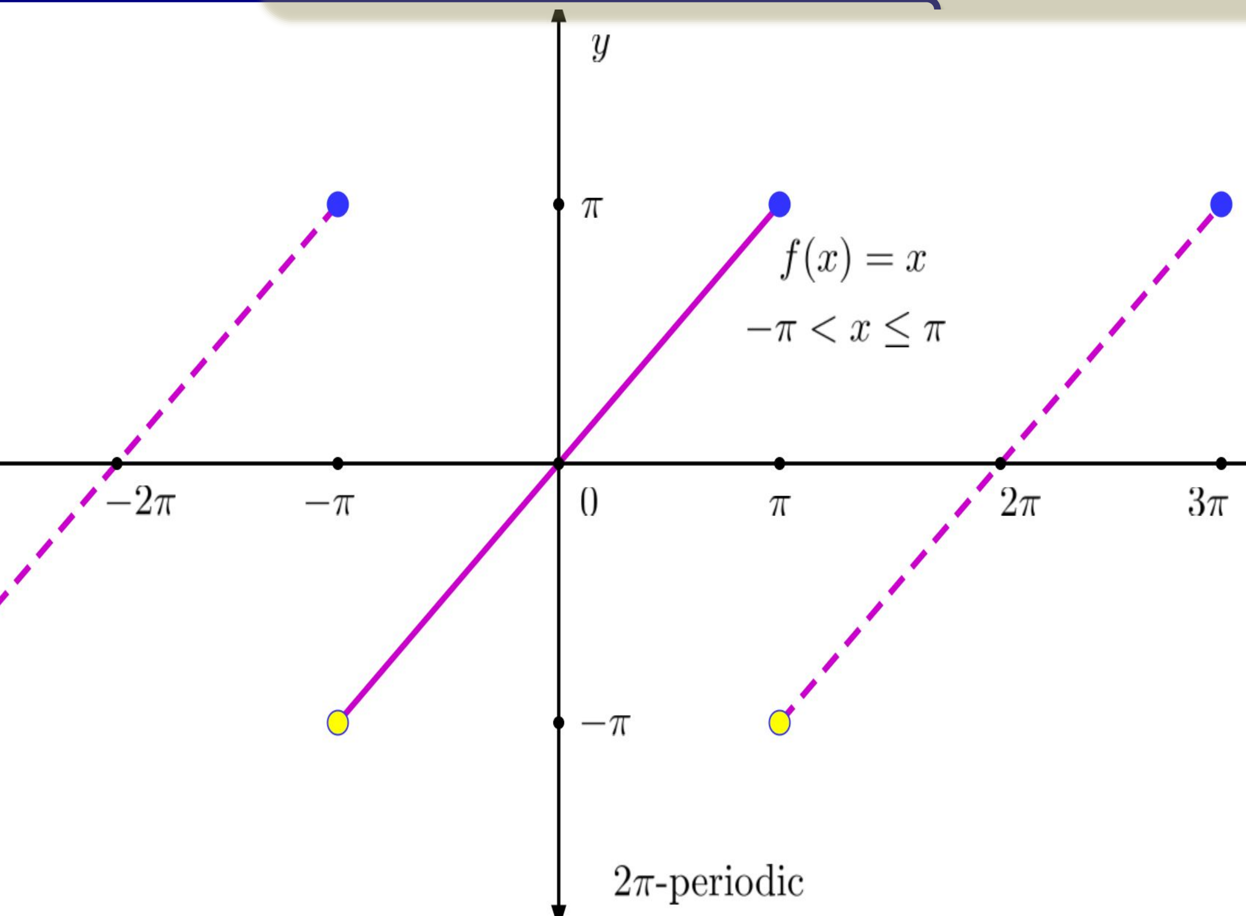
Lecture notes on

Mathematical Methods - I

PS03CMTH02



$$\frac{f(x_0^-)+f(x_0^+)}{2}$$

$$F[f]$$

P. A. Dabhi & J. G. Mehta
Department of Mathematics,
Sardar Patel University.

$$L[f]$$

Semester - III
2017-18

$$Z[(a_n)]$$

Preface and Acknowledgments

These are lecture notes of the course "Mathematical Methods - I" of Semester - III at Department of Mathematics, Sardar Patel University. They are outcome of the course lectures given the professors of the department. The notes are tailored as per the topics in the syllabus of the M.Sc. Semester - III course and do not exclusively cover all the mathematical methods. The problems incorporated in these notes serve only as an example of what different types of methods are covered during the course and by no means it is whole content of the course. Students are encouraged to solve the problems on their own and practice more problems as supplement to this material. The problems that (probably) could not be covered during the classes might be *grayed out*. Most of the problems listed as exercises in these notes were given as seminar exercises to the students during the semester.

Due to a large number of computational steps involved in the examples across the notes, it is very likely to have many typos and errors. For this reason also the students are advised to solve the problems bys their own. Should they find any corrections, feel free to point out to us.

P. A. Dabhi
Jay Mehta

Contents

SYLLABUS

PS03CMTH02: Mathematical Methods I

Unit I: Fourier series and applications to boundary value problems and summation of infinite series.

Unit II: Fourier integral representation and applications. Fourier transforms, computations of Fourier transforms of functions, properties of Fourier transforms, convolution and Fourier transform, applications to the boundary value problems involving Heat equation, Wave equation and Laplace equations..

Unit III: Laplace transform, Laplace transforms of some functions, properties of Laplace transform, inverse transform, convolution theorem, applications to solutions of ordinary differential equations, applications to the solutions of diffusion equation and wave equation.

Unit IV: Green's function and its applications, Gram-Schmidt orthonormalization method to Legendre polynomials, Hermite polynomials, Jacobi polynomials, Z-transform.

Reference Books

1. Shankar Rao, Introduction to Partial Differential Equations.
2. Courant and Hilbert; Mathematical Methods.
3. I.N. Sneddon; Special Functions of Mathematical Physics and Chemistry.
4. L.A. Pipes, Applied Mathematics for Engineers and Physicists.
5. B.S. Grewal, Higher Engineering Mathematics, Khanna Publishers, New Delhi, 2004.
6. M.D. Raisinghania, Advanced Differential Equations.

USEFUL TRIGONOMETRIC IDENTITIES

Reciprocal Identities

$$\sin\theta = \left(\frac{1}{\csc\theta}\right) \qquad \cos\theta = \left(\frac{1}{\sec\theta}\right) \qquad \tan\theta = \left(\frac{1}{\cot\theta}\right)$$

$$\csc\theta = \left(\frac{1}{\sin\theta}\right) \qquad \sec\theta = \left(\frac{1}{\cos\theta}\right) \qquad \cot\theta = \left(\frac{1}{\tan\theta}\right)$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta \qquad \csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$$

Odd-Even Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

Sum and Difference Identities

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Double Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\cos(2\theta) = 2\cos^2\theta - 1$$
$$\cos(2\theta) = 1 - 2\sin^2\theta$$

Power Reducing Formulas

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2} \qquad \tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

Half Angle Formulas

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos(\theta)}{\sin(\theta)} = \frac{\sin\alpha}{1+\cos\alpha}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$$

The signs of $\sin\left(\frac{\theta}{2}\right)$ and $\cos\left(\frac{\theta}{2}\right)$ depend on the quadrant in which $\frac{\theta}{2}$ lies.

