2 = y, \quad q_3 = z.$$

The total kinetic energy of the system is given by

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{\bar{r}}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

Therefore,

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial T}{\partial \dot{y}} = m\dot{y}, \quad \frac{\partial T}{\partial \dot{z}} = m\dot{z}$$

and so

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x}, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = m\ddot{y}, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) = m\ddot{z}$$

Also, since T is independent of x, y and z, we have

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} = \frac{\partial T}{\partial z} = 0.$$

Now, the generalized forces Q_j are given as follows:

$$Q_j = \sum_{i=1}^{N} \bar{F}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}, \qquad j = 1, 2, 3.$$

Since, in our case, we have only one particle,

$$Q_j = \bar{F} \cdot \frac{\partial \bar{r}}{\partial q_j}, \qquad j = 1, 2, 3.$$

Let the position vector of the particle be given by $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and the force on the particle be given by $\bar{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$. Then

$$Q_1 = \bar{F} \cdot \frac{\partial r}{\partial x} = (F_x\hat{i} + F_y\hat{j} + F_z\hat{k}) \cdot \hat{i} = F_x.$$

Similarly, $Q_2 = F_y$ and $Q_3 = F_z$. So, Lagrange's equations of motion can be given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = Q_1 \Rightarrow m\ddot{x} - 0 = F_x,$$

i.e. $F_x = m\ddot{x}$. Similarly, the other two equations are obtained as $F_y = m\ddot{y}$ and $F_z = m\ddot{z}$. Observe that in this case,

$$F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = m(\ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}) \Rightarrow \bar{F} = m\ddot{\bar{r}} = m\bar{a}.$$

Thus, we obtained Newton's equations of motion as Lagrange's equations of motion as in this case we have no constraints. □

1.6 Lagrange's Equations of Motion: Special Cases

In this section various types of applied forces will be considered and appropriate form of Lagrange's equations of motion will be derived.

1.6.1 Conservative Force

Suppose applied forces are conservative and derivable from a potential function depending on positions of the particles only, i.e. $V \equiv V(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N)$. In this case, applied forces are given by

$$\bar{F}_i^{(a)} = -\nabla_i V, \qquad i = 1, 2, \ldots, N, \tag{1.40}$$

where ∇_i denotes vector differential operator at the position of i^{th} particle, i.e. $\nabla_i \equiv \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_i}\right)$. Using this in Equation (1.40) we get,

$$\begin{aligned} Q_j &= \sum_{i=1}^{N} \bar{F}_i(a) \cdot \frac{\partial \bar{r}_i}{\partial q_j} \\ &= \sum_{i=1}^{N} -\nabla_i V \cdot \frac{\partial \bar{r}_i}{\partial q_j} \\ &= -\frac{\partial V}{\partial q_j} \end{aligned} \tag{1.41}$$

using these in (1.38), we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}.$$

or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0. \tag{1.42}$$

Further we subtract a vanishing term $\frac{\partial V}{\partial \dot{q}_j}$ from the first term on left hand side of above equation to get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial V}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0, \qquad j = 1, 2, \ldots, n,$$

or

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0, \qquad j = 1, 2, \ldots, n.$$

Now we define a function $L \equiv L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t)$ given by

$$L = T - V \tag{1.43}$$

Using this in above equation yields

Lagrange's equations of motion (Conservative Force)

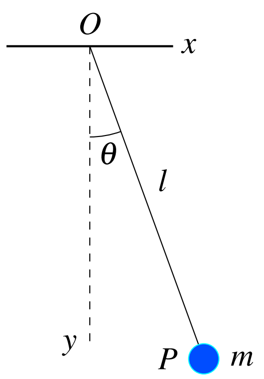
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \qquad j = 1, 2, \ldots, n. \tag{1.44}$$

These equations are referred to as Lagrange's equations of motion and the function L is called Lagrangian of the system.

Exercise 1.6.1: LEOM for Simple Pendulum

Obtain Lagrange's equations of motion for simple pendulum.

***Solution*.**



As we have seen earlier, simple pendulum is a system of one particle where the particle is suspended by a rigid weightless and inextendable string from a fixed point. The particle is allowed to move in vertical plane and motion takes place under gravity. The constraints are

1. $x^2 + y^2 + z^2 = l^2$.
2. $z = 0$.

Therefore degrees of freedom is $n = 3N - k = 1$.

Choosing the angle θ made by the pendulum with the vertical axis as the generalized coordinate, we write

$$\begin{aligned} x &= l\sin\theta, & y &= l\cos\theta, & z &= 0. \\ \dot{x} &= l\cos\theta\ \dot{\theta}, & \dot{y} &= -l\sin\theta\ \dot{\theta}, & \dot{z} &= 0. \end{aligned}$$

Therefore, the kinetic energy in terms of generalized coordinates can be written as

$$\boxed{T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{m}{2}l^2\dot{\theta}^2}.$$

Here, the force is the gravitational force which is conservative. We choose the potential $V = 0$ at the point of suspension and so potential is given by

$$V = mgh = -mgy = -mgl\cos\theta.$$

Now,

$$L = T - V = \frac{m}{2}l^2\dot{\theta}^2 + mgl\cos\theta.$$

Hence, Lagrange's equations of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0.$$

From the Lagrangian, we have

$$\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = ml^2\ddot{\theta}. \tag{1.45}$$

$$\frac{\partial L}{\partial\theta} = -mgl\sin\theta. \tag{1.46}$$

Using these values in the above LEOM, we get

$$ml(l\ddot{\theta} + g\sin\theta) = 0.$$

Therefore $ml = 0$ or $l\ddot{\theta} + g\sin\theta = 0$. But $ml = 0$ is not possible as both m and l are non-zero constants. Therefore, we have

$$\begin{aligned} l\ddot{\theta} + g\sin\theta &= 0 \\ \Rightarrow \quad \frac{d^2\theta}{dt^2} + \left(\frac{g}{l}\right)\sin\theta &= 0. \end{aligned}$$

The above equation is a non-linear differential equation. □

1.6.2 Non-conservative Force

In the previous section we discussed the case of conservative force which can be derived from a scalar potential depending on the positions of the particles in the system. In some case forces can be derived from a more general potential which may depend on velocities also. A known case of such force is of electromagnetic force. In such situations also it is possible to associate a Lagrangian function. Suppose the applied forces are derivable from a potential $U \equiv U(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$, in the following prescription

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial\dot{q}_j}\right), \qquad j = 1, 2, \ldots, n. \tag{1.47}$$

Then using these in (1.38), we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial\dot{q}_j}\right).$$

or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial U}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) = 0.$$

or

$$\frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_j}\right) - \frac{\partial (T-U)}{\partial q_j} = 0.$$

Defining Lagrangian as $L = T - U$, above equations read as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \qquad j = 1, 2, \ldots, n.$$

From (1.47) it is clear that if velocity terms are absent in the expression of U then it reduces to potential V, thus U is called general potential, it is also called velocity dependent potential.

1.6.3 Frictional Forces and Rayleigh's Dissipation Function

It is known that frictional forces are not conservative and also they are proportional to the velocity. In many cases frictional forces are derivable from a function called Rayleigh's dissipation function. His function is denoted by R. Let the frictional force on i^{th} particle be denoted by $F_i^{(d)}$, then

$$F_i^{(d)} = -\lambda_i \dot{r}_i,$$

where λ_i are constants. Here the corresponding generalized forces are given by,

$$\begin{aligned}
Q_j^{(d)} &= \sum_{i=1}^{N} F_i^{(d)} \cdot \frac{\partial \bar{r}_i}{\partial q_j} \\
&= -\sum_{i=1}^{N} \lambda_i \dot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j} \\
&= -\sum_{i=1}^{N} \lambda_i \dot{\bar{r}}_i \cdot \frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_j} && \left(\because \frac{\partial \bar{r}_i}{\partial q_j} = \frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_j}\right) \\
&= \sum_{i=1}^{N} \frac{\partial}{\partial \dot{q}_j}\left(-\frac{1}{2}\lambda_i \dot{r}_i^2\right) \\
&= \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\sum_{i=1}^{N} \lambda_i \dot{r}_i^2\right) \\
&= -\frac{\partial R}{\partial \dot{q}_j}, \qquad (1.48)
\end{aligned}$$

where

$$R = \frac{1}{2}\sum_{i=1}^{N} \lambda_i \dot{r}_i^2 \qquad (1.49)$$

is called *Rayleigh's dissipation function*.

Now suppose in a system, there are conservative applied forces as well as frictional forces derivable from a dissipation function R then using (1.48) in (1.38) along with (1.44), we get

Lagrange's equations of motion (Frictional forces)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial R}{\partial \dot{q}_j} = 0, \qquad j = 1, 2, \ldots, n. \tag{1.50}$$

1.7 Kinetic Energy in generalized Coordinates

In the previous section we have seen that Lagrange's equations of motion are written in terms of kinetic energy expressed in terms of generalized coordinates and generalized velocities. We now derive expression of kinetic energy of a system of particles in terms of generalized quantities.

The expression of kinetic energy of a system of particles is given by

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2, \tag{1.51}$$

where $\bar{v}_i$ denotes velocity of i^{th}particle. As seen earlier it is given by

$$\bar{v}_i = \sum_{j=1}^{n} \frac{\partial \bar{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \bar{r}_i}{\partial t}, \tag{1.52}$$

when this is used in the above expression gives,

$$\begin{aligned} T &= \sum_{i=1}^{N} \frac{1}{2} m_i \left(\sum_{j=1}^{n} \frac{\partial \bar{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \bar{r}_i}{\partial t} \right) \cdot \left(\sum_{k=1}^{n} \frac{\partial \bar{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \bar{r}_i}{\partial t} \right) \\ &= \sum_{i=1}^{N} \sum_{j,k} \frac{1}{2} m_i \left(\frac{\partial \bar{r}_i}{\partial q_j} \cdot \frac{\partial \bar{r}_i}{\partial q_k} \dot{q}_j \dot{q}_k + \frac{\partial \bar{r}_i}{\partial q_j} \cdot \frac{\partial \bar{r}_i}{\partial t} \dot{q}_j + \frac{\partial \bar{r}_i}{\partial q_k} \cdot \frac{\partial \bar{r}_i}{\partial t} \dot{q}_k + \frac{\partial \bar{r}_i}{\partial t} \cdot \frac{\partial \bar{r}_i}{\partial t} \right) \\ &= \sum_{i=1}^{N} \sum_{j,k} \frac{1}{2} m_i \frac{\partial \bar{r}_i}{\partial q_j} \cdot \frac{\partial \bar{r}_i}{\partial q_k} \dot{q}_j \dot{q}_k + \sum_{i=1}^{N} \sum_{j=1}^{n} m_i \frac{\partial \bar{r}_i}{\partial q_j} \cdot \frac{\partial \bar{r}_i}{\partial t} \dot{q}_j + \sum_{i=1}^{N} \frac{1}{2} m_i \frac{\partial \bar{r}_i}{\partial t} \cdot \frac{\partial \bar{r}_i}{\partial t}. \end{aligned} \tag{1.53}$$

The above equation is rewritten as,

$$T = T_2 + T_1 + T_0, \tag{1.54}$$

where T_i $(i = 1, 2, 3)$ represents a term of i^{th} degree in generalized velocities and these terms are given by,

$$\begin{aligned} T_2 &= \sum_{i=1}^{N} \sum_{j,k} \frac{1}{2} m_i \frac{\partial \bar{r}_i}{\partial q_j} \cdot \frac{\partial \bar{r}_i}{\partial q_k} \dot{q}_j \dot{q}_k, \\ T_1 &= \sum_{i=1}^{N} \sum_{j=1}^{n} m_i \frac{\partial \bar{r}_i}{\partial q_j} \cdot \frac{\partial \bar{r}_i}{\partial t} \dot{q}_j, \\ T_0 &= \sum_{i=1}^{N} \frac{1}{2} m_i \frac{\partial \bar{r}_i}{\partial t} \cdot \frac{\partial \bar{r}_i}{\partial t}. \end{aligned} \tag{1.55}$$

1.8 Configuration Space and Lagrangian

1.8.1 Configuration space and system point

Definition 1.8.1: Configuration space

Consider a system of n-degrees of freedom. Let $q_1, q_2, \ldots, q_n$ be chosen generalized coordinates. The position of the system can be determined if $q_1, q_2, \ldots, q_n$ are known. Thus a space of n-dimension, say $\mathbb{R}^n$, can be assumed to be associated with the system. For this n-dimensional space, $q_1, q_2, \ldots, q_n$ are taken as coordinates.

The n-dimensional space, associated with a system, having $q_1, q_2, \ldots, q_n$ as coordinates is called *configuration space* of the system.

Definition 1.8.2: System point

At any given time t, the position of the system can be determined using $q_1, q_2, \ldots, q_n$. Hence we can associate a point in the configuration space with the motion of the system. This point is called the *system point* in the configuration space.

1.8.2 Remarks on Lagrange's equation of motion

Remarks 1.8.3. For a system of n degrees of freedom, let $q_1, q_2, \ldots, q_n$ be the chosen generalized coordinates and let $L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t) \equiv L(q, \dot{q}, t)$ be Lagrangian. Then Lagrange's equation of motion are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \qquad j = 1, 2, \ldots, n. \tag{1.56}$$

1. In (1.56) we have a system of n second order ordinary differential equation with $q_1, q_2, \ldots, q_n$ as dependent variables and t as independent variable.

2. The solution of the system is given in the form

$$q_j \equiv q_j(t), \qquad j = 1, 2, \ldots, n. \tag{1.57}$$

3. The exact form of the above solution can be obtained using the initial condition. The solution (1.57) describes a curve in the configuration space. We will say that the system point describes the curve given in (1.57) following the motion of the system.

4. Lagrange's equation of motion form a system of non-linear second order ordinary differential equations. Such systems are very difficult to solve. However in many cases, first integrals are possible (i.e. some of the equations reduce to the first order equations) which provide some physical principles.

1.8.3 Uniqueness of Lagrangian

We know that Lagrangian of a system is given by $L = T - V$, where T is the kinetic energy and V is the potential energy of the system.

1. For a system, potential V is not unique. In fact, we have seen that, V is unique upto addition of a constant. Thus for a system another, Lagrangian can be obtained by adding a constant, i.e.
$$L = T - V + \lambda = T - (V - \lambda) = T - V_1,$$
where $V_1 = V - \lambda$ is also potential.

2. For any system, the generalized coordinates can be chosen in different ways. Then the form of Lagrangian may differ in this case.

3. Let $L \equiv L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t)$ be Lagrangian of the system. Then it satisfies
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \qquad j = 1, 2, \ldots, n.$$
Let L' be given by $L' = L + \frac{dF}{dt}$, where $F \equiv F(q_1, q_2, \ldots, q_n, t)$ is an arbitrary function. Then L' also satisfies LEOM (Prove! see Exercise 1.25), i.e.
$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} = 0, \qquad j = 1, 2, \ldots, n$$
and hence L' is also Lagrangian of the system.

These three aspects indicates that **Lagrangian** of the system **need not be unique**.

Exercises

Exercise 1.1

Determine the degrees of freedom and assign generalized coordinates (if possible) in the following systems:

1. Two particles connected by an in-extensible rod of length l.
2. Two particles connected by an in-extensible rod of length l and the center of the rod moving on a circle of radius r.
3. Simple pendulum (or a particle moving on a circle).
4. motion of a particle on a parabola or ellipse.

Exercise 1.2

Determine the degrees of freedom, assign generalized coordinates and express kinetic energy in terms of generalized coordinates for the motion of a particle in XY-plane in terms of plane polar coordinates.

Exercise 1.3

Express kinetic energy of a particle moving in space in terms of spherical coordinates.

Exercise 1.4

Distance between two points (x,y) and $(x+dx, y+dy)$ (in plane) is given by $ds^2 = dx^2 + dy^2$. Express this in terms of polar coordinates.

Exercise 1.5

Distance between two points (x,y,z) and $(x+dx, y+dy, z+dz)$ is given by $ds^2 = dx^2 + dy^2 + dz^2$. Express this in terms of spherical coordinates.

Exercise 1.6

For a particle moving on the surface of a cylinder

1. Find the constraints and classify them.
2. Determine degrees of freedom and assign generalized (cylindrical) coordinates.
3. Obtain and expression of kinetic energy in terms of generalized coordinates.

Exercise 1.7

Describe the motion of a double pendulum and determine degrees of freedom by discussing its constraints. Also assign generalized coordinates to it.

Exercise 1.8

Define a spherical pendulum. Discuss all of its constraints, determine its degrees of freedom and assign generalized coordinates.

Exercise 1.9

Show that Lagrange's equation of motion in the form of $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$ (i.e. general form) can also be written as $\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j$. These are known as Nielsen form of the Lagrange's equations.

Exercise 1.10

Obtain Lagrange's equations of motion for a particle moving in XY-plane under the effect of force $\bar{F}$ using plane polar coordinates as generalized coordinates.

Exercise 1.11

Obtain Lagrange's equations of motion for a particle moving in space under the effect of force $\bar{F}$ using cylindrical coordinates as generalized coordinates.

Exercise 1.12

Express Kinetic energy for the motion of double pendulum in terms of generalized coordinates and hence obtain its Lagrange's equations of motion.

Exercise 1.13

Express Kinetic energy of a spherical pendulum (or a particle moving on a sphere) in terms of generalized coordinates and hence obtain its Lagrange's equations of motion.

Exercise 1.14

Describe Atwood's machine with a diagram.

1. State its constraints and classify them. Determine its degrees of freedom.
2. Assign generalized coordinates and express kinetic energy in terms of generalized coordinates.
3. Obtain Lagrange's equations of motion.

Exercise 1.15

A pendulum is suspended from a point moving according to $x = a\cos\omega t$.

1. State the constraints and classify them. Determine its degrees of freedom.
2. Assign generalized coordinates and express kinetic energy in terms of generalized coordinates.
3. Obtain Lagrange's equations of motion.

Exercise 1.16

Describe Simple Harmonic Oscillator (SHO) and obtain Lagrange's equation of motion for it.

Exercise 1.17

Lagrangian of a particle in one dimension is given by $L = \frac{1}{2}m\dot{x}^2 - V + \dot{x}A$, where A and V are functions of x. Obtain Lagrange's equation of motion.

Exercise 1.18

Obtain Lagrange's equation of motion for a two dimensional isotropic oscillator. Also express it in terms of polar form (polar coordinates).

Exercise 1.19

Obtain Lagrange's equation of motion for a system for which Lagrangian is given by

$$L = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 - mglcos\theta.$$

Exercise 1.20

Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table surface and m_2 hangs suspended. Assuming m_2 moves only in vertical line, what are the generalized coordinates for the system? Write the Lagrangian equations of motion for the system.

Exercise 1.21

Define a point transformation. Show that the form of Lagrange's equations of motion are invariant under a point transformation. [Refer Goldstein's book: Derivation exercise no. 10, page no. 30].

Exercise 1.22

A particle is falling vertically under gravity. Air friction is present and it is derivable from a dissipation function $R = \frac{1}{2}kv^2$. Obtain Lagrange's equation of motion.

Exercise 1.23

For kinetic energy T of a system of n-degrees of freedom, evaluate $\sum_{j=1}^{n} \dot{q}_j \frac{\partial T}{\partial \dot{q}_j}$.

Exercise 1.24

For a system of scleronomic constraints show that $\sum_{j=1}^{n} \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T$.

Exercise 1.25

If L is given Lagrangian of a system of n-degrees of freedom satisfying Lagrange's equations of motion then show that $L' = L + \frac{dF(q_1, q_2, \ldots, q_n, t)}{dt}$ also satisfies Lagrange's equations of motion, where F is an arbitrary differentiable function of its arguments.

Exercise 1.26

A Lagrangian for a particular physical system can be written as

$$L' = \frac{m}{2}(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{k}{2}(ax^2 + 2bxy + cy^2),$$

where a, b, c are arbitrary constants but subject to condition that $b^2 - ac \neq 0$. What are Lagrange's equations of motion? Examine particularly the two cases $a = c = 0$ and $b = 0, c = -a$. What is physical system by the above Lagrangian? Show that the usual Lagrangian for this system defined by the equation

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dF}{dt}$$

is related by L' by point transformation. What is the significance of the condition on the value of $b^2 - ac \neq 0$.

CHAPTER 2

Variational principles

2.1 Hamilton's principle

2.1.1 Action Integral

Consider a system with n-degrees of freedom. Let $q_1, q_2, \ldots, q_n$ be the chosen generalized coordinates and $L \equiv L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t)$ be Lagrangian of the system, where $L = T - V$.

Suppose the system travels along a path C during the time interval $[t_1, t_2]$. Then the action integral, denoted by I or A, for the time interval $[t_1, t_2]$ along the path C in the configuration space is defined as the line integral given by

Action Integral

$$I = \int_{\substack{t_1 \\ \text{along } C}}^{t_2} L(q_1 \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)\, dt$$

or simply by

$$I = \int_{t_1}^{t_2} L\, dt.$$

Note: The line integral in the above expression may be evaluated in the configuration space by associating a system point to the system.

2.1.2 Hamilton's principle

For a system with n degrees of freedom, let $q_1, q_2, \ldots, q_n$ be the chosen generalized coordinates and $L(q, \dot{q}, t)$ be Lagrangian. Then Hamilton's principle states that,

Hamilton's principle

"among all the possible paths in the time interval $[t_1, t_2]$, the system point travels on the path on which the action integral is extremum (or stationary)."

That is, the system point takes the path on which the integral

$$I = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$$

is extremum (or stationary).

William Rowan Hamilton (1805-1865)

Here by extremum we mean maximum or minimum with respect to various cases. The condition or the equivalent form of the integral I being stationary is given as follows:

$$\delta I = \delta \int_{t_1}^{t_2} L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)\, dt = 0, \tag{2.1}$$

where δ denotes the variation due to the change in the path. Thus, the integral I is extremum if the variation δI is zero.

2.2 Calculus of Variations

Calculus of variations deals with the variational problems. For example, (2.1) is a variational problem, i.e. we need to determine a curve on which the variation is zero. In this section and in what follows, we discuss some techniques of the Calculus of Variations and its applications.

2.2.1 Condition for extremum

Consider a function $f(y, \dot{y}, x)$, where $y \equiv y(x)$, $\dot{y} = \frac{dy}{dx}$ and x is an independent variable. We state the condition for extremum of the line integral

$$J = \int_{x_1}^{x_2} f(y, \dot{y}, x)\, dx. \tag{2.2}$$

The required condition for the integral in (2.2) to be extremum. The above equation is called Euler's equation in calculus of variation.

Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0 \tag{2.3}$$

The above equation is called Euler's equation in calculus of variation. Next we shall state some extensions of this condition for extremum of certain integrals.

Certain Extensions

1. The condition for extremum of the integral of the form

$$I = \int_{x_1}^{x_2} f(y_1, y_2, \ldots, y_n, \dot{y}_1, \dot{y}_2, \ldots, \dot{y}_n, x)\, dx,$$

is given by

Euler-Lagrange equations

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}_i}\right) = 0, \qquad i = 1, 2, \ldots, n.$$

The above equations are known as Euler-Lagrange equations.

2. Suppose there are multiple integrals as follows:

$$I = \underbrace{\iint \cdots \int}_{m \text{ integrals}} f(y_1, y_2, \ldots, y_n, \dot{y}_1, \dot{y}_2, \ldots, \dot{y}_n, x_1, x_2, \ldots, x_m)\, dx_1 dx_2 \cdots dx_m,$$

where $y_i = y_i(x_1, x_2, \ldots, x_m)$, $i = 1, 2, \ldots, n$. Then the condition for extremum is given by

$$\frac{\partial f}{\partial y_i} - \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial {y_i}^{(1)}}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial {y_i}^{(2)}}\right) - \cdots - \frac{\partial}{\partial x_m}\left(\frac{\partial f}{\partial {y_i}^{(m)}}\right) = 0,$$

where $y_i^{(j)} = \frac{\partial y_i}{\partial x_j}$.

3. For the integral $I = \int f(y, \dot{y}, \ddot{y}, x)\, dx$, the condition for extremum is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial \ddot{y}}\right) = 0.$$

2.2.2 Some applications of calculus of variations

Exercise 2.2.1: Shortest distance between two points in a plane

Obtain the geodesics on a plane i.e., obtain the curve of shortest distance between two points in a plane (with Euclidean geometry).

Solution. We shall show that the curve of shortest distance between two points in a plane is a *straight line*.

Consider two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the plane. The problem is to determine the curve on which the distance between P and Q is minimum. The distance between to neighboring points (x, y) and $(x + dx, y + dy)$ on a curve is given by

$$ds^2 = dx^2 + dy^2 = dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)$$

$$\Rightarrow ds = (1+\dot{y}^2)^{\frac{1}{2}} dx,$$

where $\dot{y} = \frac{dy}{dx}$. The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the integral

$$I = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1+\dot{y}^2}\, dx. \tag{2.4}$$

Now, our problem is to determine the curve on which the integral I is minimum. We use techniques in calculus of variations by taking

$$f(y, \dot{y}, x) = (1+\dot{y}^2)^{\frac{1}{2}}. \tag{2.5}$$

Hence the condition for minimum of the integral in (2.4) is given by the Euler's equation, i.e.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0. \tag{2.6}$$

We need to solve equation (2.6) for f given in equation (2.5). Now,

$$\frac{\partial f}{\partial y} = 0.$$

Also,

$$\frac{\partial f}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}}\left(1+\dot{y}^2\right)^{\frac{1}{2}} = \frac{1}{2}(1+\dot{y}^2)^{-\frac{1}{2}} 2\dot{y} = (1+\dot{y}^2)^{-\frac{1}{2}}\dot{y}.$$

Therefore,

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = \frac{d}{dx}\left(\frac{\dot{y}}{(1+\dot{y}^2)^{\frac{1}{2}}}\right).$$

Using this values in equation (2.6), we get

$$-\frac{d}{dx}\left(\frac{\dot{y}}{(1+\dot{y}^2)^{\frac{1}{2}}}\right) = 0 \Rightarrow \frac{\dot{y}}{(1+\dot{y}^2)^{\frac{1}{2}}} = A \qquad (A \text{ is constant}).$$

Solving the above equation algebraically for $\dot{y}$, we get

$$\begin{aligned} \dot{y} &= A\,(1+\dot{y}^2)^{\frac{1}{2}} \\ \Rightarrow \quad \dot{y}^2 &= A^2(1+\dot{y}^2) \\ \Rightarrow \quad \dot{y}^2(1-A^2) &= A^2 \\ \Rightarrow \quad \dot{y}^2 &= \frac{A^2}{1-A^2} \end{aligned}$$

Therefore,

$$\dot{y} = a,$$

where $a = \left(\frac{A^2}{1-A^2}\right)^{\frac{1}{2}}$ which is a constant. Hence,

$$\boxed{y = ax+b},$$

where a and b are constants. The above equation represents a **straight line**. Thus, we have shown that the curve between two points in a plane on which the distance is minimum is a straight line joining these two points. □

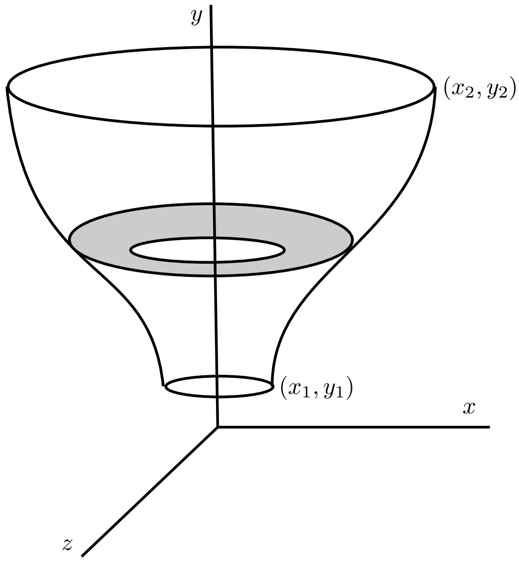
Exercise 2.2.2: Minimum surface of revolution

Obtain the curve for minimum surface of revolution.

Solution.
Suppose we form a surface by taking a curve in xy-plane passing through two fixed end points (x_1, y_1) and (x_2, y_2) and revolving the curve about y-axis (as shown in figure). Our problem is to find the curve for which the area of the surface of revolution obtained from the curve is minimum.
We shall show that this curve is given by

$$x = a\cosh\frac{y-b}{a},$$

which is the equation of a **catenary**.
(Seminar Exercise - refer Goldstein, page number $40-41$).



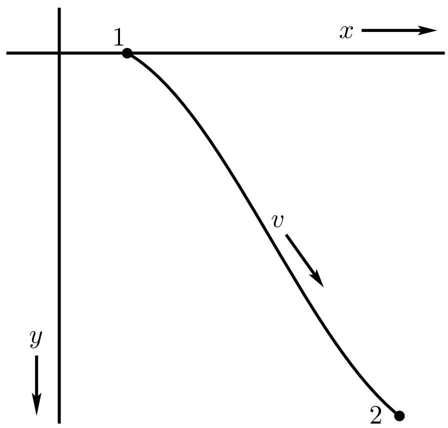
□

Exercise 2.2.3: Brachistochrone problem

Describe Brachistochrone problem (problem of least time) and obtain its solution, i.e. obtain the curve of quickest descend.

Solution.
Brachistochrone problem is a well-known problem to find a curve joining two points such that a particle at rest at the higher point falling under gravity travels to the lower point in least time, i.e. to find the curve of quickest descend.

Let t_{12} be time taken by the particle at rest to travel from a higher point 1 to a lower point 2 under gravity with velocity v. If ds is the length of the arc, then the time taken is $\frac{ds}{v}$. Then the problem is to find a minimum of the integral



$$t_{12} = \int_1^2 \frac{ds}{v}.$$

We know that $ds = \sqrt{dx^2+dy^2} = \sqrt{1+\dot{y}^2}\,dx = \sqrt{1+\dot{x}^2}\,dy$. Therefore

$$t = \int_1^2 \frac{\sqrt{1+\dot{x}^2}}{v}\,dy.$$

Suppose y is measure downwards from the higher point. Since the particle is initially at rest, kinetic energy at point 1 is $T = 0$ and its potential energy is $V = 0$. Hence, its total energy is 0. At point 2, the kinetic energy is $T = \frac{1}{2}mv^2$ and potential energy is $-mgy$. Hence total energy is $T + V = \frac{1}{2}mv^2 - mgy$. Then by law of conservation of energy, we have

$$\frac{1}{2}mv^2 - mgy = 0.$$

Therefore $v = \sqrt{2gy}$ which implies

$$t = \int_1^2 \sqrt{\frac{1+\dot{x}^2}{2gy}}\, dy.$$

By calculus of variation, we know that the condition for extremum of the above integral is given by Euler's equation, i.e.

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0,$$

where we take $f = \sqrt{\frac{1+\dot{x}^2}{2gy}}$. Now, clearly $\frac{\partial f}{\partial x} = 0$ and

$$\frac{\partial f}{\partial \dot{x}} = \frac{2\dot{x}}{2\sqrt{1+\dot{x}^2}\sqrt{2gy}} = \frac{\dot{x}}{\sqrt{(1+\dot{x}^2)2gy}}.$$

Hence, by Euler's equations, we have

$$\frac{d}{dy}\left(\frac{\partial f}{\partial \dot{x}}\right) = \frac{d}{dy}\left(\frac{\dot{x}}{\sqrt{(1+\dot{x}^2)2gy}}\right) = 0.$$

Therefore

$$\frac{\dot{x}}{\sqrt{(1+\dot{x}^2)2gy}} = c_1 \qquad \text{(where } c_1 \text{ is constant)}$$

$$\text{or} \quad \frac{\dot{x}}{\sqrt{(1+\dot{x}^2)y}} = \sqrt{2g}c_1 = c_2 \qquad \text{(where } c_2 \text{ is constant).}$$

Therefore

$$\begin{aligned}
\dot{x} &= c_2\sqrt{(1+\dot{x}^2)y} \\
\dot{x}^2 &= c_3 y + c_3 y \dot{x}^2 y \qquad (c_3 = c_2^2) \\
\dot{x}^2(1 - c_3 y) &= c_3 y \\
\dot{x}^2 &= \frac{c_3 y}{1 - c_3 y} \\
\dot{x}^2 &= \frac{y}{\frac{1}{c_3} - y} = \frac{y}{a - y} \qquad \left(\frac{1}{c_3} = a\right) \\
\dot{x} &= \sqrt{\frac{y}{a-y}}.
\end{aligned}$$

Integrating both side with respect to y, we get

$$\int dx = \int \sqrt{\frac{y}{a-y}}\, dy.$$

Take $y = a\sin^2\frac{\theta}{2}$, then $dy = a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta$. Therefore

$$\begin{aligned} x &= \int \sqrt{\frac{a\sin^2\frac{\theta}{2}}{a - a\sin^2\frac{\theta}{2}}}\, a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta \\ &= \int \frac{\sin\frac{\theta}{2}}{\sqrt{1-\sin^2\frac{\theta}{2}}}\, a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta \\ &= \int \frac{a}{2} 2\sin^2\frac{\theta}{2}\,d\theta \\ &= \int \frac{a}{2}(1-\cos\theta)\,d\theta. \end{aligned}$$

This implies, $x = \frac{a}{2}[\theta - \sin\theta] + b$. Take $\frac{a}{2} = a'$, then

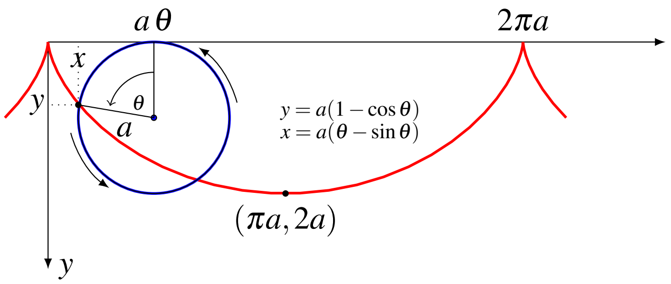
$$x = a'(\theta - \sin\theta) + b.$$

Now, $b = 0$ at point $(0,0)$. Then $x = a'(\theta - \sin\theta) + b$. Also, we obtain $y = a'(1-\cos\theta)$.

Thus, the parametric equation of the required path is

$$\begin{cases} x = a'(\theta - \sin\theta), \\ y = a'(1-\cos\theta). \end{cases}$$

which represents a **cycloid**.



□

2.3 Derivation of Lagrange's equations from Hamilton's principle

Recall that Hamilton's principle states that "among all the possible paths in the time interval $[t_1, t_2]$ the system point in the configuration space will take the path on which the action integral is extremum," i.e. the integral

$$I = \int_{x_1}^{x_2} L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t)\,dt \tag{2.13}$$

is extremum.

By calculus of variations, we know that, the condition for the integral

$$J = \int_{x_1}^{x_2} f(y_1, y_2, \ldots, y_n, \dot{y}_1, \dot{y}_2, \ldots, \dot{y}_n, x)\,dx \tag{2.14}$$

to be extremum is given by Euler-Lagrange equations, i.e.

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}_i}\right) = 0 \qquad i = 1, 2, \ldots, n. \tag{2.15}$$

Now, comparing equations (2.13) and (2.14), we consider the following transformation

$$\begin{aligned} f &\to L \\ y_i &\to q_i \qquad i = 1, 2, \dots, n \\ x &\to t. \end{aligned}$$

Using these replacements in equation (2.15), we get

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \qquad i = 1, 2, \dots, n.$$

or

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, 2, \dots, n. \tag{2.16}$$

Equations (2.16) are Lagrange's equations of motion. Thus, we have derived Lagrange's equations of motion from the Hamilton's principle.

2.4 Cyclic coordinates and Generalized momenta

Definition 2.4.1: Cyclic coordinate

Consider a system of n-degrees of freedom. Let $q_1, q_2, \dots, q_n$ be the chosen generalized coordinates and L be the Lagrangian. A coordinate q_j is said to be *cyclic coordinate* if Lagrangian does not depend explicitly on the coordinate q_j.

Lagrangian $L(q, \dot{q}, t)$ does not explicitly depend on q_j implies that $\frac{\partial L}{\partial q_j} = 0$. Therefore,

$$q_j \text{ is cyclic } \Leftrightarrow \frac{\partial L}{\partial q_j} = 0.$$

Examples 2.4.2.

1. Consider motion of a particle in XY-plane with usual Cartesian coordinates (x, y) as generalized coordinates and force derivable from a potential depending on the distance of the particle from the origin. In this case, the Lagrangian is given by

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(\sqrt{x^2 + y^2}).$$

Then, in this case, there are no cyclic coordinates, as V (and hence L) is dependent on both the generalized coordinates x and y.

2. Consider motion of a spherical pendulum. Then the Lagrangian is given by

$$L = \frac{m}{2} l^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) + mgl \cos \theta.$$

Here, ϕ is a cyclic coordinates as clearly the Lagrangian L does not depend on ϕ. Since L depends explicitly on θ, the generalized coordinate θ is non-cyclic.

Definition 2.4.3: Generalized momentum

Consider a system of n-degrees of freedom. Let $q_1, q_2, \ldots, q_n$ be the chosen generalized coordinates and $L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t) \equiv L(q, \dot{q}, t)$ be the Lagrangian. The *generalized momentum* conjugate to the generalized coordinate q_j is denoted by p_j and is given by

$$p_j = \frac{\partial L}{\partial \dot{q}_j}.$$

Generalized momentum conjugate to a generalized coordinate is sometimes also called *conjugate momentum*. Consider some examples below:

Examples 2.4.4.

1. Consider motion of a particle in XY-plane with usual Cartesian coordinates x and y as generalized coordinates, where the Lagrangian is given by

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(\sqrt{x^2 + y^2}).$$

Therefore, the generalized momenta conjugate to generalized coordinates is given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{and} \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}.$$

2. Consider the Lagrangian of a Gyroscope given by

$$L = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi} \cos \theta)^2 - mglcos\theta,$$

where θ, ϕ, ψ are generalized coordinates. Then the generalized momenta are

$$\begin{aligned} p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = I_1 \dot{\theta} \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = I_1 \dot{\phi} \sin^2 \theta + I_3 \cos \theta (\dot{\psi} + \dot{\phi} \cos \theta) \\ p_\psi &= \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\psi} + \dot{\phi} \cos \theta). \end{aligned}$$

3. We have seen that the Lagrangian in case of simple pendulum is given by

$$L = \frac{m}{2} l^2 \dot{\theta}^2 + mgl \cos \theta.$$

Then the generalized momentum corresponding to the generalized coordinate θ is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta}.$$

Theorem 2.4.5

The generalized momentum conjugate to a cyclic coordinate is conserved.

Proof. Suppose for a system a generalized coordinate q_j is cyclic. Now, Lagrange's equations of motion corresponding to q_j is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0.$$

Since q_j is cyclic, L does not depend on q_j explicitly and hence

$$\frac{\partial L}{\partial q_j} = 0.$$

Therefore, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}_j} = \text{ constant},$$

i.e. $\dot{p}_j = 0 \Rightarrow p_j =$ constant. Hence, the generalized momentum conjugate to a cyclic coordinate is conserved. □

Remark 2.4.6. We know that Lagrange's equations of motion (LEOM) are second order ordinary differential equations (ODE). If the coordinate is cyclic then the generalized momentum conjugate to that cyclic coordinate is conserved. This provides the first integral to the LEOM, i.e. we get a first order ODE. For instance, consider the following example.

Example 2.4.7. Consider the motion of a gyroscope. The Lagrangian in this case is

$$L = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 - mglcos\theta.$$

Clearly, here ϕ and ψ are cyclic coordinates as the Lagrangian does not depend on them explicitly. Therefore, the corresponding generalized linear momenta are given by

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_1\dot{\phi}\sin^2\theta + I_3\cos\theta(\dot{\psi} + \dot{\phi}\cos\theta)$$
$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\psi} + \dot{\phi}\cos\theta).$$

Notice that the above equations are first order ordinary differential equations.

2.5 Conservation theorems and Symmetry properties

2.5.1 Conservation of linear momentum in Lagrangian formalism

We have Lagrange's equations of motion given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \qquad j = 1, 2, \ldots, n.$$

But $\dfrac{\partial L}{\partial \dot{q}_j} = p_j$. Therefore the above equation can be written as

$$\frac{dp_j}{dt} = \frac{\partial L}{\partial q_j} \Rightarrow \dot{p}_j = \frac{\partial L}{\partial q_j}, \qquad j = 1, 2, \ldots, n. \tag{2.17}$$

Suppose a generalized coordinate q_j corresponds to translation of the system along a vector $\hat{n}$, i.e. change in the coordinate q_j denoted by dq_j results into translational motion of the system in the direction $\hat{n}$.