Product to Sum Formulas

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \qquad \sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \qquad \cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

Sum to Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \qquad \cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \qquad \cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

CHAPTER 1

FOURIER SERIES

1.1 Motivation

Let $\{\varphi_n\}$ be a sequence of functions on $[a,b]$ satisfying

$$\int_a^b \varphi_n(x)\varphi_m(x)\,dx = \begin{cases} 0, & m \neq n \\ \alpha_n, & m = n, \end{cases} \tag{1.1}$$

where $\alpha_n \neq 0$. Let f be a *well behaved* function on $[a,b]$ such that

$$f = \sum_n a_n \varphi_n. \tag{1.2}$$

We want to find the coefficients a_n's in the equation (1.2). Fix m. We multiply equation (1.2) by φ_m both the sides and integrate on $[a,b]$. Since f is a well behaved function, we have

$$\begin{aligned} \int_a^b f(x)\varphi_m(x)\,dx &= \int_a^b \left(\sum_n a_n\varphi_n(x)\varphi_m(x)\right)dx \\ &= \sum_n a_n \int_a^b \varphi_m(x)\varphi_n(x)\,dx \\ &= a_m \int_a^b \varphi_m(x)\varphi_m(x) = a_m\alpha_m. \end{aligned}$$

Hence, we have

$$\boxed{a_n = \frac{1}{\alpha_n}\int_a^b f(x)\varphi_n(x)\,dx, \quad \forall n.} \tag{1.3}$$

1.1.1 Orthogonality Relations

First we obtain the following important orthogonality relations:

Example 1.1.1. Evaluate
$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx$$
for non-negative integers m and n.

Solution. **Case-I:** $m = n$.
$$\begin{aligned}
\int_{-\pi}^{\pi} \cos nx \cos mx \, dx &= \int_{-\pi}^{\pi} \cos nx \cos nx \, dx \\
&= 2\int_{0}^{\pi} \cos^2 nx \, dx \qquad (\because \cos \text{ is an even function}) \\
&= 2\int_{0}^{\pi} \frac{1+\cos 2nx}{2} \, dx \\
&= \left[x + \frac{\sin 2nx}{2n}\right]_0^{\pi} = \pi.
\end{aligned}$$

Case-II: $m \neq n$.
$$\begin{aligned}
\int_{-\pi}^{\pi} \cos nx \cos mx \, dx &= \int_{-\pi}^{\pi} \frac{\cos(n+m)x + \cos(n-m)x}{2} \, dx \\
&= \frac{1}{2}\left\{2\int_{0}^{\pi} \cos(n+m)x \, dx + 2\int_{0}^{\pi} \cos(n-m)x \, dx\right\} \quad (\because \cos \text{ is even}) \\
&= \left[\frac{\sin(n+m)x}{n+m}\right]_0^{\pi} + \left[\frac{\sin(n-m)x}{n-m}\right]_0^{\pi} = 0.
\end{aligned}$$

Hence, we have
$$\boxed{\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases} \quad (m, n \in \mathbb{N} \cup \{0\}),} \tag{1.4}$$

□

Similarly, we have
$$\boxed{\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases} \quad (m, n \in \mathbb{N})} \tag{1.5}$$

and
$$\boxed{\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0 \quad (m, n \in \mathbb{N} \cup \{0\}).} \tag{1.6}$$

Therefore the set $\{1, \cos x, \cos 2x, \ldots, \sin x, \sin 2x, \ldots\}$ is orthogonal over the interval $[-\pi, \pi]$.

Let $f : [-\pi, \pi] \to \mathbb{C}$ be a *well behaved* function. Suppose that
$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$$

i.e.,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Then by using the above method and the above orthogonality relations, we have

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\,dx &= \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\,dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx\,dx \\ &= \frac{a_0}{2}[x]_{-\pi}^{\pi} + \sum_{n=1}^{\infty} a_n \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} - \sum_{n=1}^{\infty} b_n \left[\frac{\cos nx}{n}\right]_{-\pi}^{\pi} \\ &= a_0\pi.\end{aligned}$$

Therefore,

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.} \tag{1.7}$$

Also, for $m \geq 1$,

$$f(x)\cos mx = \frac{a_0}{2}\cos mx + \sum_{n=1}^{\infty} a_n \cos nx \cos mx + \sum_{n=1}^{\infty} b_n \sin nx \cos mx.$$

Integrating above on $[-\pi, \pi]$, we get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos mx\,dx &= \int_{-\pi}^{\pi} \frac{a_0}{2}\cos mx\,dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx\,dx \\ &\quad + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx\,dx \\ &= a_m \int_{-\pi}^{\pi} \cos mx \cos mx = a_m\pi.\end{aligned}$$

Therefore,

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad (n \in \mathbb{N}\cup\{0\})}. \tag{1.8}$$

Similarly,

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad (n \in \mathbb{N}).} \tag{1.9}$$

Thus, if $\int_{-\pi}^{\pi} |f(x)|\,dx < \infty$, then a_n's and b_n's exist. So for $f \in L^1[-\pi, \pi]$, we can associate the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \tag{1.10}$$

called the *Fourier series* of f.

Remark 1.1.2. The important questions in the Fourier series are the following:

1. Does the Fourier series of f converge in any of the sense: pointwise convergent, uniformly convergent, convergence in L^p-sense?
2. If it converges, then will it converge to f (in any of the above sense)?

1.2 Fourier Series

1.2.1 Definitions and Examples

Let $L^1[a,b] := \left\{ f : [a,b] \to \mathbb{C} : \int_a^b |f(x)|\,dx < \infty \right\}$.

Definition 1.2.1 (Fourier series)**.** Let $f \in L^1[-\pi,\pi]$. Then the series

$$\frac{a_0}{2} + \sum_{n=1} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \tag{1.11}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad (n \in \mathbb{N}\cup\{0\}) \tag{1.12}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad (n \in \mathbb{N}) \tag{1.13}$$

is called the *Fourier series* of f, and the scalars a_n's and b_n's are called the *Fourier coefficients* of f.

Remark 1.2.2. Notice that if the given function f is an odd function, i.e. if $f(-x) = -f(x)$ for all x in the domain of f, then the function $f(x)\cos nx$ will be an odd function (as cos is even). Therefore, the coefficients a_n's are zero. On the other hand, if the function f is an even function, i.e. if $f(-x) = f(x)$ for all x in the domain of f, then the function $f(x)\sin nx$ will be an odd function (as sin is odd). In this case, the coefficients b_n's are zero.

Thus, making this observation while deriving Fourier series of a function f saves our effort in computing the Fourier coefficients that already vanishes.

Example 1.2.3. Find the Fourier series of $f : [-\pi,\pi] \to \mathbb{C}$ defined as $f(x) = x$.

Solution. Since $f(-x) = f(x)$, for all $x \in [-\pi,\pi]$, f is an odd function and hence the coefficients $a_n = 0$ for all $n \in \mathbb{N}\cup\{0\}$. Now, for $n \in \mathbb{N}$,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx \\ &= \frac{2}{\pi}\left[-x\frac{\cos nx}{n} - \int \frac{-\cos nx}{n}dx\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} \\ &= \frac{2}{\pi n}[-\pi\cos n\pi] = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}. \end{aligned}$$

Therefore the Fourier series of f is

$$2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}.$$

□

Example 1.2.4. Compute the Fourier series of $f : [-\pi, \pi] \to \mathbb{C}$ defined as

$$f(x) = \begin{cases} 0, & -\pi \leq x \leq 0 \\ x, & 0 \leq x \leq \pi. \end{cases}$$

Solution. Notice that the function is neither odd nor even and so we compute the coefficients a_0, a_n's and b_n's. We have,

$$a_0 = \frac{1}{\pi}\int_0^\pi x\,dx = \frac{\pi}{2}.$$

Let $n \in \mathbb{N}$. Then

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^\pi x\cos nx\,dx \\ &= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - \int \frac{\sin nx}{n}dx\right]_0^\pi \\ &= \frac{1}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^\pi \\ &= \frac{1}{\pi n^2}[(-1)^n - 1] \\ &= \begin{cases} 0, & \text{if } n \text{ is even} \\ -\dfrac{2}{\pi n^2}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Also, for $n \geq 1$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^\pi x\sin nx\,dx = \frac{1}{\pi}\int_0^\pi x\sin nx\,dx \qquad (\because f(x) = 0,\ -\pi \leq x \leq 0) \\ &= \frac{1}{\pi}\left[-x\frac{\cos nx}{n} - \int \frac{-\cos nx}{n}dx\right]_0^\pi \\ &= \frac{1}{\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^\pi \\ &= \frac{1}{\pi n}[-\pi\cos n\pi] = \frac{(-1)^{n+1}}{n}. \end{aligned}$$

Therefore, the Fourier series of f is

$$\begin{aligned} &\frac{\pi}{4} - \frac{2}{\pi}\cos x - \frac{2}{\pi\cdot 3^2}\cos 3x - \frac{2}{\pi\cdot 5^2}\cos 5x - \cdots + \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\sin nx \\ &= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^\infty \frac{\cos(2n-1)x}{(2n-1)^2} + \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\sin nx. \end{aligned}$$

□

Ex Show that

1. The Fourier series of the periodic function f defined by $f(x) = \begin{cases} -1, & -\pi < x \leq 0 \\ 1, & 0 < x \leq \pi \end{cases}$ is $\dfrac{4}{\pi}\displaystyle\sum_{n=1}^\infty \frac{\sin(2n-1)x}{2n-1}$.

2. The Fourier series of the periodic function $f(x)=|x|,\ -\pi<x\leq\pi$ is $\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$.

Next, we recall the notion of periodic functions.

Definition 1.2.5 (Periodic function)**.** A function $f:\mathbb{R}\to\mathbb{C}$ is said to be *periodic* if there exist $L>0$ such that

$$f(x+L)=f(x)\ (x\in\mathbb{R}).$$

If f is periodic, then the smallest positive L such that $f(x+L)=f(x)$ $(x\in\mathbb{R})$ is called the *fundamental period* of f.

Example 1.2.6. The sine and cosine functions are periodic with fundamental period 2π.

Example 1.2.7. We compute the period of $\cos\left(\frac{x}{3}\right)$. We have,

$$\cos\left(\frac{1}{3}(x+6\pi)\right)=\cos\left(\frac{x}{3}+2\pi\right)=\cos\left(\frac{x}{3}\right).$$

Therefore, the period of $\cos\left(\frac{x}{3}\right)$ is 6π.

Ex Show that

1. The period of $\cos\left(\frac{x}{3}\right)+\sin\left(\frac{x}{5}\right)$ is 30π.
2. The period of $\cos\left(\frac{x}{3\pi}\right)+\sin\left(\frac{x}{5\pi}\right)$ is $30\pi^2$.
3. The period of $\cos\left(\frac{x}{8}\right)+\sin\left(\frac{x}{12}\right)$ is 48π.

Remark 1.2.8. Let $f:(-\pi,\pi]\to\mathbb{C}$ be a map. Then f can be extended to a 2π-periodic map on $\mathbb{R}$ by the relation $f(x+2\pi)=f(x)$. The extension of f will be denoted by f itself.

1.2.2 Dirichlet Theorem

Definition 1.2.9. A function $f:[a,b]\to\mathbb{R}$ is said to satisfy the *Dirichlet conditions* on $[a,b]$. If

1. f is bounded on $[a,b]$;
2. f has a finite number of extremas in $[a,b]$;
3. f has finite number of discontinuities in $[a,b]$.

A function $f=f_1+if_2:[a,b]\to\mathbb{C}$ is said to satisfy the *Dirichlet conditions* on $[a,b]$ if both f_1 and f_2 satisfy the Dirichlet conditions.

The following theorem answers the convergence (pointwise) of Fourier series for a large class of functions.

Theorem 1.2.10 (Dirichlet Theorem)**.** *Let $f:\mathbb{R}\to\mathbb{C}$ be a 2π periodic function. If f satisfies the Dirichlet conditions on $[-\pi,\pi]$, then the Fourier series of f converges to $\frac{f(x^+)+f(x^-)}{2}$ for every $x\in\mathbb{R}$, where $f(x^+)$ is the right limit of f at x and $f(x^-)$ is the left limit of f at x.*

Writing the Dirichlet conditions within the statement, Theorem 1.2.10 can be restated as follows:

Theorem 1.2.10 (Dirichlet Theorem). *Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function. Assume that*

1. *f is bounded on $[-\pi, \pi]$*
2. *f has finite number of discontinuities in $[-\pi, \pi]$*
3. *f has finite number of local extrema in $[-\pi, \pi]$*

Then the Fourier series of f at $x \in \mathbb{R}$ convergers to $\frac{f(x^+)+f(x^-)}{2}$, where $f(x^+)$ is the right limit of f at x and $f(x^-)$ is the left limit of f at x.

Remark 1.2.11. If f is continuous at x, then the Fourier series of f at x converges to $f(x)$.

Example 1.2.12. Consider a 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x$, $x \in (-\pi, \pi]$. We note that the set of discontinuities of f is $\{(2n+1)\pi : n \in \mathbb{Z}\}$. Also notice that f satisfies Dirichlet conditions on the interval $[-\pi, \pi]$. Now, at $x = (2n+1)\pi$,

$$f(x^+) = f(-\pi) = -\pi \quad \text{and} \quad f(x^-) = f(\pi) = \pi.$$

Therefore, by Dirichlet theorem, the Fourier series of f (found in Example 1.2.3) converges to $f(x) = x$ for $x \neq (2n+1)\pi$ and converges to $\frac{f(x^+)+f(x^-)}{2} = \frac{-\pi+\pi}{2} = 0$ at $x = (2n+1)\pi$, i.e.

$$2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n} = \begin{cases} x, & x \neq (2n+1)\pi \\ 0, & \text{otherwise} \end{cases}.$$

Example 1.2.13. Compute the Fourier series of a periodic function $f(x) = |x|$, $-\pi \leq x \leq \pi$. Hence find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

Solution. We first note the function f is continuous and it satisfies the Dirichlet conditions on $[-\pi, \pi]$. Since f is an even function $b_n = 0$ for all $n \in \mathbb{N}$. Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\, dx = \frac{2}{\pi}\int_0^{\pi} x\, dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi.$$

Also for $n \in \mathbb{N}$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\, dx \\ &= \frac{2}{\pi}\left[x\frac{\sin nx}{n} - \int \frac{\sin nx}{n}\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi} \\ &= \frac{2}{n^2\pi}[(-1)^n - 1] \\ &= \begin{cases} 0, & \text{if } n \text{ is even} \\ -\frac{4}{\pi n^2}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Therefore, the Fourier series of f is

$$\frac{\pi}{2} - \frac{4}{\pi \cdot 1^2}\cos x - \frac{4}{\pi \cdot 3^2}\cos 3x - \frac{4}{\pi \cdot 5^2}\cos 5x - \cdots$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}.$$

Since f is continuous and satisfies Dirichlet conditions on any interval of length 2π, by Dirichlet theorem we have

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}, \quad x \in \mathbb{R}.$$

Substituting $x = 0$ in the above equation, we get

$$0 = f(0) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

□

Example 1.2.14. Compute the Fourier series of a 2π-periodic function $f(x) = x + x^2$, $-\pi < x \leq \pi$. Hence find the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$, and $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$.