Suppose the constraints are scleronomic and potential depends on position only, i.e. potential is velocity independent. Then

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial (T - V)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} \qquad \left(\because \frac{\partial V}{\partial \dot{q}_j} = 0\right).$$

Thus,

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}, \qquad j = 1, 2, \ldots, n. \tag{2.18}$$

Also since the motion is translational in this case, clearly q_j cannot appear in T (i.e. T is independent of q_j) as the velocities are not affected by shifting the origin. Therefore, $\dfrac{\partial T}{\partial q_j} = 0$ and hence

$$\frac{\partial L}{\partial q_j} = \frac{\partial (T - V)}{\partial q_j} = -\frac{\partial V}{\partial q_j} = Q_j, \qquad j - 1, 2, \ldots, n. \tag{2.19}$$

Thus, for the chosen coordinate q_j, equations (2.18) and (2.19) when used in equation (2.17) and Lagrange's equations of motion gives

$$\dot{p}_j = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = -\frac{\partial V}{\partial q_j} = Q_j.$$

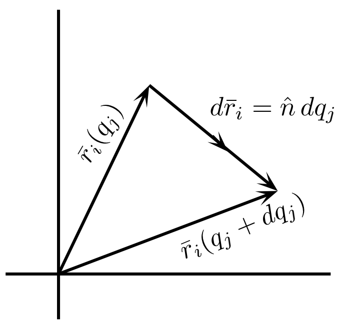
or

$$\dot{p}_j = Q_j, \qquad j = 1, 2, \ldots, n. \tag{2.20}$$

By definition of generalized forces for $j = 1, 2, \ldots, n$, we have

$$Q_j = \sum_{i=1}^{N} \bar{F}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}. \tag{2.21}$$

To evaluate $\dfrac{\partial \bar{r}_i}{\partial q_j}$, we note that it is rate of change of $\bar{r}_i$ with respect to q_j. As shown in figure



$$\begin{aligned} d\bar{r}_i &= \text{ change in } \bar{r}_i \\ &= \bar{r}_i(q_j + dq_j) - \bar{r}(q_j) \\ &= \hat{n}\, dq_j. \end{aligned}$$

Now,

$$\frac{\partial \bar{r}_i}{\partial q_j} = \lim_{dq_j \to 0} \frac{\bar{r}_i(q_j + dq_j) - \bar{r}(dq_j)}{dq_j} = \hat{n}. \tag{2.22}$$

Using equation (2.22) in (2.21), we get

$$Q_j = \sum_{i=1}^{N} \bar{F}_i \cdot \hat{n} = \hat{n} \sum_{i=1}^{N} \bar{F}_i.$$

Therefore

$$\boxed{Q_j = \hat{n} \cdot \bar{F}}, \tag{2.23}$$

where $\bar{F} = \sum_{i=1}^{N} \bar{F}_i$ is the total force. Thus, Q_j is component of the total force along $\hat{n}$. From equation (2.18)

$$\begin{aligned} p_j = \frac{\partial T}{\partial \dot{q}_j} &= \frac{\partial}{\partial \dot{q}_j} \left(\sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 \right) \\ &= \frac{\partial}{\partial \dot{q}_j} \left(\sum_{i=1}^{N} \frac{1}{2} m_i \dot{\bar{r}}_i \cdot \dot{\bar{r}}_i \right) \\ &= \sum_{i=1}^{N} m_i \dot{\bar{r}}_i \cdot \frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_j} \\ &= \sum_{i=1}^{N} m_i \bar{v}_i \frac{\partial \bar{r}_i}{\partial q_j} \qquad \text{(by law of cancellation of dots)} \\ &= \sum_{i=1}^{N} \bar{p}_i \cdot \hat{n} = \hat{n} \sum_{i=1}^{N} \bar{p}_i. \end{aligned}$$

Thus,

$$p_j = \hat{n} \cdot \bar{p},$$

where $\bar{p} = \sum_{i=1}^{N} \bar{p}_i$ is the total linear momentum of the system. Therefore, we have

$$\dot{p}_j = \frac{d}{dt} (\hat{n} \cdot \bar{p}). \tag{2.24}$$

Using equations (2.23) and (2.24) in equation (2.20), we get

$$\hat{n}\bar{F} = \frac{d}{dt} (\hat{n} \cdot \bar{p}). \tag{2.25}$$

For a single particle we know that $\bar{F} = \dot{\bar{p}}$, i.e. $\bar{F} = \frac{d\bar{p}}{dt}$. Thus, recall that, law of conservation of linear momentum for a single particle states that if $\bar{F} = 0$ then $\bar{p}$ is conserved. Here, from equation (2.25), we get law of conservation of linear momentum of a system (in Lagrangian formalism), which is stated as follows:

Law of conservation of linear momentum (Lagrangian formalism)

"Component of total linear momentum along a vector $\hat{n}$ is conserved if component of total force along that vector $\hat{n}$ is zero."

Remark 2.5.1. The derivation of law of conservation of angular momentum in Lagrangian formalism is left as a seminar exercise.

2.6 Energy Function and the Conservation of Energy

2.6.1 Energy function

Consider a system of n degrees of freedom. Let $L(q,\dot{q},t)$ be Lagrangian of the system. The total time derivative of Lagrangian is given by

$$\frac{dL}{dt} = \sum_{j=1}^{n} \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_{n=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t}. \tag{2.26}$$

Now, by Lagrange's equations of motion, we have

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right), \qquad j = 1,2,\ldots,n. \tag{2.27}$$

Using these equations in (2.26), we get

$$\frac{dL}{dt} = \sum_{j=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t}.$$

Therefore,

$$\begin{aligned}
\frac{\partial L}{\partial t} &= -\sum_{j=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{dL}{dt} \\
&= -\sum_{j=1}^{n} \left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt}\right\} + \frac{dL}{dt} \\
&= -\sum_{j=1}^{n} \frac{d}{dt}\left(\dot{q}_j\frac{\partial L}{\partial \dot{q}_j}\right) + \frac{dL}{dt} \\
&= -\frac{d}{dt}\left(\sum_{j=1}^{n} \dot{q}_j\frac{\partial L}{\partial \dot{q}_j}\right) + \frac{dL}{dt} \\
&= -\frac{d}{dt}\left\{\sum_{j=1}^{n} \dot{q}_j\frac{\partial L}{\partial \dot{q}_j} - L\right\}
\end{aligned}$$

Therefore, we write

$$\frac{\partial L}{\partial t} = -\frac{dh}{dt}, \tag{2.28}$$

where

Energy function

$$h = \sum_{j=1}^{n} \dot{q}_j\frac{\partial L}{\partial \dot{q}_j} - L. \tag{2.29}$$

The function h defined in (2.29) is called *energy function* of the system.

Corollary 2.6.1

If Lagrangian of a system does not depend on time t explicitly, then the energy function h is conserved, i.e. $\frac{\partial L}{\partial t} = 0 \Rightarrow h$ is conserved.

Proof. Here, we have $\frac{\partial L}{\partial t} = -\frac{dh}{dt} = 0$, where $h = \sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$. Hence, h is conserved. □

Exercise 2.6.2

Lagrangian of the spherical pendulum is given by

$$L = \frac{m}{2} l^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + mgl\cos\theta.$$

Evaluate the energy function using the above formula.

***Solution*.** The energy function h is given by

$$\begin{aligned} h &= \sum_j \dot{q}_j p_j - L \\ &= \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} + \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} - L \\ &= ml^2\dot{\theta}^2 + ml^2\dot{\phi}^2\sin^2\theta - \frac{ml^2}{2}(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) - mgl\cos\theta \\ &= \frac{ml^2}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mgl\cos\theta. \end{aligned}$$

□

Exercise 2.6.3

Lagrangian of a system (a Gyroscope) is given by

$$L = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 - mgl\cos\theta,$$

where θ, ϕ, ψ are generalized coordinates. Find the energy function.

***Solution*.** We know that the energy function h is given by

$$\begin{aligned} h &= \sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \\ &= \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} + \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} + \dot{\psi}\frac{\partial L}{\partial \dot{\psi}} - L \\ &= \dot{\theta}p_\theta + \dot{\phi}p_\phi + \dot{\psi}p_\psi - L. \end{aligned}$$

From given Lagrangian

$$\begin{aligned} p_\theta &= \frac{\partial L}{\partial \dot{\theta}} &&= I_1 \dot{\theta} \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} &&= I_1 \sin^2\theta \dot{\phi} + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta \\ p_\psi &= \frac{\partial L}{\partial \dot{\psi}} &&= I_3(\dot{\psi} + \dot{\phi}\cos\theta). \end{aligned}$$

Using these values in expression for h, we get

$$\begin{aligned} h &= I_1\dot{\theta}^2 + I_1\sin^2\theta\dot{\phi}^2 + I_3\dot{\phi}(\dot{\psi}+\dot{\phi}\cos\theta)\cos\theta + I_3\dot{\psi}(\dot{\psi}+\dot{\phi}\cos\theta) \\ &\quad - \frac{I_1}{2}\left[\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right] - \frac{I_3}{2}\left[\dot{\psi}+\dot{\phi}\cos\theta\right]^2 + mgl\cos\theta \\ &= \frac{I_1}{2}\left[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right] + I_3\left[\dot{\phi}\dot{\psi}\cos\theta + \dot{\phi}^2\cos^2\theta + \dot{\psi}^2 + \dot{\psi}\dot{\phi}\cos\theta\right] \\ &\quad - \frac{I_3}{2}\left[\dot{\psi}+\dot{\phi}\cos\theta\right]^2 + mgl\cos\theta \\ &= \frac{I_1}{2}\left[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right] - \frac{I_3}{2}\left[\dot{\psi}+\dot{\phi}\cos\theta\right]^2 + mgl\cos\theta. \end{aligned}$$

□

Remarks 2.6.4.

1. In many cases the energy function becomes the total energy of the system. (This is the reason for giving the name 'Energy function').
2. If constraints are scleronomic and potential does not depend on velocities then $h = T + V$ is the total energy of the system.

Example 2.6.5. Lagrangian of the spherical pendulum is given by

$$L = \frac{m}{2}l^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mgl\cos\theta.$$

Find the energy function.

Solution. Here,

$$T = \frac{m}{2}l^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) \quad \text{and} \quad V = -mgl\cos\theta.$$

Here the generalized coordinates are θ and ϕ. The constraint in spherical pendulum is clearly scleronomic. Observe that here kinetic energy T is a homogeneous function of generalized velocities of degree 2. Also the potential V is independent of generalized velocities $\dot{\theta}$ and $\dot{\phi}$. Therefore, in this case, the energy function h is the total energy, i.e.

$$h = T + V = \frac{m}{2}l^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mgl\cos\theta.$$

□

2.6.2 Conservation of energy in Lagrangian formalism

Before we prove the law of conservation of energy using Lagrangian formalism, we state the following theorem due to Euler which is used in proving the law of conservation of energy.

Theorem 2.6.6: Euler

If f is a homogeneous function of degree n in variables x_i, then

$$\sum_i x_i \frac{\partial f}{\partial x_i} = nf. \tag{2.31}$$

Theorem 2.6.7: Law of conservation of energy (in Lagrangian formalism)

If constraints are scleronomic, potential does not depend on the velocity and Lagrangian does not depend on time t explicitly then the total energy of the system is conserved.

Proof. It is known that the kinetic energy of the system of particles when expressed in terms of generalized coordinates can be decomposed in to three parts, namely

$$T = T_0 + T_1 + T_2, \tag{2.32}$$

where $T_i,\ (i = 0, 1, 2)$ contains i^{th} degree terms of generalized velocities. If the constraints are scleronomic then the transformation equations for generalized coordinates and usual coordinates do not depend on t explicitly and hence in (2.32), we will get

$$T = T_2, \tag{2.33}$$

i.e. T is a function of velocities of degree 2.

In analogy with (2.32), if we write the potential as

$$V = V_0 + V_1 + V_2,$$

then in the case of potential which is velocity independent, we get

$$V = V_0, \tag{2.34}$$

i.e. V is a function of velocity of degree zero.

Now, Lagrangian of such a system is

$$L = T - V = T_2 - V_0. \tag{2.35}$$

Using equation (2.35), the energy function is given by

$$\begin{aligned} h &= \sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \\ &= \sum_{j=1}^{n} \dot{q}_j \frac{\partial (T_2 - V_0)}{\partial \dot{q}_j} - (T_2 - V_0) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \dot{q}_j \frac{\partial T_2}{\partial \dot{q}_j} - \sum_{j=1}^{n} \dot{q}_j \frac{\partial V_0}{\partial \dot{q}_j} - (T_2 - V_0) \\
&= 2T_2 - (T_2 - V_0) \qquad \left(\because \text{ by Euler's theorem, } \frac{\partial V_0}{\partial \dot{q}_j} = 0 \right) \\
&= T_2 + V_0 \\
&= T + V = E.
\end{aligned}
$$

Therefore

$$h = T + V = E. \tag{2.36}$$

Thus, for a system with scleronomic constraints and velocity independent potential, energy function is equal to total energy of the system. Further it is known that

$$\frac{\partial L}{\partial t} = -\frac{dh}{dt}.$$

If Lagrangian does not depend on time explicitly, then h is conserved. Now, from (2.36) we get conservation of total energy E. □

Remarks 2.6.8.

1. For the conservation of total energy we need to compute the expression of total energy.
2. Since total energy is expressed in terms of first order differentials, hence conservation of energy provides first integral for the equations of motion.

Exercise 2.6.9: Simple Harmonic Oscillator (SHO)

Obtain Lagrangian for Simple Harmonic Oscillator (SHO). Is total energy conserved for a SHO?

Solution. Constraints are $y = 0,\ z = 0$. They are scleronomic. Choosing x as distance from the fixed point on the straight line as generalized coordinate.

$$T = \frac{1}{2} m\dot{x}^2 \quad \text{and} \quad V = \frac{1}{2} k x^2 \qquad (k \geq 0 \text{ constant}).$$

Note that potential V is velocity independent. Also,

$$L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2.$$

Here L does not depend on t explicitly. Since the constraints are scleronomic, potential does not depend on velocity and Lagrangian L does not depend on t, by law of conservation of energy in Lagrangian formalism, the total energy of the system (SHO) is conserved. □

Theorem 2.6.10

Consider the problem of extremum of

$$J = \int_{x_1}^{x_2} f(y, \dot{y}, x)\, dx.$$

The condition for extremum is given by Euler's equation, i.e.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0.$$

If f does not depend on x explicitly, then for $g = f - \dot{y}\dfrac{\partial f}{\partial \dot{y}}$,

$$\frac{dg}{dx} = 0.$$

In other words, if $\frac{\partial f}{\partial x} = 0$, then Euler's equation implies $\frac{dg}{dx} = 0$, where $g = f - \dot{y}\frac{\partial f}{\partial \dot{y}}$.

Proof. Here $g = f - \dot{y}\dfrac{\partial f}{\partial \dot{y}}$. Therefore,

$$\begin{aligned}\frac{dg}{dx} &= \frac{df}{dx} - \frac{d}{dx}\left(\dot{y}\frac{\partial f}{\partial \dot{y}}\right)\\ &= \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial \dot{y}}\ddot{y} - \dot{y}\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) - \ddot{y}\frac{\partial f}{\partial \dot{y}}\\ &= \dot{y}\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\right)\\ &= \dot{y}(0) = 0.\end{aligned}$$

□

Exercises

Exercise 2.1

Find the curve for minimum distance between two points in space (with Euclidean geometry).

Exercise 2.2

Using calculus of variation, determine the curve between two fixed end points such that the area of the surface of revolution obtained from the curve is minimum.

Exercise 2.3

Determine the curve of shortest distance between two points on the surface of a sphere.

Exercise 2.4

Discuss law of conservation of angular momentum of a system using Lagrangian formalism.

Exercise 2.5

State the Lagrangian in the following cases. Compute the generalized momenta and the energy function. Which of them are conserved? Justify in each of the following cases.

1. Simple pendulum (or particle moving on a circle).
2. Double pendulum.
3. Simple Harmonic Oscillator (SHO).
4. Two dimensional isotropic oscillator.

Exercise 2.6

In the each of the following cases, compute the generalized momenta and the energy function, where Lagrangian L for a system is given. Which of them are conserved? Why?

1. $L = \frac{m}{2}\left(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2\right) - \frac{k}{2}\left(ax^2 + 2bxy + cy^2\right)$.
2. $L = a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z} + g\dot{y} - k\sqrt{x^2+y^2}$.
3. $L = \dot{q}_1^2 + \frac{\dot{q_2}^2}{a + b{q_1}^2} + k_1{q_1}^2 + k_2\dot{q}_1\dot{q}_2$.

Exercise 2.7

Lagrangian of a system is given by $L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{1}{r}$. Compute all generalized momenta and energy function. Which of them are conserved? Why?

Exercise 2.8

If f does not depend on x explicitly and $F = \dot{y}\frac{\partial f}{\partial \dot{y}} - f$, then show that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0 \Rightarrow F \text{ is constant, i.e. } \frac{dF}{dx} = 0.$$

CHAPTER 3

Hamilton's Formulation

3.1 Legendre Transformations and Hamilton Equations of Motion

3.1.1 Legendre Transformation

Consider a function $f(x,y)$ which is continuous and differentiable. Then the differential of this function is

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ &= udx + vdy, \end{aligned} \tag{3.1}$$

where $u = \frac{\partial f}{\partial x}$ and $v = \frac{\partial f}{\partial y}$.

Now, we define another function g using f and its derivative $u = \frac{\partial f}{\partial x}$ as

$$g = f - ux. \tag{3.2}$$

The differential of g is given by

$$\begin{aligned} dg &= d(f - ux) \\ &= df - d(ux) \\ &= udx + vdy - (udx + xdu) \\ \therefore dg &= vdy - xdu. \end{aligned} \tag{3.3}$$

From (3.3) it is clear that g is a function of u and y, i.e. we get $g(u,y)$. Hence we can write

$$dg = \frac{\partial g}{\partial u}du + \frac{\partial g}{\partial y}dy. \tag{3.4}$$

Comparing (3.3) and (3.4), we get

$$v = \frac{\partial g}{\partial y} \text{ and } x = -\frac{\partial g}{\partial u}.$$

The function g obtained from f using (3.2) is called *Legendre transformation* of f.

Remarks 3.1.1.

1. Here the basis of description of f from (x, y) is changed to basis of description (u, y). It is possible to change the basis of description from (x, y) to (x, v) also.

2. In case of function of more variables $f(x_1, x_2, \ldots, x_m)$ by Legendre transformation we will get a new function $g(u_1, u_2, \ldots, u_k, x_{k+1}, x_{k+2}, \ldots, x_m)$, where $u_j = \frac{\partial f}{\partial x_j}$, $j = 1, 2, \ldots, k$.

   In this case, g is defined as
   $$g = f - \sum_{j=1}^{k} u_j x_j.$$

3. Inverse Legendre transformation is also a Legendre transformation.

3.1.2 Hamiltonian and Hamilton's equations of motion

Consider a system of n-degrees of freedom. Let $L(q, \dot{q}, t)$ be Lagrangian of the system. Lagrange's equation of motion are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \qquad j = 1, 2, \ldots, n.$$

or

$$\dot{p}_j = \frac{\partial L}{\partial q_j}, \tag{3.5}$$

where

$$p_j = \frac{\partial L}{\partial \dot{q}_j}.$$

Now we define a new function H given by $H(q, p, t) = \sum_{j=1}^{n} \dot{q}_j \left(\frac{\partial L}{\partial \dot{q}_j}\right) - L$, i.e.

Hamiltonian

$$H(q, p, t) = \sum_{j=1}^{n} \dot{q}_j p_j - L. \tag{3.6}$$

Now,

$$dH = \sum_{j=1}^{n} d(\dot{q}_j p_j) - dL$$

$$= \sum_{j=1}^{n} p_j d\dot{q}_j + \sum_{j=1}^{n} \dot{q}_j dp_j - \left\{ \sum_{j=1}^{n} \frac{\partial L}{\partial q_j} dq_j + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial L}{\partial t} dt \right\}$$
$$= \sum_{j=1}^{n} p_j d\dot{q}_j + \sum_{j=1}^{n} \dot{q}_j dp_j - \sum_{j=1}^{n} \dot{p}_j dq_j - \sum_{j=1}^{n} p_j d\dot{q}_j - \frac{\partial L}{\partial t} dt.$$

Therefore

$$dH = \sum_{j=1}^{n} \dot{q}_j dp_j - \sum_{j=1}^{n} \dot{p}_j dq_j - \frac{\partial L}{\partial t} dt. \tag{3.7}$$

Also since $H = H(q, p, t)$, we get

$$dH = \sum_{j=1}^{n} \frac{\partial H}{\partial q_j} dq_j + \sum_{j=1}^{n} \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt. \tag{3.8}$$

Comparing (3.7) and (3.8), we get

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

and for $j = 1, 2, \ldots, n$,

Hamilton's equations of motion

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}. \tag{3.9}$$

The expression $H(q, p, t)$ given in (3.6) is called *Hamiltonian* and (3.9) are called *Hamilton's equations of motion* (HEOM).

Remark 3.1.2. In what we saw above, we derived Hamilton's equations of motion from Lagrange's equations of motion.

3.1.3 Steps for deriving Hamilton's equation for a given system

1. Obtain Lagrangian L.
2. Obtain generalized momenta $\frac{\partial L}{\partial \dot{q}_j} = p_j$.
3. Compute $h = \sum_{j=1}^{n} \dot{q}_j p_j - L(q, \dot{q}, t)$.

   This will be a function of $q_1, q_2, \ldots, q_n$, $p_1, p_2, \ldots, p_n$, $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$ and t.
4. The generalized velocity terms can be eliminated from above expression using generalized momenta obtained in Step-2. This gives required Hamiltonian $H(q, p, t)$.
5. Obtain Hamilton's equations of motion.

Let us consider some examples (solved exercises) to obtain Hamiltonian of a system when Lagrangian is given.

Exercise 3.1.3: Hamiltonian for Simple Harmonic Oscillator

Lagrangian of SHO is given by $L = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2$. Obtain its Hamiltonian.

Solution. The generalized coordinate is x and hence the momentum conjugate to x is given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

and so $\dot{x} = \frac{p_x}{m}$. So

$$\begin{aligned} h &= \dot{x}p_x - L \\ &= \dot{x}p_x - \left(\frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2\right) \\ &= p_x\left(\frac{p_x}{m}\right) - \frac{m}{2}\left(\frac{p_x}{m}\right)^2 + \frac{k}{2}x^2 \\ &= \frac{1}{2m}\left(p_x^2 + kmx^2\right) \end{aligned}$$

Therefore

$$H = \frac{1}{2m}\left(p_x^2 + m^2w^2x^2\right),$$

where $w = \frac{k}{m}$. Thus, H is a function of x and p_x only and without generalized velocity $\dot{x}$. □

Exercise 3.1.4

Lagrangian for system of 2-degrees of freedom is given by

$$L = \dot{q}_1^2 + \frac{\dot{q}_2^2}{a + bq_1^2} + k_1q_1^2 + k_2\dot{q}_1\dot{q}_2 \tag{3.10}$$

Obtain Hamiltonian and derive Hamilton's equation of motion.