Solution. Here the given function f is neither odd nor even and so we have to compute a_n's and b_n's. Here,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x^2\,dx && (\because x \text{ is odd and } x^2 \text{ is even}) \\ &= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}. \end{aligned}$$

For $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx && (\because x\cos nx \text{ is odd and } x^2\cos nx \text{ is even}) \\ &= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - \int 2x\frac{\sin nx}{n}\,dx\right]_0^{\pi} \\ &= \frac{-4}{\pi}\left[x\frac{-\cos nx}{n} + \int\frac{\cos nx}{n}\,dx\right]_0^{\pi} \\ &= \frac{-4}{\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\,dx\right]_0^{\pi} \end{aligned}$$

$$= \frac{4\pi}{n^2\pi}\cos n\pi = \frac{4}{n^2}(-1)^n.$$

Also,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx \qquad (\because x^2\sin nx \text{ is odd and } x\sin nx \text{ is even}) \\
&= \frac{2}{\pi}\left[-x\frac{\cos nx}{n} - \int \frac{-\cos nx}{n}dx\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} \\
&= \frac{2}{\pi n}[-\pi\cos n\pi] = -\frac{2}{n}(-1)^n = 2\frac{(-1)^{n+1}}{n}.
\end{aligned}$$

Therefore the Fourier series of f is

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx.$$

The given function f is continuous on $\mathbb{R}$ except at the points of the form $(2n+1)\pi$, $n\in\mathbb{Z}$, i.e. f is continuous on $\mathbb{R}\setminus\{(2n+1)\pi : n\in\mathbb{Z}\}$. We note that

$$f((2n+1)\pi^-) = f(\pi^-) = \pi+\pi^2 \quad\text{and}\quad f((2n+1)\pi^+) = f(\pi^+) = -\pi+\pi^2.$$

The function f satisfied Dirichlet conditions on any interval of length 2π. Therefore, by Dirichlet theorem, we have

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx = \begin{cases} f(x), & x\neq(2n+1)\pi \\ \pi^2, & \text{otherwise}\end{cases}. \tag{1.14}$$

Taking $x=\pi$ in the above equation, we get

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2} = \pi^2$$

or

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.}$$

Substituting $x=0$ in equation (1.14), we get

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2} = \pi^2$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.}$$

We have,

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \text{ i.e. } 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6}$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=-\frac{\pi^2}{12}\ \text{ i.e. } -1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\cdots=-\frac{\pi^2}{12}$$
$$\text{i.e. } 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\frac{\pi^2}{12}.$$

Adding both the above series, we get

$$2\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{6}+\frac{\pi^2}{12}=\frac{\pi^2}{4}$$
$$\Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}.}$$

□

Example 1.2.15. Compute the Fourier series of a 2π-periodic function $f(x)=x\sin x$, $-\pi\le x\le\pi$. Hence evaluate $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$ and $\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}$.

Solution. We note that the function f is an even function and so $b_n=0$ for all n. Now,

$$\begin{aligned}a_0&=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin x\,dx=\frac{2}{\pi}\int_0^{\pi}x\sin x\,dx\\&=\frac{2}{\pi}\left[x(-\cos x)+\int\cos x\,dx\right]_0^{\pi}\\&=\frac{2}{\pi}\left[-x\cos x+\sin x\right]_0^{\pi}=\frac{2}{\pi}[-\pi(-1)]=2.\end{aligned}$$

For $n\ge1$,

$$\begin{aligned}a_n&=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin x\cos nx\,dx\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}x[\sin(1+n)x+\sin(1-n)x]\,dx\\&=\frac{1}{\pi}\int_0^{\pi}x\sin(n+1)x\,dx+\frac{1}{\pi}\int_0^{\pi}x\sin(1-n)x\,dx\\&=\frac{1}{\pi}\left[x\frac{-\cos(n+1)x}{n+1}+\int\frac{\cos(n+1)x}{n+1}\,dx\right]_0^{\pi}+\frac{1}{\pi}\left[x\frac{-\cos(1-n)x}{1-n}+\int\frac{\cos(1-n)x}{1-n}\,dx\right]_0^{\pi}\\&=-\frac{1}{\pi}\left[\frac{\pi(-1)^{n+1}}{n+1}\right]-\frac{1}{\pi}\left[\frac{\pi(-1)^{1-n}}{1-n}\right]\\&=(-1)^n\left[\frac{-(-1)}{n+1}+\frac{(-1)}{n-1}\right]=(-1)^n\left[\frac{n-1-n-1}{n^2-1}\right]=\frac{-2(-1)^n}{n^2-1}\qquad(n\ne1).\end{aligned}$$

Note that the above is true only when $n\ne1$ and so we have for all $n\ge2$,

$$a_n=(-1)^n\left[\frac{n-1-n-1}{n^2-1}\right]=\frac{-2(-1)^n}{n^2-1}.$$

Now, for $n = 1$

$$\begin{aligned}
a_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos x\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(2\sin x\cos x)\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} x\sin 2x\,dx \\
&= \frac{1}{\pi}\left[x\frac{-\cos 2x}{2} + \int \frac{\cos 2x}{2}\,dx\right]_0^{\pi} = \frac{1}{\pi}\left[\pi\cdot\left(-\frac{1}{2}\right)\right] = -\frac{1}{2}.
\end{aligned}$$

Therefore the Fourier series of f is

$$1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx.$$

We notice that the function f is continuous on $\mathbb{R}$ and satisfies Dirichlet conditions on $[-\pi,\pi]$ (on any interval of length 2π). Therefore by Dirichlet theorem we have

$$f(x) = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx \quad (x\in\mathbb{R}).$$

Taking $x = \pi$, we get

$$0 = f(\pi) = 1 + \frac{1}{2} - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}(-1)^n \Rightarrow \sum_{n=2}^{\infty}\frac{1}{n^2-1} = \frac{3}{4}. \tag{1.15}$$

Taking $x = 0$, we get

$$0 = f(0) = 1 - \frac{1}{2} - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1} \Rightarrow \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1} = \frac{1}{4}. \tag{1.16}$$

□

Example 1.2.16. In the above example, evaluate the sum $\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2-1}$ and $\sum_{n=1}^{\infty}\frac{1}{(2n)^2-1}$.

Solution. Rewriting equations (1.15) and (1.16) from the above example, we have

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1} = \frac{3}{4} \quad \text{i.e.,}\ \frac{1}{2^2-1} + \frac{1}{3^2-1} + \frac{1}{4^2-1} + \cdots = \frac{3}{4}$$

and

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1} = \frac{1}{4} \quad \text{i.e.,}\ \frac{1}{2^2-1} - \frac{1}{3^2-1} + \frac{1}{4^2-1} - \cdots = \frac{1}{4}.$$

Therefore subtracting equation (1.16) from (1.15), we get

$$2\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2-1} = \frac{3}{4} - \frac{1}{4} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n+1)^2-1} = \frac{1}{4}.$$

Also, adding equations (1.15) and (1.16), we get

$$2\sum_{n=1}^{\infty}\frac{1}{(2n)^2-1} = \frac{3}{4} + \frac{1}{4} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n)^2-1} = \frac{1}{2}.$$

□

Example 1.2.17. Compute the Fourier series of a periodic function

$$f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi. \end{cases}$$

Hence find the sum of the series $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots$.