Solution. Using (3.10), the generalized momenta are

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = 2\dot{q}_1 + k_2\dot{q}_2 \tag{3.11}$$

and

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} = \frac{2\dot{q}_2}{a + bq_1^2} + k_2\dot{q}_1. \tag{3.12}$$

Now,

$$\begin{aligned} h &= \sum_{j=1}^{2} \dot{q}_j p_j - L \\ &= \dot{q}_1p_1 + \dot{q}_2p_2 - L \end{aligned}$$

$$= \dot{q}_1 p_1 + \dot{q}_2 p_2 - \left\{ \dot{q}_1^2 + \frac{\dot{q}_2^2}{a + bq_1^2} + k_1 q_1^2 + k_2 \dot{q}_1 \dot{q}_2 \right\} \tag{3.13}$$

Using equations (3.11) and (3.12), to eliminate $\dot{q}_1$ and $\dot{q}_2$, we get

$$\dot{q}_1 = \frac{-k_2 p_2 (a + bq_1^2) + 2p_1}{4 - k_2^2 (a + bq_1^2)}$$

and

$$\dot{q}_2 = \frac{(-2p_2 + p_1 k_2)(a + bq_1^2)}{4 - k_2^2 (a + bq_1^2)}.$$

Now substituting these values in the expression of h above, we get Hamiltonian.
(Complete as exercise)

□

Exercise 3.1.5

$$L = a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z} - k\sqrt{x^2 + y^2},$$

where x, y, z are generalized coordinates, a, b, c, f, k are constants. Obtain Hamiltonian and derive Hamilton's equations of motion.

Solution. Generalized momenta are given by

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = 2a\dot{x} + c\dot{y} + fy^2\dot{z}. \\ p_y &= \frac{\partial L}{\partial \dot{y}} = \frac{b}{x} + c\dot{x}. \\ p_z &= \frac{\partial L}{\partial \dot{z}} = fy^2\dot{x}. \end{aligned}$$

Now,

$$\begin{aligned} h &= \dot{x}p_x + \dot{y}p_y + \dot{z}p_z - L \\ &= a\dot{x}^2 + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z} + k\sqrt{x^2 + y^2} \\ &= a\dot{x}^2 + \dot{x}(c\dot{y} + fy^2\dot{z}) + k\sqrt{x^2 + y^2} \end{aligned} \tag{3.14}$$

Therefore, from above three equations, we have

$$c\dot{y} + fy^2\dot{z} = p_x - 2a\dot{x} = p_x - \frac{2ap_z}{fy^2}.$$

Eliminating the generalized velocities $\dot{x}, \dot{y}, \dot{z}$ from the above expression of H, we get the Hamiltonian of the form (Verify!)

$$H = \frac{ap_z^2}{f^2y^4} + \frac{p_z}{fy^2}\left(p_x - \frac{2ap_z}{fy^2}\right) + k\sqrt{x^2 + y^2} = \frac{p_z}{fy^2}\left(p_x - \frac{a}{fy^2}\right) + k\sqrt{x^2 + y^2}.$$

Deduce HEOM (exercise).

□

3.1.4 Derivation of Lagrange's equations of motion from Hamilton's equations of motion

Consider a system of n-degrees of freedom. Let $q_1, q_2, \ldots, q_n$ be the generalized coordinates and $p_1, p_2, \ldots, p_n$ be corresponding generalized momenta and $H(q,p,t)$ be Hamiltonian. Hamilton's equations of motion (HEOM) are given by

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad j = 1,2,\ldots,n. \tag{3.15}$$

Consider a Legendre transformation

$$\begin{aligned} L &= \sum_{j=1}^{n} \dot{q}_j p_j - H(q,p,t) \\ &= \sum_{j=1}^{n} p_j \frac{\partial H}{\partial p_j} - H(q,p,t). && \text{(by HEOM)} \end{aligned} \tag{3.16}$$

From above equation

$$\begin{aligned} dL &= \sum_{j=1}^{n} d\left(p_j \frac{\partial H}{\partial p_j}\right) - d(H(q,p,t)) \\ &= \sum_{j=1}^{n} d(p_j \dot{q}_j) - dH(q,p,t) \\ &= \sum_{j=1}^{n} p_j d\dot{q}_j + \sum_{j=1}^{n} \dot{q}_j dp_j - \left\{ \sum_{j=1}^{n} \frac{\partial H}{\partial q_j} dq_j + \sum_{j=1}^{n} \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt \right\} \\ &= \sum_{j=1}^{n} p_j d\dot{q}_j + \sum_{j=1}^{n} \dot{q}_j dp_j + \sum_{j=1}^{n} \dot{p}_j dq_j - \sum_{j=1}^{n} \dot{q}_j dp_j - \frac{\partial H}{\partial t} dt && \text{(using HEOM in (3.15))} \\ &= \sum_{j=1}^{n} p_j d\dot{q}_j + \sum_{j=1}^{n} \dot{p}_j dq_j - \frac{\partial H}{\partial t} \end{aligned} \tag{3.17}$$

From (3.17) it is clear that $L \equiv L(q,\dot{q},t)$ then

$$L = \sum_{j=1}^{n} \frac{\partial L}{\partial q_j} dq_j + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial L}{\partial t} dt. \tag{3.18}$$

From (3.17) and (3.18), comparing the coefficients, for $j = 1,2,\ldots,n$ we get

$$\dot{p}_j = \frac{\partial L}{\partial q_j}. \tag{3.19}$$

$$p_j = \frac{\partial L}{\partial \dot{q}_j}. \tag{3.20}$$

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}. \tag{3.21}$$

From (3.19) and (3.20), we get

$$\dot{p}_j = \frac{dp_j}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \text{ and } \dot{p}_j = \frac{\partial L}{\partial q_j},$$

i.e.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \qquad j = 1,2,\ldots,n. \tag{3.22}$$

In (3.22), we have Lagrange's equations of motion.

Remark 3.1.6. Using Legendre transformation it is possible to construct functions

$$\begin{array}{rl} L'(p,\dot{p},t) & \text{(Lagrangian like) or} \\ G(\dot{q},p,t) & \text{(Hamiltonian like).} \end{array}$$

Derive equations of motion for L' and G.

3.1.5 Matrix form of Hamilton's equations of motion

Consider a system of n-degrees of freedom. Let $H(q,p,t)$ be Hamiltonian. For writing Hamilton's equations of motion we define a $2n \times 1$ matrix (or a column matrix with $2n$ roots) η defined as

$$\eta = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}. \tag{3.23}$$

This matrix consists of generalized coordinates and generalized momenta. This matrix can also be written as $\eta_j = q_j$, $\eta_{j+n} = p_j$, for $j = 1,2,\ldots,n$.

Next we define a matrix denoted by $\dot{\eta}$ given by

$$\dot{\eta} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \\ \dot{p}_1 \\ \dot{p}_2 \\ \vdots \\ \dot{p}_n \end{bmatrix}. \tag{3.24}$$

We also define another column matrix denoted by $\frac{\partial H}{\partial \eta}$ given by

$$\frac{\partial H}{\partial \eta} = \begin{bmatrix} \frac{\partial H}{\partial q_1} \\ \frac{\partial H}{\partial q_2} \\ \vdots \\ \frac{\partial H}{\partial q_n} \\ \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ \vdots \\ \frac{\partial H}{\partial p_n} \end{bmatrix}. \tag{3.25}$$

Hamliton's equations of motion are given by

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad j = 1, 2, \ldots, n. \tag{3.26}$$

To represent these equations in matrix form we finally define a $2n \times 2n$ matrix denote by J and given by

$$\begin{aligned} J &= \left[\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ \hline -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right]. \end{aligned} \tag{3.27}$$

We note that the matrix J is formed using $n \times n$ identity matrices and $n \times n$ zero entries. Now, using this HEOM in (3.26) are given in matrix form by by

Matrix form of Hamilton's equations of motion

$$\dot{\eta} = J \frac{\partial H}{\partial \eta} \tag{3.28}$$

The elements in J are given by, for $i = 1, 2, \ldots, n$,

$$\begin{aligned} J_{ij} &= 0 \\ J_{i(j+n)} &= \delta_{ij} \\ J_{(i+n)j} &= -\delta_{ij} \\ J_{(i+n)(j+n)} &= 0 \end{aligned}$$

(3.29)

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. For $n = 3$ we have

$$J = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right]$$

Exercise 3.1.7: Matrix form of HEOM for $n = 2$

Verify that equation (3.28) gives Hamilton's equations of motion for $n = 2$.

Solution. For $n = 2$,

$$\eta = \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{bmatrix}, \qquad \dot{\eta} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix}.$$

Also,

$$\frac{\partial H}{\partial \eta} = \begin{bmatrix} \frac{\partial H}{\partial q_1} \\ \frac{\partial H}{\partial q_2} \\ \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Then right hand side of equation (3.28) is

$$\begin{aligned} J\frac{\partial H}{\partial \eta} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q_1} \\ \frac{\partial H}{\partial q_2} \\ \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ -\frac{\partial H}{\partial q_1} \\ -\frac{\partial H}{\partial q_2} \end{bmatrix} \\ &= \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \dot{\eta}. \end{aligned}$$

□

Remark 3.1.8. If J' denote the transpose of the matrix J then observe that

$$J' = -J$$

i.e. J is a skew-symmetric matrix.

Exercise 3.1.9

Show that $J^{-1} = J' = -J$ or $JJ' = I = -J^2$, where I is a $2n \times 2n$ identity matrix.

3.2 Cyclic coordinates and Conservation Theorems

3.2.1 Cyclicity of a generalized coordinate in Hamiltonian

Theorem 3.2.1

If H is Hamiltonian of a system, then

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}.$$

Proof. We have

$$\begin{aligned}\frac{dH}{dt} &= \sum_j \frac{\partial H}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial H}{\partial p_j}\dot{p}_j + \frac{\partial H}{\partial t}\\ &= \sum_j (-\dot{p})\dot{q}_j + \sum_j \dot{q}_j\dot{p}_j + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}.\end{aligned}$$

□

Corollary 3.2.2

If Hamiltonian does not depend on time t explicitly, then it is conserved i.e.

$$\frac{\partial H}{\partial t} = 0 \Rightarrow \frac{dH}{dt} = 0.$$

Corollary 3.2.3

If Lagrangian does not depend on time t explicitly, then Hamiltonian is conserved i.e.

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{dH}{dt} = 0.$$

Proof. Recall that derivation of Hamilton's equations of motion from Lagrange's equations of motion yields

$$-\frac{\partial L}{\partial t} = \frac{dH}{dt} \Rightarrow \frac{dH}{dt} = 0.$$

Hence, H is conserved. □

Theorem 3.2.4

If a coordinate q_j is cyclic in L i.e. $\frac{\partial L}{\partial q_j} = 0$ then it is cyclic in H also, i.e. $\frac{\partial H}{\partial q_j} = 0$.

Proof. Exercise. □

3.2.2 Ignorable coordinate

It is known that a generalized coordinate q_j is called cyclic if H does not depend on q_j explicitly. Suppose q_j is cyclic coordinate then

$$\frac{\partial H}{\partial q_j} = 0 \Rightarrow \dot{p}_j = 0 \Rightarrow p_j = \text{constant} = \alpha_j \text{ (say)}.$$

In this case the from of Hamiltonian is

$$H(q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n, p_1, \ldots, p_{j-1}, \alpha_j, p_{j+1}, \ldots, p_n, t).$$

Thus, H is a function of $(n-1)$ generalized coordinates and $(n-1)$ generalized momenta (as p_j is replaced by α_j). In other words, the problem reduces to a problem with $(n-1)$ degrees of freedom, thus the coordinate which is cyclic is now ignored. Hence, a cyclic coordinate in Hamiltonian formalism is ignorable.

Remark 3.2.5. Note that, it can be easily seen, a coordinate is cyclic with respect to Lagrangian formalism if and only if it is cyclic in Hamiltonian formalism. However, we call the cyclic coordinate ignorable in Hamiltonian but not in Lagrangian. The reason for this is the following.

Lagrangian of the system is given by

$$L(q_1, q_2 \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t).$$

Thus, even if a generalized coordinate, say q_j is cyclic with respect to Lagrangian, it cannot be ignored as Lagrangian is also a function of the corresponding generalized velocity, i.e. L still depends on $\dot{q}_j$. On the other hand, observe that Hamilton of a system is given as

$$H(q_1, q_2 \ldots, q_n, p_1, p_2, \ldots, p_n, t).$$

Thus, if a generalized coordinate q_j is cyclic in Hamiltonian formalism then, as seen before, the generalized momenta p_j conjugate to q_j are constant and so the cyclic coordinate q_j can be ignored.

Thus, a cyclic coordinate cannot be completely ignored in Lagrangian but it is ignorable in Hamiltonian. This fact is used in obtaining a function, called Routhian, of a system and deriving Routhian equations of motion, as given in Section 3.3 below.

3.3 Routh's Procedure

In Routhian procedure, cyclic coordinates and non-cyclic coordinates are dealt with separately. It is known that in the Hamiltonian formalism a cyclic coordinate is ignorable. In other words, in Hamiltonian formalism, cyclic coordinates are handled efficiently. Non-cyclic coordinates are equally handled efficiently by Lagrangian formalism (and Hamiltonian formalism). In Routh's procedure both the formalism are combined.

Consider a system of n-degrees of freedom. Let $q_1, q_2, \dots, q_n$ be chosen generalized coordinates. Suppose $q_1, q_2, \dots, q_s$ are non-cyclic coordinates and $q_{s+1}, q_{s+2}, \dots, q_n$ are cyclic coordinates.

Now, using Legendre transformation, we replace the generalized velocities corresponding to cyclic coordinates. Thus we define a new function called *Routhian* which is denoted by R and given by

Routhian

$$R = \sum_{j=s+1}^{n} \dot{q}_j p_j - L. \tag{3.30}$$

It can be check that (Verify!) equation (3.30) gives the function R described as

$$R(q_1, q_2, \dots, q_s, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_s, p_{s+1}, p_{s+2}, \dots, p_n, t).$$

It is also clear from equation (3.30) that, we can write

$$R = H_{\text{cyclic}} - L_{\text{non-cyclic}}. \tag{3.31}$$

i.e. Routhian is Hamiltonian for cyclic coordinates and Lagrangian for non-cyclic coordinates. The equations of motion for the system in terms of R are called Routhian equations of motion (REOM). Thus, Routhian equations of motion are like LEOM for non-cyclic coordinates and HEOM for cyclic coordinates. They are given by

Routhian equations of motion

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_j}\right) - \frac{\partial R}{\partial q_j} = 0, \quad j = 1, 2, \dots, s \quad \text{(Non-cyclic)} \tag{3.32}$$

and

$$\frac{\partial R}{\partial q_j} = -\dot{p}_j, \quad \frac{\partial R}{\partial p_j} = \dot{q}_j, \quad j = s+1, s+2, \dots, n \quad \text{(Cyclic).} \tag{3.33}$$

Let us consider one example to see how to derive Routhian equations of motion for a system whose Lagrangian is given.

Exercise 3.3.1

Obtain Routhian equations of motion for a system with Lagrangian

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{k}{r}. \tag{3.34}$$

Solution. Clearly, here θ is a cyclic coordinate and r is a non-cyclic coordinate. Thus,

$$R = \dot{\theta} p_\theta - L = \dot{\theta} p_\theta - \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{k}{r}. \tag{3.35}$$

Since $p_\theta = \dfrac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$, we have

$$\dot{\theta} = \frac{p_\theta}{mr^2}. \tag{3.36}$$

Equation (3.35) gives

$$\begin{aligned} R &= mr^2\dot{\theta}^2 - \frac{m}{2}\dot{r}^2 - \frac{m}{2}r^2\dot{\theta}^2 - \frac{k}{r} \\ &= \frac{mr^2\dot{\theta}^2}{2} - \frac{m}{2}\dot{r}^2 - \frac{k}{r} \\ &= \frac{m}{2}r^2\frac{p_\theta^2}{m^2r^4} - \frac{m}{2}\dot{r}^2 - \frac{k}{r} && \text{(using (3.36))}. \end{aligned}$$

Therefore, Routhian of the system is

$$R = \frac{p_\theta^2}{2mr^2} - \frac{m}{2}\dot{r}^2 - \frac{k}{r}. \tag{3.37}$$

Now, Routhian equations of motion are

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) - \frac{\partial R}{\partial r} = 0 \;\text{ and} \\ &\frac{\partial R}{\partial \theta} = -\dot{p}_\theta, \qquad \frac{\partial R}{\partial P_\theta} = \dot{\theta}. \end{aligned}$$

This gives

$$\begin{aligned} m\ddot{r} - \frac{{p_\theta}^2}{r^3m} + \frac{k}{r} = 0, \;\text{and} \\ \dot{p}_\theta = 0 \Rightarrow p_\theta = \text{constant}, \qquad \frac{p_\theta}{mr^2} = \dot{\theta}. \end{aligned}$$

□

Note: Notice that LEOM are two equations of 2^{nd} order. HEOM are four equations of 1^{st} order, while REOM are three equations of which one is 2^{nd} order and other two are 1^{st} order.

3.4 Phase space and Canonical variables

In Hamiltonian formalism, generalized coordinates and generalized momenta are put on equal footing as these two types of variables are used to describe Hamiltonian and equations of motion are given in terms of these variables.

Definition 3.4.1: Phase space

For a system of n-degrees of freedom, let $q_1, q_2, \ldots, q_n$ be chosen generalized coordinates and $p_1, p_2, \ldots, p_n$ be generalized momenta corresponding to $q_1, q_2, \ldots, q_n$ respectively. For such a system a $2n$-dimensional space can be associated in which

$$(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n) \equiv (q, p)$$

represent coordinates of a point in this space. Following the motion of the system, coordinates (q, p) will change.

The $2n$-dimensional space associated with the system is called *Phase space* of the system and the point representing the system is called *system point* of the phase space.

Definition 3.4.2: Canonical variables

Coordinates $(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n) \equiv (q, p)$ are also called *canonical variables*.

3.5 Derivation of Hamilton's equations from a variational principle

3.5.1 Hamilton's modified principle

The motion of a system in time interval $[t_1, t_2]$ is characterized by Hamilton's modified principle.

Consider a system of n-degrees of freedom. Let (q, p) be canonical variables. We can associate a system point in the phase space which traces a curve according to the motion of the system. In the time interval $[t_1, t_2]$ the system point traces a curve. To find this curve, Hamilton's modified principle is used which states that

Hamilton's modified principle

"Among all possible paths for the interval $[t_1, t_2]$, a system point in the phase space travels on the curve on which the integral

$$I = \int_{t_1}^{t_2} \left(\sum_{j=1}^{n} p_j \dot{q}_j - H(q, p, t) \right) dt$$

is extremum."

In other words, the path on which system point travels is such that

$$\delta I = \delta \int_{t_1}^{t_2} (\sum_{j=1}^{n} p_j \dot{q}_j - H(q,p,t))\, dt = 0.$$

3.5.2 Derivation of Hamilton's equations from Hamilton's modified principle

Consider a system of n-degrees of freedom. Let $H(q,p,t)$ be Hamiltonian, then Hamilton's modified principle states that on the actual path

$$\delta I = \delta \int_{t_1}^{t_2} (\sum_{j=1}^{n} p_j \dot{q}_j - H(q,p,t))\, dt = 0. \tag{3.38}$$

Recall that, Euler-Lagrange's equations for the function of type $f(y_1, y_2, \ldots, y_n, \dot{y}_1 . \dot{y}_2 \ldots, \dot{y}_n, x)$ are given by

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}_i}\right) = 0, \qquad i = 1, 2, \ldots, n. \tag{3.39}$$

Also, Euler-Lagrange's equations for the function of the type $f(y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n, \dot{y}_1 . \dot{y}_2 \ldots, \dot{y}_n, \dot{z}_1 . \dot{z}_2 \ldots, \dot{z}_n, x)$ are given by equations (3.39) and also

$$\frac{\partial f}{\partial z_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{z}_i}\right) = 0, \qquad i = 1, 2, \ldots, n. \tag{3.40}$$

From equation (3.38), in our case, the function is

$$f(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, \dot{p}_1, \dot{p}_2, \ldots, \dot{p}_n, t) = \sum_{i=1}^{n} (\dot{q}_i p_i - H(q,p,t)). \tag{3.41}$$

From equation (3.41) it is clear that $\dot{p}_1, \dot{p}_2, \ldots, \dot{p}_n$ do not appear in the expression of f. Also there are $2n$ dependent variables $q_1, q_2, \ldots, q_n$ and $p_1, p_2, \ldots, p_n$. Now, Euler-Lagrange equations are of the form

$$\frac{\partial f}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_j}\right) = 0, \quad j = 1, 2 \ldots, n. \tag{3.42}$$

$$\frac{\partial f}{\partial p_j} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{p}_j}\right) = 0, \quad j = 1, 2 \ldots, n. \tag{3.43}$$

From (3.41), we have

$$\frac{\partial f}{\partial \dot{q}_j} = p_j - \frac{\partial H}{\partial \dot{q}_j} = p_j.$$

Therefore,

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_j}\right) = \dot{p}_j.$$

Using this in (3.42), we get

$$-\frac{\partial H}{\partial q_j} - \dot{p}_j = 0 \Rightarrow \frac{\partial H}{\partial q_j} = -\dot{p}_j, \quad j = 1, 2, \ldots, n. \tag{3.44}$$

From (3.41), we also have for $j = 1, 2, \ldots, n$

$$\frac{\partial f}{\partial p_j} = \dot{q}_j - \frac{\partial H}{\partial p_j} \quad \text{and} \quad \frac{\partial f}{\partial \dot{p}_j} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{p}_j}\right) = 0.$$

Using these in (3.43), we get

$$\dot{q}_j - \frac{\partial H}{\partial p_j} = 0 \Rightarrow \frac{\partial H}{\partial p_j} = \dot{q}_j, \quad j = 1, 2, \ldots, n. \tag{3.45}$$

Equations (3.44) and (3.45) are Hamilton's equations of motion.