Solution. Here, we note that the given function f is neither odd nor even. Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}[-\cos x]_0^{\pi} = -\frac{1}{\pi}[-1-1] = \frac{2}{\pi}.$$

Now,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx\,dx \\
&= \frac{1}{2\pi}\int_0^{\pi} [\sin(n+1)x + \sin(1-n)x]\,dx \\
&= \frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{cos(n-1)x}{n-1}\right]_0^{\pi} \\
&= \frac{1}{2\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \\
&= \frac{1}{2\pi}\left[\frac{(-1)^{n}}{n+1} - \frac{(-1)^{n}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \\
&= \frac{1}{2\pi}\left[\frac{(-1)^{n}+1}{n+1} - \frac{(-1)^{n}+1}{n-1}\right] = \frac{(-1)^n+1}{2\pi}\left[\frac{n-1-n-1}{n^2-1}\right] \\
&= \frac{(-1)^n+1}{2\pi}\cdot\frac{(-2)}{n^2-1} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{-2}{\pi(n^2-1)}, & \text{if } n \text{ is even} \quad (n \ge 2) \end{cases}.
\end{aligned}$$

For $n = 1$,

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cos x\,dx = \frac{1}{2\pi}\int_0^{\pi} \sin 2x\,dx = -\frac{1}{2\pi}\left[\frac{\cos 2x}{2}\right]_0^{\pi} = -\frac{1}{4\pi}[1-1] = 0.$$

Also, for $n \ge 2$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx\,dx \\
&= -\frac{1}{2\pi}\int_0^{\pi} [\cos(n+1)x - \cos(1-n)x]\,dx \\
&= -\frac{1}{2\pi}\left[\frac{\sin(n+1)x}{n+1} - \frac{\sin(1-n)x}{1-n}\right]_0^{\pi} = 0.
\end{aligned}$$

For $n = 1$,

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2 x\,dx = \frac{1}{\pi}\int_0^{\pi}\left[\frac{1-\cos 2x}{2}\right]dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}.$$

Therefore, the Fourier series of f is

$$\begin{aligned} &\frac{1}{\pi}-\frac{2}{\pi(2^2-1)}\cos 2x-\frac{2}{\pi(4^2-1)}\cos 4x-\frac{2}{\pi(6^2-1)}\cos 6x-\cdots+\frac{1}{2}\sin x \\ =&\frac{1}{\pi}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n)^2-1}\cos(2n)x+\frac{1}{2}\sin x. \end{aligned}$$

Notice that the function f is continuous and satisfies Dirichlet conditions on any interval of length 2π. Therefore by Dirichlet theorem, we have

$$f(x)=\frac{1}{\pi}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n)^2-1}\cos(2n)x+\frac{1}{2}\sin x.$$

Taking $x=0$, we get

$$\begin{aligned} \frac{1}{\pi}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n)^2-1}&=f(0)=0 \\ \Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n)^2-1}&=\frac{1}{\pi}\cdot\frac{\pi}{2}=\frac{1}{2} \\ \text{i.e.,}\quad \frac{1}{2^2-1}+\frac{1}{4^2-1}+\frac{1}{6^2-1}+\cdots&=\frac{1}{2} \\ \text{i.e.,}\quad \frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\cdots&=\frac{1}{2} \\ \text{i.e.,}\quad \frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots&=\frac{1}{2}. \end{aligned}$$

□

Ex

1. Find the Fourier series of the following periodic functions.

   (a) $f(x)=\begin{cases}1, & -\pi<x\le 0\\ 0, & 0<x\le\pi.\end{cases}$

   (b) $f(x)=\begin{cases}-\pi/4, & -\pi<x<0\\ 0, & x=0\\ \pi/4, & 0<x\le\pi.\end{cases}$

   (c) $f(x)=\begin{cases}0, & -\pi<x\le \pi/2\\ 1, & \pi/2<x\le\pi.\end{cases}$

   (d) $f(x)=\begin{cases}-1, & -\pi<x\le \pi/2\\ 1, & \pi/2<x\le\pi.\end{cases}$

   (e) $f(x)=\begin{cases}0, & -\pi<x\le 0\\ -1, & 0\le x\le \pi/2\\ 1, & \pi/2<x\le\pi.\end{cases}$

   (f) $f(x)=\begin{cases}0, & -\pi<x\le 0\\ x, & 0<x\le\pi.\end{cases}$

   (g) $f(x)=1+x, -\pi<x\le\pi.$

   (h) $f(x)=\begin{cases}x+\pi, & -\pi<x\le 0\\ 0, & 0<x\le\pi.\end{cases}$

   (i) $f(x)=\begin{cases}x+\pi, & -\pi<x\le 0\\ -x, & 0<x\le\pi.\end{cases}$

   (j) $f(x)=x-x^2,\ -\pi<x\le\pi.$

   (k) $f(x)=x^2,\ -\pi<x\le\pi.$

   (l) $f(x)=|\sin x|,\ -\pi<x\le\pi.$

2. Find the Fourier series of the periodic function $f(x)=\begin{cases}0, & -\pi<x\le 0\\ x, & 0<x\le\pi\end{cases}$. Hence find the sum of the series $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$.

3. Find the Fourier series of the periodic function $f(x) = \begin{cases} 0, & -\pi < x \leq 0 \\ \sin x, & 0 < x \leq \pi \end{cases}$.
   Hence find the sum of the series $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots$.
4. Find the Fourier series of $f(x) = e^x$, $-\pi < x \leq \pi$ and hence derive a series for $\frac{\pi}{\sinh \pi}$.
5. Obtain the Fourier series of $f(x) = e^{-x}, -\pi < x \leq \pi$.
6. Find the Fourier series of $f(x) = \begin{cases} 0, & -\pi < x \leq 0 \\ \frac{1}{4}\pi x, & 0 < x \leq \pi. \end{cases}$ Hence find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.
   (a) Find the period of the function $f(x) = \cos(x/3) + \cos(x/4)$.
   (b) Show that, if the function $f(x) = \cos(\omega_1 x) + \cos(\omega_2 x)$ is periodic with a period T, then the ratio ω_1/ω_2 must be a rational number.
7. Show that if $f(x+P) = f(x)$, then

$$\int_{a-P/2}^{a+P/2} f(x)dx = \int_{-P/2}^{P/2} f(x)dx, \quad \int_{P}^{P+a} f(x)dx = \int_{0}^{a} f(x)dx.$$

1.2.3 Half range Fourier series

Let $f \in L^1(0,\pi)$. Extend f as an even function on $(-\pi,\pi)$ (giving the value 0 at $x=0$). Denote the extension by f itself. Then the extension is an even function. So, $b_n = 0$ for all $n \geq 1$ and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx.$$

Definition 1.2.18. Let $f \in L^1(0,\pi)$. Then the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

is called the *half range Fourier cosine series* of f, where

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx \qquad (n \in \mathbb{N}\cup\{0\}).$$

Definition 1.2.19. Let $f \in L^1(0,\pi)$. Then the series

$$\sum_{n=1}^{\infty} b_n \sin nx$$

is called the *half range Fourier sine series* of f, where

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx \qquad (n \in \mathbb{N}).$$

When $f:(0,\pi)\to\mathbb{C}$ satisfies the Dirichlet condition on $(0,\pi)$, it also satisfies the Dirichlet condition on $(-\pi,\pi)$. Hence the half range Fourier cosine series of f converges to $\frac{f(x^+)+f(x^-)}{2}$ for every $x \in \mathbb{R}$ (we have extended the map as a 2π periodic map).

Example 1.2.20. Compute the half range Fourier cosine series of $f(x) = x$, $0 < x < \pi$.

Solution. Here,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi.$$

For $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx \\ &= \frac{2}{\pi}\left[x\frac{\sin nx}{n} - \int \frac{\sin nx}{n}dx\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} \\ &= \frac{2}{\pi n^2}[(-1)^n - 1] = \begin{cases} -\frac{4}{\pi n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Therefore the Fourier cosine series of f is $\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$. □

Example 1.2.21. Compute the half range sine series of $f(x) = x$, $0 < x < \pi$.

Solution. For $n \geq 1$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx \\ &= \frac{2}{\pi}\left[-x\frac{\cos nx}{n} - \int \frac{-\cos nx}{n}dx\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} \\ &= \frac{2}{\pi n}[-\pi\cos n\pi] = \frac{2(-1)^{n+1}}{n}. \end{aligned}$$

Therefore the half range Fourier sine series of f is $2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$. □

Example 1.2.22. Find the half range Fourier cosine series of the function

$$f(x) = \begin{cases} 1, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi. \end{cases}$$

Solution. Here

$$a_0 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} 1\,dx = 1.$$

For $n \geq 1$,

$$a_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos nx\,dx = \frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\frac{\pi}{2}}.$$

If $n = 1, 5, 9, 13, \ldots$, i.e. $n = 4k+1$ then $\sin\frac{n\pi}{2} = 1$.
If $n = 3, 7, 11, 15, \ldots$, i.e. $n = 4k+3$ then $\sin\frac{n\pi}{2} = -1$.
Thus for $n \geq 1$,

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{2}{n\pi}, & \text{if } n \text{ is odd and of the form } 4k+1 \\ -\frac{2}{n\pi}, & \text{if } n \text{ is odd and of the form } 4k+3 \end{cases}$$

Therefore the half range Fourier cosine series of f is

$$\begin{aligned} &\frac{1}{2} + \frac{2}{\pi}\cos x - \frac{2}{3\pi}\cos 3x + \frac{2}{5\pi}\cos 5x - \frac{2}{7\pi}\cos 7x + \cdots \\ = &\frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\cos(2n+1)x}{2n+1}. \\ \text{or } &\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(2n-1)x}{2n-1}. \end{aligned}$$

□

Example 1.2.23. Using the half range Fourier cosine series of $f(x) = x$, $0 < x < \pi$ compute the sum of the series $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$.

Solution. By Example 1.2.20 the half range Fourier cosine series of f is $\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$ or we can write

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}.$$

Here f satisfies Dirichlet conditions on the interval $(0, \pi)$. Therefore it will satisfy Dirichlet condition on the interval $(-\pi, \pi)$ when it is extended as an even function giving the 0 at $x = 0$. Notice that the function f is note defined at the end points $-\pi$ or π. So to extend it as a 2π-periodic function on $\mathbb{R}$, we assign the value π at $x = \pi$. Then f is continuous function on $\mathbb{R}$ as

$$f((2n+1)\pi^-) = \pi = f(\pi) = f((2n+1)\pi^+).$$

Then by Dirichlet theorem, the Fourier series of f at $x \in \mathbb{R}$ converges to $f(x)$, i.e.

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2} = f(x).$$

By taking $x = 0$, we get $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$. □

Ex Find the half range Fourier sine and half range Fourier cosine series of the following functions.

1. $f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 \leq x < \pi. \end{cases}$
2. $f(x) = \begin{cases} 0, & 0 < x < \pi/2 \\ \pi/2, & \pi/2 \leq x < \pi \end{cases}$ Hence find the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$.

3. $f(x) = e^x$, $0 < x < \pi$
4. $f(x) = x^2 + x$, $0 < x < \pi$
5. $f(x) = x(\pi - x)$, $0 < x < \pi$

1.2.4 Complex form of the Fourier series

In this subsection, we wish to express f as $f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}$.

Example 1.2.24. Evaluate the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\,dx.$, $n \in \mathbb{Z}$.

Solution. For $n = 0$, the integral becomes $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} 1\,dx = 1$. For $n \neq 0$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\,dx = \frac{1}{2\pi}\left[\frac{e^{inx}}{in}\right]_{-\pi}^{\pi} = \frac{1}{2in\pi}\left[e^{in\pi} - e^{-in\pi}\right] = \frac{1}{2in\pi}[(-1)^n - (-1)^n] = 0.$$

Thus,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\,dx = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

□

Also, for $m, n \in \mathbb{Z}$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}e^{-imx}\,dx = \begin{cases} 0, & m \neq n \\ 1, & m = n. \end{cases}$$

This orthogonality relation inspires the following definition of the *complex form* of the Fourier series.

Definition 1.2.25 (Complex Fourier series)**.** Let $f \in L^1[-\pi,\pi]$. Then the series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \quad (n \in \mathbb{Z}),$$

is called the (complex) Fourier series of f.

Example 1.2.26. Compute the complex Fourier series of a periodic function $f(x) = \begin{cases} x, & -\pi < x \leq 0 \\ \pi - x, & 0 < x \leq \pi. \end{cases}$

Solution. We have $c_n = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$. Then,

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left\{\int_{-\pi}^{0} x\,dx + \int_{0}^{\pi}(\pi - x)\,dx\right\}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\left\{\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \left[\pi x - \frac{x^2}{2}\right]_{0}^{\pi}\right\} \\
&= \frac{1}{2\pi}\left\{-\frac{\pi^2}{2} + \pi^2 - \frac{\pi^2}{2}\right\} = 0.
\end{aligned}
$$

For $n \neq 0$, $n \in \mathbb{Z}$,

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \\
&= \frac{1}{2\pi}\left\{\int_{-\pi}^{0} xe^{-inx}\,dx + \int_{0}^{\pi}(\pi - x)e^{-inx}\,dx\right\} \\
&= \frac{1}{2\pi}\left\{\left[x\frac{e^{-inx}}{-in} - \int \frac{e^{-inx}}{-in}\right]_{-\pi}^{0} + \left[(\pi - x)\frac{e^{-inx}}{-in} - \int (-1)\frac{e^{-inx}}{-in}\right]_{0}^{\pi}\right\} \\
&= \frac{1}{2\pi}\left\{\left[x\frac{e^{-inx}}{-in} + \frac{e^{-inx}}{n^2}\right]_{-\pi}^{0} + \left[(\pi - x)\frac{e^{-inx}}{-in} - \frac{e^{-inx}}{n^2}\right]_{0}^{\pi}\right\} \\
&= \frac{1}{2\pi}\left\{\left[0 + \frac{1}{n^2} - (-\pi)\frac{(-1)^n}{-in} - \frac{(-1)^n}{n^2}\right] + \left[0 - \frac{(-1)^n}{n^2} - \frac{\pi}{-in} + \frac{1}{n^2}\right]\right\} \\
&= \frac{1}{2\pi}\left\{\frac{2}{n^2} - \frac{\pi(-1)^n}{in} + \frac{\pi}{in} - \frac{2(-1)^n}{n^2}\right\} \\
&= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{1}{\pi}\left(\frac{\pi}{in} + \frac{2}{n^2}\right), & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}
$$

Hence, the complex Fourier series of f is

$$
\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\left(\frac{\pi}{i(2n-1)} + \frac{2}{(2n-1)^2}\right)e^{i(2n-1)x}.
$$

□

Example 1.2.27. Compute the complex Fourier series of a 2π-periodic function $\cos 2x + \sin 2x$.

Solution. Here,

$$
\begin{aligned}
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}(\cos 2x + \sin 2x)\,dx \\
&= \frac{1}{2\pi}\left[\frac{\sin 2x}{2} - \frac{\cos 2x}{2}\right]_{-\pi}^{\pi} = \frac{1}{4\pi}[-1+1] = 0.
\end{aligned}
$$

Hence,

$$
\cos 2x + \sin 2x = \frac{e^{2ix} + e^{-2ix}}{2} + \frac{e^{2ix} - e^{-2ix}}{2i} = \underbrace{\left(\frac{1}{2} - \frac{1}{2i}\right)}_{c_{-2}}e^{-2ix} + \underbrace{\left(\frac{1}{2} + \frac{1}{2i}\right)}_{c_2}e^{2ix}
$$

which is the complex Fourier series of f, i.e. the complex Fourier series of f can be written as

$$c_{-2}e^{-i2x} + c_2 e^{i2x},$$

where $c_{-2} = \left(\frac{1}{2} - \frac{1}{2i}\right)$ and $c_2 = \left(\frac{1}{2} + \frac{1}{2i}\right)$. □

Next, we shall derive the relation between the complex Fourier coefficients and the Fourier coefficients by comparing the Fourier series and the complex Fourier series.