Exercises

Exercise 3.1

Assuming the form of Lagrangian in the following systems, obtain Hamiltonian and hence derive Hamilton's equations of motion.

1. Simple pendulum (or particle moving on a circle).
2. Spherical pendulum (or particle moving on a sphere).
3. Double pendulum.
4. Simple Harmonic Oscillator (SHO).
5. Two dimensional isotropic oscillator.

Exercise 3.2

In the each of the following systems, compute Hamiltonian and hence derive Hamilton's equations of motion from the given Lagrangian L.

1. $L = \frac{m}{2}\left(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2\right) - \frac{k}{2}\left(ax^2 + 2bxy + cy^2\right)$.
2. $L = a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z} + g\dot{y} - k\sqrt{x^2 + y^2}$.
3. $L = \dot{q}_1^2 + \frac{\dot{q_2}^2}{a + b{q_1}^2} + k_1{q_1}^2 + k_2\dot{q}_1\dot{q}_2$.
4. $L = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 - mgl\cos\theta$.
5. $L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$.

Exercise 3.3

Derive Hamilton's equations of motion for a system with Hamiltonian

$$H = \frac{1}{2m}\left(P_r^2 + \frac{P_\theta^2}{2} + \frac{P_\theta^2}{r^2\sin^2\theta}\right) + \frac{k}{2}r^2,$$

where r and θ are generalized coordinates.

Exercise 3.4

Using Legendre transformation, obtain Lagrangian corresponding to

$$H = \frac{1}{2m}\left(P_r^2 + \frac{P_\theta^2}{2} + \frac{P_\theta^2}{r^2\sin^2\theta}\right) + V(r).$$

Also derive Lagrange's equation of motion.

Exercise 3.5

A Hamiltonian of one degree of freedom has the form

$$H = \frac{p^2}{2\alpha} - bqpe^{-\alpha t} + \frac{bq}{2}q^2e^{-\alpha t}(\alpha + be^{-\alpha}t) + \frac{kq^2}{2}$$

where q, b, α, k are constants and find the Lagrangian corresponding to Hamiltonian.

Exercise 3.6

For given Hamiltonian $H = \frac{1}{2m}\left(p^2 + m^2\omega^2q^2\right)$, obtain the corresponding Lagrangian and hence derive Lagrange's equations of motion.

Exercise 3.7

For given Hamiltonian $H = \frac{p^2}{2m} - mAtx$, find the corresponding Lagrangian and derive Lagrange's equations of motion.

Exercise 3.8

A Hamiltonian like formalism can be set up in which $\dot{q}_i$ and $\dot{p}_i$ are the independent variable with a Hamiltonian $G(\dot{q}, \dot{p}, t)$. [Here p_i are defined in terms of $\dot{q}_i$, $\dot{q}_i$ in the usual manner]. Starting from the Lagrangian formulation show in details how to construct $G(\dot{q}_i, \dot{p}_i, t)$ and derive the corresponding Hamilton's equations of motion.

Exercise 3.9

Using appropriate Legendre transformation and Hamiltonian $H(q, p, t)$, find the equations of motion for the function $L(p, \dot{p}, t)$.

Exercise 3.10

If a coordinate q_j is cyclic in L i.e. $\frac{\partial L}{\partial q_j} = 0$ then it is cyclic in H also, i.e. $\frac{\partial H}{\partial q_j} = 0$.

Exercise 3.11

Obtain Routhian equations of motion (REOM) for a heavy symmetrical top with one point fixed for which Lagrangian is given as below.

$$L = \frac{I_1}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 - mgl\cos\theta.$$

CHAPTER 4

Canonical Transformations

4.1 Canonical Transformation

Canonical transformation is related with canonical variables, i.e. for a system of particles with generalized coordinates and generalized momenta (q,p).

It is known that choice of generalized coordinates and momenta is arbitrary. Thus another choice of variables to represent canonical variables is also possible.

Suppose for a system (q,p) are chosen canonical variables. Consider transformation of these variables. Let (Q,P) be new set of variables. Let the transformation $(q,p) \to (Q,P)$ be given by

$$\begin{aligned} Q_i &\equiv Q_i(q,p,t) \\ P_i &\equiv P_i(q,p,t), \qquad i = 1,2,\ldots,n. \end{aligned} \tag{4.1}$$

In principle, this transformation is invertible (non-singular), i.e. we may write

$$\begin{aligned} q_i &\equiv q_i(Q,P,t) \\ p_i &\equiv p_i(Q,P,t), \qquad i = 1,2,\ldots,n. \end{aligned} \tag{4.2}$$

Let $H(q,p,t)$ be Hamiltonian for the system. Then Hamilton's equations of motion are given by

$$\dot{q}_i = \frac{\partial H}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1,2,\ldots,n. \tag{4.3}$$

Using the transformation given in equation (4.2), it is possible to transform H in terms of (Q,P,t). Let the transformed function H be denoted by $K(Q,P,t)$.

Definition 4.1.1: Canonical transformation

For a system of n-degrees of freedom, let $H(q,p,t)$ be Hamiltonian. Consider a transformation $(q,p) \to (Q,P)$. Let $K(Q,P,t)$ be the transformed Hamiltonian. The transformation $(q,p) \to (Q,P)$ is said to be *canonical* if Hamilton's equations of motion are satisfied by

$K(Q,P,t)$ in terms of new variables (Q,P). In other words, $(q,p) \to (Q,P)$ is canonical if

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i}.$$

Note: Alternatively, $(q,p) \to (Q,P)$ is canonical if K behaves as Hamiltonian.

Einstein's Summation Convention and Kronecer Delta function

For a summation of indexed terms, Einstein observed that the index on which the sum is taken is repeated in a term with product.

Examples 4.1.2. 1. $\bar{u} = (u_1, u_2, \ldots, u_n)$ and $\bar{v} = (v_1, v_2, \ldots, v_n)$. Then

$$\bar{u} \cdot \bar{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{j=1}^{n} u_j v_j = u_j v_j.$$

2. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$. Then $C = AB = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$, where

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + a_{i3} b_{3j} = \sum_{k=1}^{3} a_{ik} b_{kj} = a_{ik} b_{kj}.$$

Thus, Einstein's summation convention states that the repeated index is to be summed over from the context (i.e. the summation sign is dropped).

3. For matrix B in the above example 2,

$$b_{ii} = b_{11} + b_{22} + b_{33} = \text{tr}(B).$$

Kronecker Delta:

Kronecker delta is a two indexed notation, denoted by δ_{ij}, defined as

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Note that symbols δ_{ij} form identity matrix.

Examples 4.1.3. 1. $\delta_{ij} u_j = u_i$.

$$\begin{aligned} \delta_{2j} u_j &= \delta_{21} u_1 + \delta_{22} u_2 + \cdots + \delta_{2n} u_n \\ &= \delta_{22} u_2 = u_2. \end{aligned}$$

2. $\delta_{ii} = \delta_{11} + \delta_{22} + \cdots + \delta_{nn} = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = n.$

4.1.1 Condition for a transformation to be Canonical

Consider a system of n-degrees of freedom with $H(q,p,t)$ as Hamiltonian. Let H transforms to $K(Q,P,t)$ under a transformation $(q,p) \to (Q,P)$. Since $H(q,p,t)$ is Hamiltonian, by Hamilton's modified principle, we have

$$\delta \int_{t_1}^{t_2} (\dot{q}_i p_i - H(q,p,t))\, dt = 0. \tag{4.4}$$

(Note that Einstein summation convention is used above.)

The transformation is canonical if the function $K(Q,P,t)$ is also Hamiltonian. Thus, the transformation is canonical if Hamilton's modified principle can be written for $K(Q,P,t)$, i.e.

$$\delta \int_{t_1}^{t_2} (\dot{Q}_i P_i - K(Q,P,t))\, dt = 0. \tag{4.5}$$

From equations (4.4) and (4.5), we get

$$\begin{aligned} \delta \int_{t_1}^{t_2} (\dot{q}_i p_i - H(q,p,t))\, dt &= \delta \int_{t_1}^{t_2} (\dot{Q}_i P_i - K(Q,P,t))\, dt \\ \Rightarrow \int_{t_1}^{t_2} (\dot{q}_i p_i - H(q,p,t))\, dt &= \int_{t_1}^{t_2} (\dot{Q}_i P_i - K(Q,P,t))\, dt \end{aligned}$$

The above two integrals are equal if

$$\dot{q}_i p_i - H(q,p,t) = \dot{Q}_i P_i - K(Q,P,t) + \frac{dF}{dt} \tag{4.6}$$

for an arbitrary function F of $q_1, q_2, \ldots, q_n$, $p_1, p_2, \ldots, p_n$, $Q_1, Q_2, \ldots, Q_n$, $P_1, P_2, \ldots, P_n$. This a transformation is canonical if condition (4.6) is satisfied.

4.1.2 Alternative form of condition for Canonical Transformation

We have seen that a condition for a transformation to be canonical is given by

$$\dot{q}_i p_i - H(q,p,t) = \dot{Q}_i P_i - K(Q,P,t) + \frac{dF}{dt}. \tag{4.7}$$

In the above condition when (Q,P) is replaced by (q,p) on the right hand side of the above expression of K, we get the function $H(q,p,t)$. Thus, equation (4.7) can be rewritten as

$$\dot{q}_i p_i - \dot{Q}_i P_i = \frac{dF}{dt}$$

or

Condition for a transformation to be canonical

$$p_i dq_i - P_i dQ_i = dF. \tag{4.8}$$

Thus, the condition for a transformation to be canonical is

$$p_i dq_i - P_i dQ_i$$

is an exact differential form.

Exercise 4.1.4

Show that the identity transformation is canonical, i.e. $Q_i = q_i$ and $P_i = p_i$ for all $i = 1, 2, \ldots, n$.

Solution. Here, $dQ_i = dq_i$. Therefore

$$p_i dq_i - P_i dQ_i = p_i dq_i - p_i dq_i = 0 = d(\text{constant}).$$

Thus, here the function F = constant and the transformation is canonical. □

Exercise 4.1.5

Is the following transformation canonical?

$$Q_1 = q_1, \quad P_1 = p_1 - 2p_2, \quad Q_2 = p_2, \quad P_2 = -2q_1 - q_2.$$

Solution. Here, $dQ_1 = dq_1$ and $dQ_2 = dp_2$. Now,

$$\begin{aligned} p_1 dq_1 + p_2 dq_2 - P_1 dQ_1 - P_2 dQ_2 &= p_1 dq_1 + p_2 dq_2 - (p_1 - 2p_2) dq_1 - (-2q_1 - q_2) dp_2 \\ &= p_1 dq_1 + p_2 dq_2 - p_1 dq_1 + 2p_2 dq_1 + 2q_1 dp_2 + q_2 dp_2 \\ &= p_2 dq_2 + 2p_2 dq_1 + 2q_1 dp_2 + q_2 dp_2 \\ &= d(p_2 q_2) + d(2p_2 q_1) \\ &= d(p_2 q_2 + 2p_2 q_1) \\ &= d(p_2(q_2 + 2q_1)) \end{aligned}$$

Taking $F = p_2(q_2 + 2q_1) = -p_2 P_2$, we get that

$$p_1 dq_1 + p_2 dq_2 - P_1 dQ_1 - P_2 dQ_2 = dF.$$

Hence, the transformation is canonical. □

Exercise 4.1.6

Show that the transformation

$$Q = \log(1 + \sqrt{q}\cos p), \qquad P = 2(1 + \sqrt{q}\cos p)\sqrt{q}\sin p$$

is canonical.

Solution. Here,

$$\begin{aligned} dQ &= d\left(\log(1+\sqrt{q}\cos p)\right) \\ &= \frac{1}{(1+\sqrt{q}\cos p)}\left(-\sqrt{q}\sin p\, dp + \frac{\cos p}{2\sqrt{q}}dq\right). \end{aligned}$$

Now,

$$\begin{aligned} p\,dq - PdQ &= p\,dq - 2(1+\sqrt{q}\cos p)(\sqrt{q}\sin p)\cdot\frac{1}{(1+\sqrt{q}\cos p)}\left(-\sqrt{q}\sin p\, dp + \frac{\cos p}{2\sqrt{q}}dq\right) \\ &= p\,dq - 2\frac{(\sqrt{q}\sin p)}{2\sqrt{q}}(\cos p\, dq - 2q\sin p\, dp) \\ &= p\,dq - \sin p\,(\cos p\, dq - 2q\sin p\, dp) \\ &= p\,dq - \sin p\,(\cos p\, dq - q\sin p\, dp - q\sin p\, dp) \\ &= p\,dq - \sin p\,(d(q\cos p) - q\sin p\, dp) \\ &= p\,dq - \sin p\, d(q\cos p) + q\sin^2 p\, dp \\ &= p\,dq - \sin p\, d(q\cos p) + q(1-\cos^2 p)\, dp \\ &= p\,dq + q\,dp - \sin p\, d(q\cos p) - q\cos^2 p\, dp \\ &= d(pq) - (\sin p\, d(q\cos p) + q\cos p\, d(\sin p)) \\ &= d(pq) - d(q\sin p\,\cos p) \\ &= d(q(p - \sin p\,\cos p)) = dF. \end{aligned}$$

Here we find the function $F = q(p - \sin p\,\cos p)$ and hence the given transformation is canonical.

Alternative method: Sometimes it is difficult to find a function F such that

$$p\,dq = P\,dQ = dF.$$

In such cases one can check by the method of differential equations that whether the given differential form is exact or not. We know that the form $Mdx + Ndy$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Therefore, here we have

$$p\,dq = P\,dQ = (p - \sin p\cos p)\,dq + 2q\sin^2 p\,dp.$$

Taking $M = (p - \sin p\cos p)$ and $N = 2q\sin^2 p$, then we verify that $\frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$.

$$\begin{aligned} \frac{\partial}{\partial p}(p - \sin p\cos p) &= 1 - (-\sin^2 p + \cos^2 p) \\ &= 1 + \sin^2 p - \cos^2 p \\ &= \sin^2 p + \sin^2 p = 2\sin^2 p \\ &= \frac{\partial}{\partial q}(2q\sin^2 p). \end{aligned}$$

Hence, $p\,dq - P\,dQ$ is an exact differential form. □

4.2 Generating Function and Canonical Transformations

4.2.1 Generating Function

In the condition for canonical transformation an arbitrary function F appears which is a function of old and new variables. This function F is in fact useful to generate the transformation. It is convenient if F can be expressed partly in terms of the new variables and partly in terms of the old variables. In this case the function is called a *generating function*.

4.2.2 Generating Function and Canonical Transformations

Definition 4.2.1: Generating function of canonical transformations

Recall that a transformation $(q,p) \to (Q,P)$ is said to be canonical if

$$\dot{q}_i p_i - H(q,p,t) = \dot{Q}_i P_i - K(Q,P,t) + \frac{dF}{dt}, \tag{4.9}$$

where H and K denote the Hamiltonian in variables (q,p) and (Q,P) respectively and F is an arbitrary function of old and new variables. The function F appearing on the right hand side of equation (4.9) is useful in obtaining the exact form of the canonical transformation. This is possible only when the function F is expressed in terms of half of the old set of variables ($q_1,q_2,\dots,q_n$ only or $p_1,p_2,\dots,p_n$) and half of the new set of variables ($Q_1,Q_2,\dots,Q_n$ or $P_1,P_2,\dots,P_n$). Thus, it acts as a bridge between the old and the new variables and it is called a *generating function* of the transformation.

Thus, there are four types of generating functions and they are as follows:

$$F_1(q,Q,t), \qquad F_2(q,P,t), \qquad F_3(p,Q,t), \qquad F_4(p,P,t).$$

4.2.3 Canonical transformation generated by $F_1(q,Q,t)$

The condition for a transformation $(q,p) \to (Q,P)$ to be canonical is

$$\dot{q}_i p_i - H = \dot{Q}_i P_i - K + \frac{dF}{dt}, \tag{4.10}$$

This condition determines F. Take

$$F = F_1(q,Q,t). \tag{4.11}$$

Substituting F given in (4.11) in equation (4.10), we get

$$\begin{aligned}
\dot{q}_i p_i - H &= \dot{Q}_i P_i - K + \frac{d}{dt}\left(F_1(q,Q,t)\right) \\
&= \dot{Q}_i P_i - K + \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i + \frac{\partial F_1}{\partial t} \\
&= \left(P_i + \frac{\partial F_1}{\partial Q_i}\right)\dot{Q}_i + \frac{\partial F}{\partial q_i}\dot{q}_i - K + \frac{\partial F}{\partial t}.
\end{aligned}$$

Since the old and new generalized coordinates are separately independent, above equation holds only if each of the coefficients of $\dot{q}_i$ and $\dot{p}_i$ vanish. Thus, comparing the coefficients, we get

$$p_i = \frac{\partial F_1}{\partial q_i}, \qquad P_i + \frac{\partial F_1}{\partial Q_i} = 0, \qquad -H = -K + \frac{\partial F_1}{\partial t}$$

or

$$p_i = \frac{\partial F_1}{\partial q_i}, \qquad P_i = -\frac{\partial F_1}{\partial Q_i}, \qquad K = H + \frac{\partial F_1}{\partial t}.$$

The transformation is thus generated from $F_1(q,Q,t)$.

4.2.4 Canonical transformation generated by $F_2(q,P,t)$

The condition for a transformation $(q,p) \to (Q,P)$ to be canonical is

$$\dot{q}_i p_i - H = \dot{Q}_i P_i - K + \frac{dF}{dt}, \tag{4.10}$$

This condition determines F. Take

$$F = F_2(q,P,t) - Q_i P_i. \tag{4.12}$$

Substituting F given in (4.12) in equation (4.10), we get

$$\begin{aligned}
\dot{q}_i p_i - H &= \dot{Q}_i P_i - K + \frac{d}{dt}\left(F_2(q,P,t) - Q_i P_i\right) \\
&= \dot{Q}_i P_i - K + \frac{\partial F_2}{\partial q_i} + \frac{\partial F_2}{\partial P_i} + \frac{\partial F_2}{\partial t} - Q_i \frac{dP_i}{dt} - P_i \frac{dQ_i}{dt} \\
&= \frac{\partial F_2}{\partial q_i}\dot{q}_i + \frac{\partial F_2}{\partial t} - \left(Q_i - \frac{\partial F_2}{\partial P_i}\right)\dot{P}_i - K.
\end{aligned}$$

Comparing the coefficients, we get

$$p_i = \frac{\partial F_2}{\partial q_i}, \qquad Q_i = \frac{\partial F_2}{\partial P_i}, \qquad K = H + \frac{\partial F_2}{\partial t}.$$

The transformation is thus generated from $F_2(q,P,t)$.

Exercise 4.2.2

Similarly, find the generating functions of the type F_3 and F_4, i.e. show that

1. For $F = F_3(p,Q,t) + q_i p_i$, we have

$$q_i = -\frac{\partial F_3}{\partial p_i}, \qquad P_i = -\frac{\partial F_3}{\partial Q_i}.$$

2. For $F = F_4(p,P,t) + q_i p_i - Q_i P_i$, we have

$$q_i = -\frac{\partial F_4}{\partial p_i}, \qquad Q_i = \frac{\partial F_4}{\partial P_i}.$$

Exercise 4.2.3

Obtain the transformation generated by $F_2(q,P,t) = q_j P_j$.

Solution. The transformation generated by a generating function of type $F_2(q,P,t)$ is given by

$$Q_i = \frac{\partial F_2}{\partial P_i}, \qquad p_i = \frac{\partial F_2}{\partial q_i}.$$

In our case

$$\begin{aligned} Q_i &= \frac{\partial}{\partial P_i}(q_j P_j) \\ &= q_j \frac{\partial P_j}{\partial P_i} \\ &= q_i. \end{aligned}$$

and

$$\begin{aligned} p_i &= \frac{\partial}{\partial q_i}(q_j P_j) \\ &= P_j \frac{\partial q_j}{\partial q_i} \\ &= P_i. \end{aligned}$$

Thus, the transformation generated by $F_2 = q_j P_j$ gives

$$Q_i = q_i, \qquad P_i = p_i$$

or it is identity transformation. □

Observation: Identity transformation is also a canonical transformation.