Lemma 1.2.28. *The relation between the Fourier coefficients and the complex Fourier coefficients is given by*

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}).$$

Proof. We have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} c_n e^{inx} &= \sum_{n=-\infty}^{\infty} c_n(\cos nx + i \sin nx) \\
&= \sum_{n=-\infty}^{\infty} \cos nx + i \sum_{n=-\infty}^{\infty} \sin nx \\
&= \sum_{n=-\infty}^{-1} \cos nx + \sum_{n=0}^{\infty} \cos nx + i \sum_{n=-\infty}^{-1} \sin nx + i \sum_{n=1}^{\infty} \sin nx \quad (\because \sin 0 = 0) \\
&= c_0 + \sum_{n=1}^{\infty} (c_n + c_{-n}) \cos nx + i \sum_{n=1}^{\infty} (c_n - c_{-n}) \sin nx. \qquad (\because \sin(-nx) = -\sin nx)
\end{aligned}$$

Comparing the (complex) Fourier series with the Fourier series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$$

we get

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}).$$

□

Definition 1.2.29. We say that a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ satisfies Dirichlet conditions if both real part of f and imaginary part of f do so.

Theorem 1.2.30. *Let $f : (-\pi, \pi] \to \mathbb{C}$. Extend f as a 2π periodic function on $\mathbb{R}$. If f satisfies the Dirichlet conditions, then the (complex) Fourier series of f converges to $\frac{f(x^+)+f(x^-)}{2}$ for every $x \in \mathbb{R}$.*

Thus, the Dirichlet theorem also holds in case of complex Fourier series.

Ex Find the (complex) Fourier series of

1. $f(x) = e^{-\pi x}$, $-\pi < x \leq \pi$;
2. $f(x) = \cos ax$, $-\pi < x \leq \pi$;
3. $f(x) = \sin^4 x + \cos^3 x - \sin 2x$, $-\pi < x \leq \pi$.

1.2.5 Parseval's Identity

Theorem 1.2.31. *Let H be a Hilbert space and let $\{e_\alpha\} \subseteq H$. Then the following are equivalent:*

1. $\{e_\alpha\}$ *is an orthonormal basis*
2. $x = \sum_\alpha \langle x, e_\alpha \rangle e_\alpha \ (x \in H)$
3. $\|x\|^2 = \sum_\alpha |\langle x, e_\alpha \rangle|^2 \ (x \in H)$
4. $\langle x, e_\alpha \rangle = 0$ *for all* α*, then* $x = 0$

Let $L^2[-\pi,\pi] = \left\{ f : [-\pi,\pi] \to \mathbb{C} : \int_{-\pi}^{\pi} |f|^2 < \infty \right\}$. Then $L^2[-\pi,\pi]$ is a Hilbert space with inner product defined as

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx \qquad (f, g \in L^2[-\pi,\pi]).$$

The set $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}} : n \in \mathbb{N} \right\}$ is an orthonormal basis of $L^2[-\pi,\pi]$.

Theorem 1.2.32 (Parseval's identity)**.** *Let $f \in L^2[-\pi,\pi]$. Then*

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2),$$

where $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$ *and* $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$.

Proof. We know that the set $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}} : n \in \mathbb{N} \right\}$ is an orthonormal basis of the Hilbert space $L^2[-\pi,\pi]$. Therefore

$$\begin{aligned}
\|f\|^2 &= \left| \left\langle f, \frac{1}{\sqrt{2\pi}} \right\rangle \right|^2 + \sum_{n=1}^{\infty} \left| \left\langle f, \frac{\cos nx}{\sqrt{\pi}} \right\rangle \right|^2 + \sum_{n=1}^{\infty} \left| \left\langle f, \frac{\sin nx}{\sqrt{\pi}} \right\rangle \right|^2 \\
&= \left| \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\, dx \right|^2 + \sum_{n=1}^{\infty} \left| \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \right|^2 + \sum_{n=1}^{\infty} \left| \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \right|^2 \\
&= \frac{|a_0|^2 \pi}{2} + \sum_{n=1}^{\infty} \pi |a_n|^2 + \sum_{n=1}^{\infty} \pi |b_n|^2,
\end{aligned}$$

where $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$ and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$. Therefore

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2).$$

□

Theorem 1.2.33 (Parseval's identity in complex form). *Let $f \in L^2[-\pi,\pi]$. Then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \sum_{n=-\infty}^{\infty} |c_n|^2,$$

where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$.

Proof. We know that the set $\left\{\frac{e^{inx}}{\sqrt{2\pi}} : n \in \mathbb{Z}\right\}$ is an orthonormal basis of the Hilbert space $L^2[-\pi,\pi]$. If $f \in L^2[-\pi,\pi]$, then

$$\begin{aligned}\|f\|^2 &= \sum_{n=-\infty}^{\infty} \left|\left\langle f, \frac{e^{inx}}{\sqrt{2\pi}}\right\rangle\right|^2 \\ &= \sum_{n=-\infty}^{\infty} \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx\right|^2 \\ &= 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2,\end{aligned}$$

where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$. Therefore,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

□

Example 1.2.34. Compute the Fourier series of a 2π-periodic function $f(x) = x^2$. Hence, evaluate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Solution. Since, the function f is even, $b_n = 0$ for all $n \geq 1$. Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2\pi^2}{3}.$$

For $n \geq 1$,

$$\begin{aligned}a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx \\ &= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - 2\int x\frac{\sin nx}{n}\,dx\right]_0^{\pi} \\ &= \frac{2}{n\pi}\left[x^2\sin nx - 2x\frac{-\cos nx}{n} + 2\int \frac{-\cos nx}{n}\,dx\right]_0^{\pi} \\ &= \frac{4}{n\pi}\left[\pi\frac{(-1)^n}{n}\right] = \frac{4(-1)^n}{n^2}.\end{aligned}$$

Therefore the Fourier series of f is $\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$. By Parseval's identity, we have

Now,

$$\int_{-\pi}^{\pi}|f(x)|^2dx=\int_{-\pi}^{\pi}x^4dx=\frac{2\pi^5}{5}.$$

Therefore, from Parseval's formula, it follows that

$$\begin{aligned}\frac{1}{\pi}\frac{2\pi^5}{5} &= \frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2dx\\ &= \frac{|a_0|^2}{2}+\sum_{n=1}^{\infty}(|a_n|^2+|b_n|^2)\\ &= \frac{2\pi^4}{9}+\sum_{n=1}^{\infty}\frac{16}{n^4}.\end{aligned}$$

Hence, $\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$. □

Example 1.2.35. Compute the Fourier series of a 2π-periodic function $f(x)=|x|$, $-\pi\le x\le\pi$. Use the Parseval's formula to find the sum of the series $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$.

Solution. Since, the function f is even, $b_n=0$ for all $n\ge 1$. Now,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|\,dx=\frac{2}{\pi}\int_0^{\pi}x\,dx=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}=\pi.$$

For $n\ge 1$,

$$\begin{aligned}a_n&=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx\\ &=\frac{2}{\pi}\left[x\frac{\sin nx}{n}-\int\frac{\sin nx}{n}\right]_0^{\pi}\\ &=\frac{2}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi}\\ &=\frac{2}{n^2\pi}[(-1)^n-1]=\begin{cases}0, & \text{if } n \text{ is even}\\ -\frac{4}{\pi n^2}, & \text{if } n \text{ is odd.}\end{cases}\end{aligned}$$

Therefore, the Fourier series of f is $\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$.