4.3 Symplectic condition for canonical transformation

4.3.1 Matrix form of condition for canonical transformation

Consider the matrix form of Hamilton's equations of motion in terms of canonical variables given by

$$\dot{\eta} = J \frac{\partial H}{\partial \eta}, \tag{4.13}$$

where $\eta = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$ and J is a $2n \times 2n$ matrix given by $J = \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right]$. Now consider a transformation $(q,p) \to (Q,P)$ or $\eta \to \zeta$, where ζ is the column matrix given by $\zeta = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \\ P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix}$, and $\dot{Q}_i = (Q_i(q,p,t))$, $\dot{P}_i = (P_i(q,p,t))$. By chain rule

$$\dot{\zeta}_i = \frac{\partial \zeta_i}{\partial \eta_j}\dot{\eta}_j. \tag{4.14}$$

Here, we have taken the transformation not depending on t. Equations (4.14) are written in matrix form as

$$\dot{\zeta} = M\dot{\eta}, \tag{4.15}$$

where $M = (M_{ij}) = \left(\frac{\partial \zeta_i}{\partial \eta_j}\right)$ is the Jacobian matrix for the transformation, i.e.

$$M = \begin{bmatrix} \frac{\partial Q_1}{\partial q_1} & \frac{\partial Q_1}{\partial q_2} & \cdots & \frac{\partial Q_1}{\partial p_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial P_n}{\partial q_1} & \frac{\partial P_n}{\partial q_2} & \cdots & \frac{\partial P_n}{\partial p_n} \end{bmatrix}_{2n\times 2n}.$$

Also, for Hamiltonian H, we can write

$$\frac{\partial H}{\partial \eta_i} = \frac{\partial H}{\partial \zeta_j}\frac{\partial \zeta_j}{\partial \eta_i}.$$

Therefore, we have

$$\frac{\partial H}{\partial \eta} = M'\frac{\partial H}{\partial \zeta}, \tag{4.16}$$

where M' denotes the transpose of the matrix M written above. From equation (4.15), we have

$$\dot{\zeta} = M\dot{\eta} = MJ\frac{\partial H}{\partial \eta} \qquad \text{(using (4.13))}.$$

$$\dot{\zeta} = MJM'\frac{\partial H}{\partial \zeta} \qquad \text{(using (4.16))}.$$

Thus, we have

$$\dot{\zeta} = MJM'\frac{\partial H}{\partial \zeta}. \tag{4.17}$$

If the transformation $\eta \to \zeta$ is canonical then H satisfies Hamilton's equations of motion in terms of ζ, i.e.

$$\dot{\zeta} = J\frac{\partial H}{\partial \zeta}. \tag{4.18}$$

From equations (4.17) and (4.18), we get

Symplectic condition for a transformation to be canonical

$$J = MJM'. \tag{4.19}$$

Thus, a transformation $\eta \to \zeta$ is canonical if the corresponding *Jacobian matrix* M satisfies the condition (4.19). The condition in equation (4.19) is also called *symplectic condition* for canonical transformation.

Exercise 4.3.1

Show that the transformation

$$Q = \log\left(\frac{\sin p}{q}\right), \qquad P = q\cot p$$

is canonical using symplectic condition.

Solution. The problem is of 1-degrees of freedom and the Jacobian matrix for given transformation is

$$M = \begin{bmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{bmatrix}.$$

Now,

$$\begin{aligned}
\frac{\partial Q}{\partial q} &= \frac{q}{\sin p}(\sin p)\left(-\frac{1}{q^2}\right) = -\frac{1}{q}.\\
\frac{\partial Q}{\partial p} &= \frac{q}{\sin p}\cdot\frac{\cos p}{q} = \cot p.\\
\frac{\partial P}{\partial q} &= \cot p.\\
\frac{\partial P}{\partial p} &= -q\operatorname{cosec}^2 p.
\end{aligned}$$

So, we have

$$M = \begin{bmatrix} -\frac{1}{q} & \cot p \\ \cot p & -q\operatorname{cosec}^2 p \end{bmatrix} \quad \text{and so} \quad M' = \begin{bmatrix} -\frac{1}{q} & \cot p \\ \cot p & -q\operatorname{cosec}^2 p \end{bmatrix} = M.$$

Now,

$$MJ = \begin{bmatrix} -\frac{1}{q} & \cot p \\ \cot p & -q\operatorname{cosec}^2 p \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\cot p & -\frac{1}{q} \\ q\operatorname{cosec}^2 p & \cot p \end{bmatrix}.$$

Then,

$$\begin{aligned} MJM' &= \begin{bmatrix} -\cot p & -\frac{1}{q} \\ q\operatorname{cosec}^2 p & \cot p \end{bmatrix} \begin{bmatrix} -\frac{1}{q} & \cot p \\ \cot p & -q\operatorname{cosec}^2 p \end{bmatrix} \\ &= \begin{bmatrix} \frac{\cot p}{q} - \frac{\cot p}{q} & -\cot^2 p + \operatorname{cosec}^2 p \\ -\operatorname{cosec}^2 p + \cot^2 p & \frac{\cot p}{q} - \frac{\cot p}{q} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = J. \end{aligned}$$

Thus, the transformation is canonical. □

Note: Here

$$\begin{aligned} dQ &= \frac{\partial Q}{\partial q} dq + \frac{\partial Q}{\partial p} dp \\ &= -\frac{1}{q} dq + \cot p \, dp. \end{aligned}$$

Definition 4.3.2

A $2n \times 2n$ matrix is said to be a simplectic matrix if it satisfies

$$MJM' = J.$$

Theorem 4.3.3

Product of two symplectic matrices is also a symplectic matrix.

Proof. Let M and N be two symplectic matrices, i.e.

$$MJM' = J \qquad \text{and} \qquad NJN' = J.$$

We want to show that MN is a symplectic matrix, i.e. to prove that $(MN)J(MN)' = J$. Now,

$$\begin{aligned} (MN)J(MN)' &= (MN)J(N'M') \\ &= M(NJN')M' \\ &= MJM' \\ &= J \end{aligned}$$

□

Exercise 4.3.4

If M is a symplectic matrix then show that $\det(M) \neq 0$.

Solution. Since M is a symplectic matrix, $MJM' = J$. Therefore

$$\det(MJM') = \det(M)\det(J)\det(M') = \det(J).$$

Therefore,

$$(\det(M))^2 = 1 \Rightarrow \det(M) = \pm 1 \neq 0.$$

Thus, $\det(M) \neq 0$ if M is symplectic. □

Exercise 4.3.5

If M is symplectic matrix then M is non-singular (or invertible matrix) and $|M| = \pm 1$.

Solution. Same as above exercise. □

Exercise 4.3.6

Prove that If M is symplectic then M^{-1} is also symplectic.

Theorem 4.3.7

Show that transpose of a symplectic matrix is symplectic, i.e. $MJM' = J$ if and only if $M'JM = J$.

Proof. Suppose the symplectic condition $MJM' = J$ holds. Then by above theorem the matrix M and hence M' is invertible. Now,

$$\begin{aligned}
& MJM' = J \\
\Rightarrow\ & MJ = J(M')^{-1} \\
\Rightarrow\ & JMJJ = JJ(M')^{-1}J \\
\Rightarrow\ & (JM)(JJ) = (JJ)((M')^{-1}J) \\
\Rightarrow\ & JM(-1) = (-1)(M')^{-1}J \qquad (\because J^2 = -I) \\
\Rightarrow\ & JM = (M')^{-1}J \\
\Rightarrow\ & M'JM = J.
\end{aligned}$$

For converse part, replace M by M'. □

Remark 4.3.8. Thus, the condition $MJM' = J$ or equivalently $M'JM = J$ is the symplectic condition for a canonical transformation.

Theorem 4.3.9

(The set of all $2n \times 2n$) symplectic matrices form a group under usual matrix multiplication.

Proof. (`Exercise`) Show the following:

1. M, N are symplectic implies that their product MN is also symplectic.
2. Associativity (this is trivial since usual matrix multiplication is associative).
3. Identity matrix I is symplectic.
4. M is symplectic then M^{-1} is also symplectic.

□

Corollary 4.3.10

The set of all canonical transformations form a group under usual composition.

4.4 Canonical transformation depending on time t

A transformation depending on time t can be regarded as evolving in time, i.e. the transformation takes place in small intervals of time. Thus the symplectic condition can be obtained in small intervals of time. In a small interval of time, the change in canonical variables is small. Thus, we will show that symplectic condition is satisfied for infinitesimal transformation.

By group property, it can be shown that the transformation (or Jacobian) matrix for a time dependent canonical transformation is a symplectic matrix. More precisely, we shall prove the following result.

4.4.1 Infinitesimal canonical transformation

Theorem 4.4.1

The matrix for infinitesimal canonical transformation is symplectic.

In other words, for an infinitesimal transformation, symplectic condition is satisfied.

Proof. An infinitesimal transformation is obtained by a small (infinitesimal) change in a variable. Thus, the transformation $(q, p) \to (Q, P)$ is given by

$$Q_i = q_i + \delta q_i, \qquad P_i = p_i + \delta p_i, \tag{4.20}$$

where δ denotes infinitesimal (or small) change. The transformation (4.20) can also be written as

$$\zeta = \eta + \delta \eta. \tag{4.21}$$

The transformation (4.20) (or (4.21)) is obtained by small deviation in the identity transformation. Also, (as seen in Example 4.2.4) identity transformation can be generated by a generating function $F_2(q,P,t) = q_i P_i$. Thus, a suitable generating function for the transformation given in (4.20) is of type F_2 and is given by

$$F_2(q,P,t) = q_i P_i + \varepsilon\, G(q,P,t), \tag{4.22}$$

where ε is an infinitesimal parameter and G is an arbitrary differentiable function. We know that the canonical transformation given by F_2 is

$$p_j = \frac{\partial F_2}{\partial q_j} \quad \text{and} \quad Q_j = \frac{\partial F_2}{\partial P_j}.$$

Using F_2 given in (4.22), we get

$$p_j = \frac{\partial F_2}{\partial q_j} = P_j + \varepsilon\,\frac{\partial G}{\partial q_j}.$$

Therefore by equation (4.20), we write

$$\delta p_j = P_j - p_j = -\varepsilon\,\frac{\partial G}{\partial q_j} \tag{4.23}$$

Similarly,

$$Q_j = \frac{\partial F_2}{\partial P_j} = q_j + \varepsilon\,\frac{\partial G}{\partial P_j}.$$

Therefore,

$$\delta q_j = \varepsilon\,\frac{\partial G}{\partial P_j} \tag{4.24}$$

Since (by equation (4.20)) P differs from p only by infinitesimal, it is consistent in the first order to replace P_j by p_j in the derivative function. We may then consider G as a function of q and p only (and possibly t), i.e. $G(q,p,t)$ and G can be referred as the generating function of the infinitesimal canonical transformation. In this case, the above equation can be rewritten as

$$\delta q_j = \varepsilon\,\frac{\partial G}{\partial p_j} \tag{4.25}$$

From equations (4.23) and (4.25), we get

$$\delta \eta = \varepsilon J \frac{\partial G}{\partial \eta} \tag{4.26}$$

and hence in equation (4.21), we get

$$\zeta = \eta + \delta\eta = \eta + \varepsilon J \frac{\partial G}{\partial \eta}. \tag{4.27}$$

From (4.27), the Jacobian (or the transformation) matrix is

$$M = \frac{\partial \zeta}{\partial \eta} = I + \varepsilon J \frac{\partial^2 G}{\partial \eta \partial \eta}.$$

Here the matrix $\dfrac{\partial^2 G}{\partial\eta\partial\eta}$ is a square matrix formed by second order derivative of G with respect to canonical variables, i.e. it is a $2n \times 2n$ matrix with entries (or elements) of the form

$$\left(\frac{\partial^2 G}{\partial\eta\partial\eta}\right)_{ij} = \frac{\partial^2 G}{\partial\eta_i\partial\eta_j} = \frac{\partial^2 G}{\partial\eta_j\partial\eta_i}$$

Thus, the matrix $\dfrac{\partial^2 G}{\partial\eta\partial\eta}$ is a symmetric matrix. Now the transpose of M is given by

$$\begin{aligned} M' &= \left(I + \varepsilon J \frac{\partial^2 G}{\partial\eta\partial\eta}\right)' \\ &= I + \varepsilon \left(J \frac{\partial^2 G}{\partial\eta\partial\eta}\right)' \\ &= I + \varepsilon \left(\frac{\partial^2 G}{\partial\eta\partial\eta}\right)' J' \\ &= I - \varepsilon \frac{\partial^2 G}{\partial\eta\partial\eta} J \qquad \left(\because \frac{\partial^2 G}{\partial\eta\partial\eta} \text{ is symmetric and } J' = -J\right) \end{aligned} \tag{4.28}$$

Now,

$$\begin{aligned} MJM' &= \left(I + \varepsilon J \frac{\partial^2 G}{\partial\eta\partial\eta}\right) J \left(I - \varepsilon \frac{\partial^2 G}{\partial\eta\partial\eta} J\right) \\ &= \left(J + \varepsilon J \frac{\partial^2 G}{\partial\eta\partial\eta} J\right) \left(I - \varepsilon \frac{\partial^2 G}{\partial\eta\partial\eta} J\right) \\ &= J - \varepsilon J \frac{\partial^2 G}{\partial\eta\partial\eta} J + \varepsilon J \frac{\partial^2 G}{\partial\eta\partial\eta} J - \varepsilon^2 J \frac{\partial^2 G}{\partial\eta\partial\eta} J \frac{\partial^2 G}{\partial\eta\partial\eta} J. \end{aligned}$$

Since ε is an infinitesimal parameter, ε^2 term can be neglected. Hence,

$$MJM' = J.$$

Thus, infinitesimal transformation is canonical. □

4.5 Poisson Brackets and Other Canonical Invariants

4.5.1 Poisson brackets

Definition 4.5.1: Poisson brackets

Let $u(q, p, t)$ and $v(q, p, t)$ be two functions of canonical variables and time. The *Poisson bracket* of u and v with respect to canonical variables (q, p) is denoted by $[u, v]_{q,p}$ and is defined as

$$[u, v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}.$$

Here, summation convention is used.

Exercise 4.5.2

Evaluate Poisson bracket $[u,v]_{q,p}$ for $u = p_1 - 2p_2$ and $v = -2q_1 - q_2$.

Solution. Here the system is of 2-degrees of freedom. Hence, Poisson bracket is given by

$$[u,v]_{q,p} = \frac{\partial u}{\partial q_1}\frac{\partial v}{\partial p_1} + \frac{\partial u}{\partial q_2}\frac{\partial v}{\partial p_2} - \frac{\partial u}{\partial p_1}\frac{\partial v}{\partial q_1} - \frac{\partial u}{\partial p_2}\frac{\partial v}{\partial q_2}.$$

Now,

$$\frac{\partial u}{\partial q_1} = 0 = \frac{\partial u}{\partial q_2} \quad \text{and} \quad \frac{\partial v}{\partial p_1} = 0 = \frac{\partial v}{\partial p_2}.$$

Also,

$$\frac{\partial u}{\partial p_1} = 1, \quad \frac{\partial u}{\partial p_2} = -2, \quad \frac{\partial v}{\partial q_1} = -2, \quad \frac{\partial v}{\partial q_2} = -1.$$

Therefore,

$$[u,v]_{q,p} = 0 + 0 - 1(-2) - (-2)(-1) = 2 - 2 = 0.$$

□

4.5.2 Fundamental Poisson Brackets

If the functions u and v are taken to be canonical variables then their Poisson bracket is called *Fundamental Poisson Bracket*. In other words, Poisson brackets of canonical variables are called Fundamental Poisson Brackets. There are four types of Fundamental Poisson brackets. They are

1. $[q_j, q_k]_{q,p} = \dfrac{\partial q_j}{\partial q_i}\dfrac{\partial q_k}{\partial p_i} - \dfrac{\partial q_j}{\partial p_i}\dfrac{\partial q_k}{\partial q_i} = 0.$

2. $[p_j, p_k]_{q,p} = \dfrac{\partial p_j}{\partial q_i}\dfrac{\partial p_k}{\partial p_i} - \dfrac{\partial p_j}{\partial p_i}\dfrac{\partial p_k}{\partial q_i} = 0.$

3. $[q_j, p_k]_{q,p} = \dfrac{\partial q_j}{\partial q_i}\dfrac{\partial p_k}{\partial p_i} - \dfrac{\partial q_j}{\partial p_i}\dfrac{\partial p_k}{\partial q_i} = \delta_{ji}\delta ik = \delta_{jk}.$

4. $[p_j, q_k]_{q,p} = \dfrac{\partial p_j}{\partial q_i}\dfrac{\partial q_k}{\partial p_i} - \dfrac{\partial p_j}{\partial p_i}\dfrac{\partial q_k}{\partial q_i} = -\delta_{ji}\delta_{ik} = -\delta_{jk}.$

4.5.3 Matrix form of Poisson brackets

Canonical variables are expressed in matrix form by a column matrix η. The Poisson bracket of u and v with respect to η is given by

$$[u,v]_\eta = \left(\frac{\partial u}{\partial \eta}\right)' J \left(\frac{\partial v}{\partial \eta}\right),$$

where $J = \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right]$.

Matrix of Fundamental Poisson Brackets:

Matrix form of fundamental Poisson brackets

$$[\eta,\eta]_\eta = J,$$

where $[\eta,\eta]_\eta = ([\eta_i,\eta_j])_{ij}$ is a $2n\times 2n$ matrix .

Theorem 4.5.3

A coordinate transformation is a canonical transformation if and only if fundamental Poisson brackets are invariant (under the transformation).

Proof. Recall that fundamental Poisson brackets are of the form

$$[\eta,\eta]_\eta = J. \tag{4.29}$$

Consider a transformation $\eta \to \zeta$. Now,

$$[\zeta,\zeta]_\eta = \left(\frac{\partial \zeta}{\partial \eta}\right)' J \left(\frac{\partial \zeta}{\partial \eta}\right) = M'JM. \tag{4.30}$$

From equations (4.29) and (4.30), it is clear that the transformation is canonical if and only if fundamental Poisson brackets are invariant (under the transformation), i.e.

$$[\zeta,\zeta]_\eta = [\eta,\eta]_\eta \Leftrightarrow M'JM = J.$$

□

Theorem 4.5.4

A coordinate transformation is canonical if and only if all Poisson brackets are invariant.

Proof. Consider a canonical transformation $\eta \to \zeta$ and M be the corresponding Jacobian. Then M is symplectic matrix, i.e. $MJM' = J$. By chain rule, we can write

$$\frac{\partial u}{\partial \eta} = M'\frac{\partial u}{\partial \zeta}, \qquad \frac{\partial v}{\partial \eta} = M'\frac{\partial v}{\partial \zeta}$$

where M is the Jacobian matrix, i.e. the matrix of transformation. Now,

$$\begin{aligned}[u,v]_\eta &= \left(\frac{\partial u}{\partial \eta}\right)' J \frac{\partial v}{\partial \eta} \\ &= \left(M'\frac{\partial u}{\partial \zeta}\right)' J \left(M'\frac{\partial v}{\partial \zeta}\right) \\ &= \left(\frac{\partial u}{\partial \zeta}\right)' MJM' \frac{\partial v}{\partial \zeta}\end{aligned}$$

$$\begin{aligned} &= \left(\frac{\partial u}{\partial \zeta}\right)' J \frac{\partial v}{\partial \zeta} \\ &= [u,v]_\zeta . \end{aligned}$$

Thus, if the transformation is canonical, then all the Poisson brackets are invariant.

Conversely, assume that all the Poisson brackets are invariant under the transformation. Then in particular, the fundamental Poisson brackets are invariant. Therefore, by above theorem, we conclude that the transformation is canonical. □

Remark 4.5.5. 1. As an application of above theorem, one can check whether a transformation is canonical or not by verifying the invariance of Poisson brackets.

2. Since, by above theorem, the Poisson brackets are canonical invariants, from now onward we do not specify the chosen canonical variables as a suffixes.

Exercise 4.5.6

Show that the transformation

$$Q = \log\left(\frac{\sin p}{q}\right), \qquad P = q \cot p$$

is canonical using fundamental Poisson brackets.

Solution. By definition of fundamental Poisson brackets, we have

$$[Q,Q]_{Q,P} = 0 = [P,P]_{Q,P} \qquad \text{and} \qquad [Q,P]_{Q,P} = 1 = -[P,Q]_{Q,P}.$$

As computed earlier in Example 4.3.1, we have

$$\begin{aligned} \frac{\partial Q}{\partial q} &= \frac{q}{\sin p}(\sin p)\left(-\frac{1}{q^2}\right) = -\frac{1}{q}. \\ \frac{\partial Q}{\partial p} &= \frac{q}{\sin p}\cdot\frac{\cos p}{q} = \cot p. \\ \frac{\partial P}{\partial q} &= \cot p. \\ \frac{\partial P}{\partial p} &= -q\,\mathrm{cosec}^2\, p. \end{aligned}$$

Then observe that

$$\begin{aligned} [Q,P]_{q,p} &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \left(-\frac{1}{q}\right)(-q\,\mathrm{cosec}^2\, q) - (\cot p)(\cot p) = 1 = [Q,P]_{Q,P} \\ [P,Q]_{q,p} &= \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} = -1 = [P,Q]_{Q,P} \\ [Q,Q]_{q,p} &= \frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial q} = 0 = [Q,Q]_{Q,P} \\ [P,P]_{q,p} &= \frac{\partial P}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial P}{\partial q} = 0 = [P,P]_{Q,P} \end{aligned}$$

Thus, in the given transformation $\eta(q,p) \to \zeta(Q,P)$, the fundamental Poisson brackets are invariant under the transformation, i.e.

$$[\zeta,\zeta]_\eta = [\zeta,\zeta]_\zeta.$$

Hence, the transformation is canonical. □

Remark 4.5.7. Thus, by far, we have seen three different ways of showing that (or determining whether) a given transformation is canonical or not. We showed that the same transformation given by

$$Q = \log\left(\frac{\sin p}{q}\right), \qquad P = q\cot p$$

is canonical by these three different ways, which are, by direct method (i.e. finding the function F), by symplectic condition and by fundamental Poisson brackets in Example 4.1, Example 4.3.1 and Example 4.5.3 respectively.

4.5.4 Properties of Poisson brackets

Anti-Symmetry of Poisson brackets

1. Poisson bracket is *anti-symmetric*, i.e. $[u,v] = -[v,u]$.

Proof.

$$[u,v] = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i} = -\left[\frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i} - \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i}\right] = -[v,u].$$

□

Corollary 4.5.8

$[u,u] = 0.$

Bilinearity of Possion brackets

2. Poisson bracket is *bilinear*. In other words, Poisson brackets are linear in both the arguments i.e.

$$[au+bv,w] = a[u,w]+b[v,w].$$

Similarly,

$$[u,av+bw] = a[u,v]+b[u,w].$$

Proof. We show that Poisson bracket is linear in first variable, i.e.

$$[au+bv,w] = a[u,w]+b[v,w].$$

$$\begin{aligned}
[au+bv,w] &= \frac{\partial(au+bv)}{\partial q_i}\frac{\partial w}{\partial p_i} - \frac{\partial(au+bv)}{\partial p_i}\frac{\partial w}{\partial q_i} \\
&= \left(a\frac{\partial u}{\partial q_i}+b\frac{\partial v}{\partial q_i}\right)\frac{\partial w}{\partial p_i} - \left(a\frac{\partial u}{\partial p_i}+b\frac{\partial v}{\partial p_i}\right)\frac{\partial w}{\partial q_i} \\
&= \left[a\left\{\frac{\partial u}{\partial q_i}\frac{\partial w}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial w}{\partial q_i}\right\}\right] + \left[b\left\{\frac{\partial v}{\partial q_i}\frac{\partial w}{\partial p_i} - \frac{\partial v}{\partial p_i}\frac{\partial w}{\partial q_i}\right\}\right] \\
&= a[u,w]+b[v,w].
\end{aligned}$$

Similarly, it can be shown that $[u,av+bw] = a[u,v]+b[u,w]$, i.e. Poisson bracket is linear in second argument also. □

Product rule of Poisson brackets

3. *Product rule*, i.e. $[uv,w] = u[v,w]+v[u,w]$.
Similarly, $[u,vw] = w[u,v]+v[u,w]$.

Proof. We have

$$\begin{aligned}
[uv,w] &= \frac{\partial(uv)}{\partial q_i}\frac{\partial w}{\partial p_i} - \frac{\partial(uv)}{\partial p_i}\frac{\partial w}{\partial q_i} \\
&= \left(u\frac{\partial v}{\partial q_i}+v\frac{\partial u}{\partial q_i}\right)\frac{\partial w}{\partial p_i} - \left(u\frac{\partial v}{\partial p_i}+v\frac{\partial u}{\partial p_i}\right)\frac{\partial w}{\partial q_i} \\
&= u\left(\frac{\partial v}{\partial q_i}\frac{\partial w}{\partial p_i} - \frac{\partial v}{\partial p_i}\frac{\partial w}{\partial q_i}\right) + v\left(\frac{\partial u}{\partial q_i}\frac{\partial w}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial w}{\partial q_i}\right) \\
&= u[v,w]+v[u,w].
\end{aligned}$$

Similarly, we can show $[u,vw] = w[u,v]+v[u,w]$. □

Jacobi's identity

4. *Jacobi Identity* for Poisson brackets.

If u,v and w are three functions with continuous second order derivatives then

$$[u,[v,w]]+[v,[w,u]]+[w,[u,v]] = 0;$$

i.e., the sum of cyclic permutations of the double Poisson brackets of three functions is zero.

Proof. The proof can be given using matrix notation of Poisson brackets, i.e.

$$[u,v] = \left(\frac{\partial u}{\partial \eta}\right)' J \left(\frac{\partial v}{\partial \eta}\right) = u_i J_{ij} v_j,$$

where $u_i = \frac{\partial u}{\partial \eta_i}$ and $v_{ij} = \frac{\partial^2 v}{\partial \eta_i \partial \eta_j}$. The proof is left as an exercise (given in book for reference). □

4.5.5 Poincare's Integral

Poincare's integral is defined as

Poincare's integeral

$$J = \underbrace{\iint \cdots \int}_{2n \text{ intervals}} dq_1\, dq_2 \cdots dq_n\, dp_1\, dp_2 \cdots dp_n\,.$$

Since $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$ are coordinates in phase space, $dq_1, dq_2, \ldots, dq_n, dp_1, dp_2, \ldots, dp_n$ gives volume in the phase space. It is denoted by $(d\eta)$.

Now, we shall prove that J is canonical invariant. Thus, we have to prove that

$$\int (d\eta) = \int (d\zeta).$$

By elementary calculus, we have

$$(d\eta) = |\det(M)|\, (d\zeta). \tag{4.31}$$

Now, since $\eta \to \zeta$ is a canonical transformation M is symplectic matrix, i.e.

$$M'JM = J.$$

Taking determinant on both sides, we get

$$\det(M'JM) = \det(M')\det(J)\det(M) = \det(J)$$
$$(\det(M))^2 = 1.$$

Therefore, $\det(M) = \pm 1$. Substituting this in equation 4.31, we get

$$(d\eta) = (d\zeta).$$

4.5.6 Lagrange Brackets

Definition 4.5.9: Lagrange brackets

Let $u(q,p,t)$ and $v(q,p,t)$ be two dynamical quantities associated with a system of n-degrees of freedom. The Lagrange bracket of u with v with respect to canonical variables

(q,p) is denoted by $\{u,v\}_{q,p}$ and defined by

$$\{u,v\}_{q,p} = \frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u}\frac{\partial q_i}{\partial v}.$$

Matrix form of Lagrange brackets:

Matrix form of Lagrange brackets

$$\{u,v\}_{\eta} = \left(\frac{\partial \eta}{\partial u}\right)' J \frac{\partial \eta}{\partial v}.$$

4.5.7 Properties of Lagrange brackets

1. Lagrange brackets are *invariant under a canonical transformation*, i.e.

$$\{u,v\}_{q,p} = \{u,v\}_{Q,P},$$

where the transformation $(q,p) \to (P,Q)$ is canonical.

Proof. Exercise (same as in case of invariance of Poisson brackets). □

2. *Anti-symmetric*: $\{u,v\} = -\{v,u\}$.
3. *Linearity*: $\{au+bv,w\} = a\{u,w\} + b\{v,w\}$.

Theorem 4.5.10

If $u(q,p,t)$ is a dynamical quantity associated with a system of n-degrees of freedom and $H(q,p,t)$ be Hamitonian of the system then

$$\frac{du}{dt} = [u,H] + \frac{\partial u}{\partial t}.$$

Proof. By Hamilton's equations of motion, we have

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j}. \tag{4.35}$$

Also, for $u(q,p,t)$ we have

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial q_j}\dot{q}_j + \frac{\partial u}{\partial p_j}\dot{p}_j + \frac{\partial u}{\partial t} \\ &= \frac{\partial u}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial H}{\partial q_j} + \frac{\partial u}{\partial t} && \text{(by (4.35))} \\ &= [u,H] + \frac{\partial u}{\partial t}.\end{aligned}$$

□

Corollary 4.5.11

If u is a constant of motion then
$$\frac{\partial u}{dt} = [H,u].$$

Proof. If u is a constant of motion then $\frac{du}{dt} = 0$. Then by above theorem
$$[u,H] + \frac{\partial u}{\partial t} = 0$$
$$\Rightarrow \frac{\partial u}{\partial t} = -[u,H] = [H,u].$$
□

Exercise 4.5.12

Show that converse of above corollary is also true.

4.5.8 Equations of motion in Poisson bracket form

Let $H(q,p,t)$ be Hamiltonian of a system of n-degrees of freedom. Then Hamilton's equations of motion are given by
$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j}. \tag{4.36}$$
Since q_j and p_j do not depend on t explicitly, we get
$$\dot{q}_j = [q_j,H], \qquad \dot{p}_j = [p_j,H]. \tag{4.37}$$
From equations (4.36) and (4.37),
$$[q_j,H] = \frac{\partial H}{\partial p_j}, \qquad [p_j,H] = -\frac{\partial H}{\partial q_j}. \tag{4.38}$$
In matrix form equation (4.38) can be written as

Hamilton's equations of motion in Poisson bracket form

$$[\eta,H] = J\frac{\partial H}{\partial \eta}.$$

Theorem 4.5.13: Poisson's theorem

If $u = u(q,p,t)$ and $v = v(q,p,t)$ are constants of motion then $[u,v]$ is also a constant of motion.

In other words, Poisson bracket of any two constants of motion is also a constant of motion.

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Proof. Since u and v are constants of motion, by above corollary

$$[H,u] = \frac{\partial u}{\partial t}, \qquad [H,v] = \frac{\partial v}{\partial t}. \tag{4.39}$$

We want to prove that $w = [u,v]$ is a constant of motion. To prove this consider Jacobi identity in the form

$$\begin{aligned}
&[H,[u,v]] + [u,[v,H]] + [v,[H,u]] = 0\\
\Rightarrow\ &[H,[u,v]] + \left[u, -\frac{\partial v}{\partial t}\right] + \left[v, \frac{\partial u}{\partial t}\right] = 0\\
\Rightarrow\ &[H,[u,v]] - \left[u, \frac{\partial v}{\partial t}\right] + \left[v, \frac{\partial u}{\partial t}\right] = 0.
\end{aligned} \tag{4.40}$$

Now,

$$\begin{aligned}
\left[u, \frac{\partial v}{\partial t}\right] &= \frac{\partial u}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial v}{\partial t}\right) - \frac{\partial u}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial v}{\partial t}\right)\\
&= \frac{\partial u}{\partial q_i}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p_i}\right) - \frac{\partial u}{\partial p_i}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial q_i}\right).
\end{aligned}$$

Similarly,

$$\left[v, \frac{\partial u}{\partial t}\right] = \frac{\partial v}{\partial q_i}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial p_i}\right) - \frac{\partial v}{\partial p_i}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial q_i}\right).$$

Hence,

$$\begin{aligned}
\left[v, \frac{\partial u}{\partial t}\right] - \left[u, \frac{\partial v}{\partial t}\right] &= \left\{\frac{\partial v}{\partial q_i}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial p_i}\right) - \frac{\partial v}{\partial p_i}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial q_i}\right)\right\} - \left\{\frac{\partial u}{\partial q_i}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p_i}\right) - \frac{\partial u}{\partial p_i}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial q_i}\right)\right\}\\
&= \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial q_i}\frac{\partial u}{\partial p_i}\right) - \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p_i}\frac{\partial u}{\partial q_i}\right)\\
&= -\frac{\partial}{\partial t}[u,v].
\end{aligned} \tag{4.41}$$

Substituting (4.41) in (4.40), we get

$$[H,[u,v]] - \frac{\partial}{\partial t}[u,v] = 0 \Rightarrow [H,[u,v]] = \frac{\partial}{\partial t}[u,v]$$

or

$$[H,w] = \frac{\partial w}{\partial t}.$$

Hence, (by above corollary) $w = [u,v]$ is a constant of motion. □

Note: If u and v are constants of motion and do not depend on t explicitly then the proof of Poisson's theorem becomes simpler. Consider the following result.

Theorem 4.5.14

If u and v do not depend on t explicitly and they are constants of motion then $[u,v]$ is a constant of motion.

Proof. We know that

$$\frac{du}{dt} = [u,H] + \frac{\partial u}{\partial t}.$$

Since u is a constant of motion, $\frac{du}{dt} = 0$. Also since u does not depend on time explicitly $\frac{\partial u}{\partial t} = 0$. Therefore,

$$[u,H] = 0 \Rightarrow [H,u] = 0.$$

Similarly,

$$[v,H] = 0.$$

Then from Jacobi identity

$$[H,[u,v]] + [u,[v,H]] + [v,[H,u]] = 0$$

we have

$$[H,[u,v]] = 0.$$

Also, if u and v does not depend explicitly on t then $[u,v]$ also do not depend on t explicitly. Hence

$$\frac{d[u,v]}{dt} = [[u,v],H] + \frac{\partial [u,v]}{\partial t} = 0.$$

Therefore, $[u,v]$ is a constant of motion. □

Exercise 4.5.15

For a system of 2-degrees of freedom Hamiltonian is given by

$$H = q_1 p_1 - q_2 p_2 - a q_1^2 + b q_2^2.$$

Show that the functions $F_1 = \frac{p_1 - a q_1}{q_2}$ and $F_2 = q_1 q_2$ are constants of motion. Are there any other constants of motion obtained using Poisson brackets?

Solution. Clearly, F_2 does not depend on t explicitly and so $\frac{\partial F_2}{\partial t} = 0$. Thus, to show that F_2 is constant of motion it suffices to show that $[F_2,H] = 0$. Now,

$$\begin{aligned}[F_2,H] &= \frac{\partial F_2}{\partial q_1}\frac{\partial H}{\partial p_1} - \frac{\partial F_2}{\partial p_1}\frac{\partial H}{\partial q_1} + \frac{\partial F_2}{\partial q_2}\frac{\partial H}{\partial p_2} - \frac{\partial F_2}{\partial p_2}\frac{\partial H}{\partial q_2} \\ &= (q_2)(q_1) - 0 + (q_1)(-q_2) - 0 = 0.\end{aligned}$$

Thus, F_2 is a constant of motion. Similarly, show that $[F_1,H] = 0$. Thus, F_1 is also a constant of motion. Now,

$$\begin{aligned}[F_1,F_2] &= \frac{\partial F_1}{\partial q_1}\frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial q_1} + \frac{\partial F_1}{\partial q_2}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial q_2} \\ &= 0 - \left(\frac{1}{q_2}\right)(q_2) + 0 - 0 \\ &= -1 \neq 0.\end{aligned}$$

If u and v are constants of motion then (by an earlier result) $[u,v]$ is also a constant of motion. Hence, further, $[u,[u,v]]$ and $[v,[u,v]]$ are also constants of motion.

A system of $2n$ first order ordinary differential equations admits at most $2n$ independent constants. Therefore, there are no constants of motion obtained using Poisson brackets. □

Theorem 4.5.16

If u is a dynamical quantity which does not depend on t explicitly, then

$$u(t) = u_0 + t[u,H]_0 + \frac{t^2}{2!}[[u,H],H]_0 + \cdots$$

where suffix 0 denotes the value of a quantity at $t = 0$.

Proof. By Maclaurin series expansion, we have

$$u(t) = u_0 + t\left(\frac{du}{dt}\right)_{t=0} + \frac{t^2}{2!}\left(\frac{d^2u}{dt^2}\right)_{t=0} + \frac{t^3}{3!}\left(\frac{d^3u}{dt^3}\right)_{t=0} + \cdots. \tag{4.42}$$

Now since u does not depend on t explicitly $\frac{\partial u}{\partial t} = 0$ and hence

$$\frac{du}{dt} = [u,H].$$

Differentiating again with respect to t (and replacing u by $[u,H]$ in above), we get

$$\frac{d^2u}{dt^2} = \frac{d}{dt}\left(\frac{du}{dt}\right) = \frac{d}{dt}[u,H] = [[u,H],H].$$

Similarly, higher order time derivatives can be obtained in terms of Poisson brackets with H. Using them in equation (4.42), we get

$$u(t) = u_0 + t[u,H]_0 + \frac{t^2}{2!}[[u,H],H]_0 + \cdots.$$

□

Remark 4.5.17. Above theorem when used for canonical variables (q,p) gives formal solution to a mechanical problem in terms of Poisson brackets. That is, formal solution is given by

$$q_i(t) = q_{i0} + t[q_i,H]_0 + \frac{t^2}{2!}[[q_i,H],H]_0 + \cdots$$

and

$$p_i(t) = p_{i0} + t[p_i,H]_0 + \frac{t^2}{2!}[[p_i,H],H]_0 + \cdots.$$

Exercise 4.5.18

Hamiltonian for a particle moving on a curve with constant acceleration is given by

$$H = \frac{p^2}{2m} - max \quad \texttt{(verify this)},$$

where m is the mass of the particle, x is the generalized coordinate, p is momentum conjugate to x and a is the acceleration. Solve this problem using Poisson brackets subject to the conditions $x = 0$, $p_0 = mv_0$ at time $t = 0$ (or $x = 0$, $\frac{dx}{dt} = \frac{p_0}{m}$ at $t = 0$).

Solution. For x and p (by above theorem or remark), we can write

$$x(t) = x_0 + t[x,H]_0 + \frac{t^2}{2!}[[x,H],H]_0 + \cdots \tag{4.43}$$

and

$$p(t) = x_0 + t[p,H]_0 + \frac{t^2}{2!}[[p,H],H]_0 + \cdots. \tag{4.44}$$

Now,

$$[x,H] = \frac{\partial x}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial x}{\partial p}\frac{\partial H}{\partial x} = \frac{p}{m} \qquad \left(\because \frac{\partial x}{\partial p} = 0\right).$$

Therefore

$$\begin{aligned}[][[x,H],H] &= \left[\frac{p}{m}, H\right] \\ &= \frac{1}{m}\left\{\frac{\partial p}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial p}{\partial p}\frac{\partial H}{\partial x}\right\} \\ &= \frac{1}{m}(ma) = a \text{ (constant)} \qquad \left(\because \frac{\partial p}{\partial x} = 0\right).\end{aligned}$$

Then

$$[[[x,H],H],H] = [a,H] = 0.$$

Hence, all other higher order Poisson brackets with x vanish. Now,

$$\begin{aligned}[][p,H] &= \frac{\partial p}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial p}{\partial p}\frac{\partial H}{\partial x} \\ &= -(-ma) = ma \text{ (constant)} \qquad \left(\because \frac{\partial p}{\partial x} = 0\right).\end{aligned}$$

Hence,

$$[[p,H],H] = [a,H] = 0$$

and all other higher order Poisson brackets with x vanish. Using these values in equations (4.43) and (4.44), we get

$$x = 0 + t\frac{p_0}{m} + \frac{t^2}{2!}a = v_0 t + \frac{1}{2}at^2$$

and

$$p = p_0 + t(ma) = mv_0 + mat.$$

(or $mv = mv_0 + mat \Rightarrow v = v_0 + at$). □

Exercise 4.5.19

Consider Hamiltonian given in above exercise. Obtain Hamilton's equations of motion and solve them. Verify that the solution thus obtained is same as the solution obtained in above exercise.

Exercises

Exercise 4.1

Show that the transformation

$$Q = \log\left(\frac{\sin p}{q}\right), \qquad P = q\cot p$$

is canonical by direct method, i.e. by directly finding the function.

Exercise 4.2

Find the generating functions of the type F_3 and F_4, i.e. show that

1. For $F = F_3(p,Q,t) + q_i p_i$, we have

$$q_i = -\frac{\partial F_3}{\partial p_i}, \qquad P_i = -\frac{\partial F_3}{\partial Q_i}.$$

2. For $F = F_4(p,P,t) + q_i p_i - Q_i P_i$, we have

$$q_i = -\frac{\partial F_4}{\partial p_i}, \qquad Q_i = \frac{\partial F_4}{\partial P_i}.$$

Exercise 4.3

Determine the canonical transformation generated by $F_1 = q_j Q_j$.

Exercise 4.4

Determine the transformation generated by $F_3 = -(e^Q - 1)^2 \tan p$.

Exercise 4.5

For a system of 1-degree of freedom, a generating function is given by

$$F_1 = \frac{mwq^2}{2}\cot Q.$$

Obtain the canonical transformation generated by F_1.

Exercise 4.6

Show that the transformation

$$Q = \tan^{-1}\left(\frac{\alpha q}{p}\right), \qquad P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha q^2}\right)$$

is canonical by verifying that it satisfies symplectic condition (i.e. matrix method).

Exercise 4.7

Prove that If M is symplectic then M^{-1} is also symplectic.

Exercise 4.8

Let $u = q_1 p_1 - q_2 p_2 - a q_1^2 + b q_2^2$ and $v = \dfrac{p_1 - a q_1}{q_2}$. Evaluate $[u, v]_{q,p}$.

Exercise 4.9

Verify the matrix form of Poisson brackets for a system of 2-degrees of freedom.

Exercise 4.10

Using fundamental Poisson brackets show that the transformation

$$Q = \tan^{-1}\left(\frac{q}{p}\right), \qquad P = \frac{q^2}{2}\left(1 + \frac{p^2}{q^2}\right)$$

is canonical.

Exercise 4.11

Find under what conditions

$$Q = \frac{\alpha p}{x}, \qquad P = \beta x^2,$$

where α and β are constants, represents a canonical transformation for a system of one degree of freedom.

Exercise 4.12

Determine whether the transformation

$$\begin{aligned} Q_1 &= q_1 q_2, \qquad & P_1 &= \frac{p_1 - p_2}{q_2 - q_1} + 1 \\ Q_2 &= q_1 + q_2, \qquad & P_2 &= \frac{q_2 p_2 - q_1 p_1}{q_2 - q_1} - (q_2 + q_1) \end{aligned}$$

is canonical.

Exercise 4.13

By any method, prove that the following transformation is canonical:

$$\begin{aligned} Q_1 &= q_1^2, & Q_2 &= q_2 \sec p_2 \\ P_1 &= \frac{p_1 \cos p_2 - 2q_2}{2q_1 \cos p_2}, & P_2 &= \sin p_2 - 2q_1 \end{aligned}$$

Exercise 4.14

Using fundamental Poisson brackets find the values of α and β for which the equations

$$Q = q^\alpha \cos \beta p, \qquad P = q^\alpha \sin \beta p$$

represent a canonical transformation.

Exercise 4.15

Check whether the following transformations are canonical or not.

1. $P = \log \sin p, \ Q = q \tan p$
2. $P = qp^2, \ Q = \frac{1}{p}$
3. $P = q^2 \sin 2p, \ Q = q^2 \cos 2p$

Exercise 4.16

Prove Jacobi's identity for Poisson brackets.

Exercise 4.17

Show that a transformation is a canonical transformation if and only if fundamental Lagrange brackets are invariant.

Exercise 4.18

Show that

$$D = \frac{pq}{2} - Ht$$

is a constant of motion for a system with Hamiltonian given by

$$H = \frac{p^2}{2} - \frac{1}{2q^2}.$$

Exercise 4.19

Show that $\dfrac{d[u,v]}{dt} = \left[\dfrac{du}{dt}, v\right] + \left[u, \dfrac{dv}{dt}\right].$

Answers to Exercises

CHAPTER 1

***Solution* 1.1 (Page 43).**

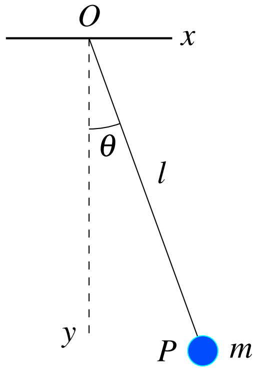
1. Let $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ be coordinates of two particles. Since they are connected by a rod of length l, the distance between them remains constant l during motion. This is the only constraint of the system and it is a holonomic constraint expressed by
$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2=l^2$$
which is a holonomic constraint. Hence, degrees of freedom of the system is
$$3N-k=3(2)-1=5.$$

2. Let $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ be coordinates of two particles. Since they are connected by an in-extensible rod of length l one constraint of the system is expressed by
$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2=l^2.$$
Further more the center of the rod which is $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$ moves on a circle (i.e. on a circle in plane) of radius r. Therefore, we have two more holonomic constraints which are expressed by:
   1. $\frac{z_1+z_2}{2}=0$ i.e. $z_1+z_2=0$ or $z_2=-z_1$.
   2. $\left(\frac{x_1+x_2}{2}\right)^2+\left(\frac{y_1+y_2}{2}\right)^2=r^2$.

   Thus, there are three constraints and all are holonomic. Hence, degrees of freedom of the system is
$$3N-k=3(2)-3=3.$$

3. Simple pendulum is a system of one particle where the particle is suspended by a rigid weightless and inextendable string from a fixed point. The particle is allowed to move in vertical plane and motion takes place under gravity. The constraints are
   1. $x^2+y^2+z^2=l^2$.
   2. $z=0$.

   Therefore degrees of freedom is $n=3N-k=1$.



Since l is constant, we choose the angle θ made by the pendulum with the vertical axis as the generalized coordinate.

4. Here, the number of particles, $N = 1$. Since the particle moves on a parabola or an ellipse, it satisfies the equation of the parabola or the ellipse on which it moves. Therefore, the constraint is
$$y^2 = 4ax \text{ or } x^2 = 4ay$$
if the particle moves on a parabola and the constraint is
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
if the particle is moving on an ellipse. In each case, the number of constraint $k = 1$ and it is holonomic constraint. Hence, the degrees of freedom is $n = 3N - k = 2$.
In case of a parabola $y^2 = 4ax$, we choose x (or y) as a generalized coordinate. In case of an ellipse, we choose either x or y as a generalized coordinate.

***Solution* 1.2 (Page 43).** For a particle moving in Xy-plane, $N = 1$. The only constraint is $z = 0$ which is a holonomic constraint. Thus, $k = 1$ and hence degrees of freedom is $n = 3N - k = 2$. Since, we have to choose plane polar coordinates, we have

$$x = r\cos\theta \text{ and } y = r\sin\theta,$$

where $r \in \mathbb{R}$ and $\theta \in [0, 2\pi)$ or $\theta \in [-\pi, \pi)$. Since, degrees of freedom of the system is 2, in terms of plane polar coordinates, the generalized coordinates are $q_1 = r$ and $q_2 = \theta$.

Note that in terms of Cartesian coordinates, the generalized coordinates are $q_1 = x$ and $q_2 = y$.

***Solution* 1.3 (Page 44).**

***Solution* 1.4 (Page 44).** Seminar Exercise

***Solution* 1.7 (Page 44).** Seminar exercise.

***Solution* 1.8 (Page 44).** Here, degrees of freedom $n = 3N = k = 3 - 1 = 2$. Writing the coordinates

$$\begin{aligned} x &= l\sin\theta\cos\phi \\ y &= l\sin\theta\sin\phi \\ z &= l\cos\theta \end{aligned} \tag{1.14}$$

Therefore, the kinetic energy is given by

$$\begin{aligned} T &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &= \frac{ml^2}{2}(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2). \end{aligned}$$

Also, the potential energy is given by

$$V = -mgl\cos\theta.$$

Then the Lagrangian is given by

$$\begin{aligned} L &= T - V \\ &= \frac{ml^2}{2}(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) + mgl\cos\theta. \end{aligned}$$

Now, Lagrange's equations of motion are given by

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} &= 0. \end{aligned}$$

which gives

$$\ddot{\theta} = -\frac{1}{l}\sin\theta(g - \dot{\phi}^2\, l\cos\theta) \text{ and } \ddot{\phi} = -2\frac{\dot{\phi}\dot{\theta}\cos\theta}{\sin\theta}.$$

***Solution* 1.12 (Page 44).** A *Double Pendulum* is a system of two particles $P(x_1, y_1, z_2)$ and $Q(x_2, y_2, z_2)$ having masses m_1 and m_2 respectively, where P is suspended from origin by a rod of length l_1 and second particle Q of mass m_2 is suspended from P by a rod of length l_2.

Constraints of double pendulum are as follows:

$$\begin{aligned} x_1^2 + y_1^2 &= l_1^2 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 &= l_2^2 \\ z_1 &= 0 \\ z_2 &= 0. \end{aligned}$$

Thus, there are $k = 4$ constraints in case of double pendulum and therefore the degrees of freedom is

$$n = 3N - k = 3(2) - 4 = 2.$$

Now, we assign generalized coordinates to the double pendulum. Choosing θ_1, the angle made by $\overline{OP}$ with vertical line and θ_2, the angle made by $\overline{PQ}$ with the vertical line as the generalized coordinates.

As shown in figure, in $\triangle OSP$, we have

$$\sin\theta_1 = \frac{x_1}{l_1} \Rightarrow x_1 = l_1\sin\theta_1.$$

Also,

$$\cos\theta_1 = \frac{y_1}{l_1} \Rightarrow y_1 = l_1\cos\theta_1.$$

Similarly, from $\triangle PQR$, we have

$$x_2 = l_1\sin\theta_1 + l_2\sin\theta_2 \quad \text{and} \quad y_2 = l_1\cos\theta_1 + l_2\cos\theta_2.$$

Now to obtain Lagrange's equations of motion we first find the Lagrangian for double pendulum. The Lagrangian L is given by $L = T - V$, where T is kinetic energy and V is potential energy. The kinetic energy T is given by

$$\begin{aligned} T &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \\ &= \frac{1}{2}(\dot{x_1}^2 + \dot{y_1}^2 + \dot{z_1}^2) + \frac{1}{2}(\dot{x_2}^2 + \dot{y_2}^2 + \dot{z_2}^2) \\ &= \frac{1}{2}m_1(l_1^2\dot{\theta_1}^2) + \frac{1}{2}m_2(l_1^2\dot{\theta_1}^2 + l_2^2\dot{\theta_2}^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)). \end{aligned}$$

Also, potential V is given by

$$\begin{aligned} V &= -mgy_1 - mgy_2 \\ &= -mgl_1(\cos\theta_1) - mg(l_1\cos\theta_1 + l_2\cos\theta_2). \end{aligned}$$

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m_1l_1^2\dot{\theta_1}^2 + \frac{1}{2}m_2(l_1^2\dot{\theta_1}^2 + l_2^2\dot{\theta_2}^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)) \\ &\quad + mgl_1(\cos\theta_1) + mg(l_1\cos\theta_1 + l_2\cos\theta_2). \end{aligned}$$

Now Lagrange's equations of motion are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = 0 \qquad \text{and} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) - \frac{\partial L}{\partial\theta_2} = 0.$$

This gives (after computation),

$$\ddot{\theta}_1 = \frac{-m_2l_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) - m_2l_2\dot{\theta_2}^2\sin(\theta_1 - \theta_2) - (m_1 + m_2)g\sin\theta_1}{(m_1 + m_2)l_1}.$$

and

$$\ddot{\theta}_2 = \frac{1}{l_2}\left[-l_1\ddot{\theta}_1\cos(\theta_1 - \theta_2) + l_1\dot{\theta}_1^2\sin(\theta_1 - \theta_2) - g\sin\theta_2\right].$$

***Solution* 1.13 (Page 45).** For a spherical pendulum (or a particle moving on a sphere), the degrees of freedom is 2. We use $q_1 = \theta$ and $q_2 = \phi$ as generalized coordinates (in terms of spherical coordinates). We know that

$$\begin{aligned} x &= l\sin\theta\cos\phi \\ y &= l\sin\theta\sin\phi \\ z &= l\cos\theta \end{aligned}$$

Therefore

$$\begin{aligned} \dot{x} &= l[\sin\theta(-\sin\phi)\dot{\phi} + \cos\phi\cos\theta\dot{\theta}] \\ \dot{y} &= l[\sin\theta(\cos\phi)\dot{\phi} + \sin\phi\cos\theta\dot{\theta}] \\ \dot{z} &= -l\sin\theta\dot{\theta} \end{aligned}$$

Now, kinetic energy of spherical pendulum is given by

$$\begin{aligned}
T &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) \\
&= \frac{1}{2}ml^2[(-\sin\theta\sin\phi\dot{\phi}+\cos\theta\cos\phi\dot{\theta})^2+(\sin\theta\cos\phi\dot{\phi}+\cos\theta\sin\phi\dot{\theta})^2+(-\sin\theta\dot{\theta})^2] \\
&= \frac{1}{2}ml^2\left[\sin^2\theta\sin^2\phi\dot{\phi}^2-2\sin\theta\sin\phi\cos\theta\cos\phi\dot{\theta}\dot{\phi}+\cos^2\theta\cos^2\phi\dot{\theta}^2+\sin^2\theta\cos^2\phi\dot{\phi}^2\right. \\
&\qquad \left.+\,2\sin\theta\sin\phi\cos\theta\cos\phi\dot{\theta}\dot{\phi}+\sin^2\phi\cos^2\theta\dot{\theta}^2+\sin^2\theta\dot{\theta}^2\right] \\
&= \frac{1}{2}ml^2[\dot{\phi}^2\sin^2\theta(\sin^2\phi+\cos^2\phi)+\dot{\theta}^2\cos^2\theta(\cos^2 phi+\sin^2\phi)+\sin^2\theta\dot{\theta}^2] \\
&= \frac{1}{2}ml^2[\dot{\phi}^2\sin^2\theta+\dot{\theta}^2].
\end{aligned}$$

***Solution* 1.16 (Page 45).** A simple harmonic oscillator is an oscillator that is neither driven nor damped. The motion of simple harmonic oscillator is called simple harmonic motion, which is motion on a straight line. It consists of a mass m which experiences a single force $\bar{F}$, which pulls the mass m in the direction of the point $x=0$ and depends only on the position x. Here x is the only generalized coordinate. Hence, degrees of freedom is 1.

Now, the force is directly proportional to the negative of the distance of the particle from a fixed point on the line of motion, i.e. $F \propto -x$. So, $F=-kx$ $(k>0)$ or $\bar{F}=kx\hat{i}$. We know that $\bar{F}=-\nabla V$. Therefore,

$$-kx=-\left(\frac{\partial V}{\partial x},\frac{\partial V}{\partial y},\frac{\partial V}{\partial z}\right)\Rightarrow V=\frac{kx^2}{2}+f(y,z)\Rightarrow V=\frac{kx^2}{2}\quad(\because f(y,z)=0).$$

So the potential energy stored in SHO at position x is $V=\frac{1}{2}kx^2$. The kinetic energy of SHO is given by

$$T=\frac{1}{2}mv^2=\frac{1}{2}m\dot{x}^2.$$

Therefore, Lagrangian of SHO is

$$L=T-V=\frac{1}{2}m\dot{x}^2-\frac{1}{2}kx^2,$$

where m is the mass, k is constant and x is the position which is generalized coordinate. Now, Lagrange's equation of motion for SHO with degrees of freedom 1 is given by

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0 \\
\Rightarrow\ &\frac{d}{dt}(m\dot{x})-(-kx)=0 \\
\Rightarrow\ &\boxed{m\ddot{x}+kx=0} \\
\Rightarrow\ &\ddot{x}+\omega^2x=0\Rightarrow x=A\cos(\omega t+B) \qquad \left(\omega^2=\frac{k}{m}\right).
\end{aligned}$$

***Solution* 1.25 (Page 46).** We have $L' = L + \frac{dF}{dt}$. Then differentiating above equation with respect to q_i and $\dot{q}_i$ respectively, we get

$$\frac{\partial L'}{\partial q_i} = \frac{\partial L}{\partial q_i} + \frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right) \tag{1.58}$$

and

$$\frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial}{\partial \dot{q}_i}\left(\frac{dF}{dt}\right). \tag{1.59}$$

Now, for the function $F(q_1, q_2, \ldots, q_n, t)$, we have

$$\frac{dF}{dt} = \sum_j \frac{\partial F}{\partial q_j}\dot{q}_j + \frac{\partial F}{\partial t}.$$

Therefore,

$$\begin{aligned}
\frac{\partial}{\partial \dot{q}_i}\left(\frac{dF}{dt}\right) &= \frac{\partial}{\partial \dot{q}_i}\left(\sum_j \frac{\partial F}{\partial q_j}\dot{q}_j + \frac{\partial F}{\partial t}\right)\\
&= \sum_j \frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial F}{\partial q_j}\dot{q}_j\right) + \frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial F}{\partial t}\right)\\
&= \sum_j \left(\frac{\partial F}{\partial q_j}\right)\frac{\partial \dot{q}_j}{\partial \dot{q}_i} + 0 = \frac{\partial F}{\partial q_i}.
\end{aligned}$$

Substituting this value in equation (1.59), we get

$$\frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial F}{\partial q_i}.$$

Differentiating above equation with respect to t, we have

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d}{dt}\left(\frac{\partial F}{\partial q_i}\right). \tag{1.60}$$

Now, from equations (1.58) and (1.60), using the fact that F is differentiable and Lagrangian L satisfies Lagrange's equations of motion, we conclude that

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) - \frac{\partial L'}{\partial q_i} = \left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}\right\} + \left\{\frac{d}{dt}\left(\frac{\partial F}{\partial q_i}\right) - \frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right)\right\} = 0.$$

Thus, L' also satisfies Lagrange's equations of motion.

CHAPTER 2

***Solution* 2.1 (Page 64).** Consider two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in space. Our problem is to determine the curve in space on which the distance between P and Q is minimum.

The distance between to neighboring points (x, y, z) and $(x + dx, y + dy, z + dz)$ on a curve in space is given by

$$ds^2 = dx^2 + dy^2 + dz^2 = dx^2\left(1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right)$$

$$\therefore ds = \sqrt{1+\dot{y}^2+\dot{z}^2}\,dx,$$

where $\dot{y} = \frac{dy}{dx}$ and $\dot{z} = \frac{dz}{dx}$.

Thus, the distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the integral

$$I = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1+\dot{y}^2+\dot{z}^2}\,dx. \tag{2.7}$$

The curve of shortest distance can be obtained by solving Euler-Lagrange equations for the above integral, i.e. we need to solve the following equations

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0 \tag{2.8}$$

$$\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{z}}\right) = 0, \tag{2.9}$$

where $f \equiv f(y, z, \dot{y}, \dot{z}, x) = (1+\dot{y}^2+\dot{z}^2)^{\frac{1}{2}}$. Now,

$$\frac{\partial f}{\partial y} = 0 \quad \text{and} \quad \frac{\partial f}{\partial z} = 0.$$

Also,

$$\frac{\partial f}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}}\left(1+\dot{y}^2+\dot{z}^2\right)^{\frac{1}{2}} = \frac{1}{2}(1+\dot{y}^2+\dot{z}^2)^{-\frac{1}{2}}(2\dot{y}) = (1+\dot{y}^2+\dot{z}^2)^{-\frac{1}{2}}\dot{y}.$$

Therefore,

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = \frac{d}{dx}\left(\frac{\dot{y}}{(1+\dot{y}^2+\dot{z}^2)^{\frac{1}{2}}}\right).$$

Using this values in equation (2.8), we get

$$-\frac{d}{dx}\left(\frac{\dot{y}}{(1+\dot{y}^2+\dot{z}^2)^{\frac{1}{2}}}\right) = 0 \Rightarrow \frac{\dot{y}}{(1+\dot{y}^2+\dot{z}^2)^{\frac{1}{2}}} = a_1 \qquad (a_1 \text{ is constant}).$$

which gives

$$({a_1}^2-1)\dot{y}^2 + {a_1}^2\dot{z}^2 = -{a_1}^2. \tag{2.10}$$

Similarly, from equation (2.9), we get $\frac{\dot{z}}{(1+\dot{y}^2+\dot{z}^2)^{\frac{1}{2}}} = b_1$ for some constant b_1 (say) and hence

$$({b_1}^2-1)\dot{z}^2 + {b_1}^2\dot{y}^2 = -{b_1}^2. \tag{2.11}$$

Solving equations (2.10) and (2.11) (by elimination method or Cramer's method), we get

$$\dot{y} = \left(\frac{{a_1}^2}{1-{a_1}^2-{b_1}^2}\right)^{\frac{1}{2}} = a_2 \text{ and } \dot{z} = \left(\frac{{b_1}^2}{1-{a_1}^2-{b_1}^2}\right)^{\frac{1}{2}} = b_2,$$

where a_2 and b_2 are constants. Hence, solving above equations, we have

$$\begin{cases} y = a_2 x + a_3 \\ z = b_2 x + b_3 \end{cases} \tag{2.12}$$

which are (individually) equations of plane. However, equation (2.12) together represents a straight line in space. Thus, the curve of shortest between two points in space is a straight line joining these two points.

***Solution* 2.2 (Page 64).** Curve

***Solution* 2.3 (Page 64).** Great circle.

***Solution* 2.5 (Page 64).**

1. **Simple pendulum:**

***Solution* 2.6 (Page 65).**

1. $L = \frac{m}{2}\left(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2\right) - \frac{k}{2}\left(ax^2 + 2bxy + cy^2\right).$

***Solution* 2.7 (Page 65).** Here, r and θ are generalized coordinates. The generalized momenta p_r and p_θ conjugate to r and θ respectively are given by

$$\boxed{p_r = \frac{\partial L}{\partial \dot{r}} = \dot{r}} \text{ and } \boxed{p_\theta = \frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta}}. \tag{2.30}$$

The energy function h is given by

$$\begin{aligned} h &= \dot{r}p_r + \dot{\theta}p_\theta - L \\ &= \dot{r}^2 + r^2\dot{\theta}^2 - L \\ &= \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{r} \end{aligned}$$

Since the Lagrangian does not depend on θ (i.e. since θ is cyclic coordinate), the generalized momenta p_θ is conserved.

Since Lagrangian does not depend on time explicitly, i.e. $\frac{\partial L}{\partial t} = 0$ and we know that $\frac{\partial L}{\partial t} = -\frac{dh}{dt}$, we have $\frac{dh}{dt} = 0$ and hence h is conserved.

***Solution* 2.8 (Page 65).** By Theorem 2.6.2.

CHAPTER 3

***Solution* 3.1 (Page 82).**

1. **Simple pendulum:**

***Solution* 3.2 (Page 82).**

1. $L = \frac{m}{2}\left(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2\right) - \frac{k}{2}\left(ax^2 + 2bxy + cy^2\right).$

***Solution* 3.11 (Page 83).** Seminar exercise.

***Solution* 4.1 (Page 112).** Seminar exercise (Find $F = d(pq + q\cot p)$).

***Solution* 4.3 (Page 112).** The canonical transformation is $Q_i = p_i$ and $P_i = -q_i$.

***Solution* 4.17 (Page 114).** Lagrange brackets are given in matrix form as

$$\{u,v\}_\eta = \left(\frac{\partial \eta}{\partial u}\right)' J \frac{\partial \eta}{\partial v}. \tag{4.32}$$

Therefore Fundamental Lagrange brackets can be written in matrix form as

$$\{\eta,\eta\}_\eta = J. \tag{4.33}$$

Consider a transformation $\eta \to \zeta$. Now,

$$\{\zeta,\zeta\}_\eta = \left(\frac{\partial \zeta}{\partial \eta}\right)' J \left(\frac{\partial \zeta}{\partial \eta}\right) = M'JM. \tag{4.34}$$

From equations (4.33) and (4.34), it is clear that the transformation is canonical if and only if fundamental Poisson brackets are invariant (under the transformation), i.e.

$$\{\zeta,\zeta\}_\eta = \{\eta,\eta\}_\eta \Leftrightarrow M'JM = J.$$

***Solution* 4.18 (Page 114).** Show that $[H,D] = \dfrac{\partial D}{\partial t}$ (`Left as exercise`).

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