By Parseval's identity, we have

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx &= \frac{|a_0|^2}{2}+\sum_{n=1}^{\infty}(|a_n|^2+|b_n|^2)\\ \Rightarrow\quad \frac{1}{\pi}\int_{-\pi}^{\pi}|x|^2\,dx &= \frac{\pi^2}{2}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\\ \Rightarrow\quad \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} &= \frac{\pi^2}{2}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\end{aligned}$$

$$\Rightarrow \quad \frac{2}{\pi}\cdot\frac{\pi^3}{3}-\frac{\pi^2}{2} = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$$

Therefore
$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^2}{16}\left(\frac{4\pi^2-3\pi^2}{6}\right) = \frac{\pi^4}{96}.$$

□

Example 1.2.36. Compute the Fourier series of a 2π-periodic function
$$f(x) = \begin{cases} 0, & -\pi < x \leq 0 \\ 1, & 0 < x \leq \pi \end{cases}$$

Evaluate the sum of the series $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$ using Parseval's identity.

Solution. Here
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} 1\,dx = 1.$$

For $n \geq 1$,
$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} 1\cdot\cos nx\,dx \\ &= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = 0. \end{aligned}$$

Also, for $n \geq 1$
$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} 1\cdot\sin nx\,dx \\ &= \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi} \\ &= -\frac{1}{n\pi}[(-1)^n-1] = \begin{cases} 0, & n \text{ is even} \\ \frac{2}{n\pi}, & n \text{ is odd.} \end{cases} \end{aligned}$$

Therefore Fourier series of f is $\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)x.$

By using Parseval's identity, we have
$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx &= \frac{|a_0|^2}{2}+\sum_{n=1}^{\infty}(|a_n|^2+|b_n|^2) \\ \Rightarrow \quad \frac{1}{\pi}\int_{-\pi}^{\pi}1^2\,dx &= \frac{1}{2}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \\ \Rightarrow \quad 1-\frac{1}{2} &= \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}. \end{aligned}$$

Therefore $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$ □

Example 1.2.37. Compute the Fourier series of a 2π-periodic function $f(x) = x(\pi - x)$, $-\pi < x \leq \pi$. Hence evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Solution. Here,

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x(\pi - x)\,dx \\
&= \frac{1}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3} - \frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = -\frac{2\pi^3}{3}\cdot\frac{1}{\pi} = -\frac{2\pi^2}{3}.
\end{aligned}$$

For $n \geq 1$,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x(\pi - x)\cos nx\,dx \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{\pi} \pi x\cos nx\,dx - \int_{-\pi}^{\pi} x^2\cos nx\,dx\right\} \\
&= 0 - \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx \\
&= -\frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - \int (2x)\frac{\sin nx}{n}\,dx\right]_0^{\pi} \\
&= \frac{4}{n\pi}\left[x\frac{-\cos nx}{n} + \int \frac{\cos nx}{n}\,dx\right]_0^{\pi} \\
&= -\frac{4}{n^2\pi}[\pi\{(-1)^n - 0\}] = \frac{4}{n^2}(-1)^{n+1}.
\end{aligned}$$

Now,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} \pi x\sin nx\,dx - \int_{-\pi}^{\pi} x^2\sin nx\,dx\right] \\
&= 2\int_{0}^{\pi} x\sin nx\,dx \\
&= 2\left[x\frac{-\cos nx}{n} + \int \frac{\cos nx}{n}\,dx\right]_0^{\pi} \\
&= -\frac{2}{n}[\pi(-1)^n] = \frac{2\pi}{n}(-1)^{n+1}.
\end{aligned}$$

Therefore the Fourier series of f is

$$-\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx + 2\pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx.$$

We note that f is continuous on $\mathbb{R}$ except the points of the form $(2n+1)\pi$. The average of the left limit and the right limit of f at these points is $\frac{f(\pi^+)+f(\pi^-)}{2} = \frac{0-2\pi^2}{2} = -\pi^2$.

Also, the function f satisfies Dirichlet conditions on any interval of length 2π. Therefore by Dirichlet theorem, we have

$$-\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\cos nx + 2\pi \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx = \begin{cases} f(x), & x \neq (2n+1)\pi \\ -\pi^2, & \text{otherwise.} \end{cases}$$

Taking $x = \pi$ in the above equation, we get

$$-\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{1}{n^2} = -\pi^2 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Now by Parseval's identity, we get

$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx &= \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2) \\ \Rightarrow \quad \frac{1}{\pi}\int_{-\pi}^{\pi} x^2(\pi - x)^2\, dx &= \frac{2\pi^2}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4} + 4\pi^2 \sum_{n=1}^{\infty}\frac{1}{n^2} \\ \Rightarrow \quad \frac{2}{\pi}\int_{0}^{\pi} (x^2\pi^2 + x^4)\, dx &= \frac{2\pi^2}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4} + 4\pi^2 \cdot \frac{\pi^2}{6} \\ \Rightarrow \quad \frac{2}{\pi}\left[\frac{\pi^5}{3} + \frac{\pi^5}{5}\right] - \frac{8\pi^4}{9} &= 16\sum_{n=1}^{\infty}\frac{1}{n^4} \\ \Rightarrow \quad \sum_{n=1}^{\infty}\frac{1}{n^4} = \left(\frac{16\pi^4}{15} - \frac{8\pi^4}{9}\right)\frac{1}{16} &= \frac{\pi^4}{90}. \end{aligned}$$

□

Example 1.2.38. Compute the half-range Fourier cosine series of $f(x) = x - x^2$, $0 < x < \pi$.

Solution. Here,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx = \frac{2}{\pi}\int_0^{\pi}(x - x^2)\, dx = \frac{2}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^3}{3}\right] = \pi - \frac{2\pi^2}{3}.$$

For $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi}(x - x^2)\cos nx\, dx \\ &= \frac{2}{\pi}\left[\int_0^{\pi} x\cos nx\, dx - \int_0^{\pi} x^2\cos nx\, dx\right] \\ &= \frac{2}{\pi}\left\{\left[x\frac{\sin nx}{n} - \int \frac{\sin nx}{n}\, dx\right]_0^{\pi} - \left[x^2\frac{\sin nx}{n} - \int (2x)\frac{\sin nx}{n}\, dx\right]_0^{\pi}\right\} \\ &= \frac{2}{n\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi} + \frac{4}{n\pi}\left[x\frac{-\cos nx}{n} + \int \frac{\cos nx}{n}\, dx\right]_0^{\pi} \\ &= \frac{2}{n\pi}\left[\frac{(-1)^n - 1}{n}\right] - \frac{4}{n\pi}\left[\pi \cdot \frac{(-1)^n}{n}\right] \end{aligned}$$

$$= \frac{2}{n^2\pi}[(-1)^n - 1] - \frac{4}{n^2}(-1)^n$$
$$= \begin{cases} -\frac{4}{n^2}, & \text{if } n \text{ is even} \\ -\frac{4}{n^2\pi} + \frac{4}{n^2}, & \text{if } n \text{ is odd.} \end{cases}$$

Therefore, the half-range Fourier cosine series of f is

$$\frac{\pi}{2} - \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{1}{(2n)^2}\cos(2nx) - 4\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\left(\frac{1}{\pi} - 1\right)\cos(2n-1)x$$
$$\Rightarrow \frac{\pi}{2} - \frac{\pi^2}{3} - \sum_{n=1}^{\infty} \frac{1}{n^2}\cos(2nx) - 4\left(\frac{1}{\pi} - 1\right)\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos(2n-1)x.$$

□

Example 1.2.39. Compute the half-range Fourier sine series of $f(x) = \pi x - x^2$, $0 < x < \pi$. Use Parseval's identity to evaluate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$.

Solution. For $n \geq 1$,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx\,dx$$
$$= \frac{2}{\pi}\left[\pi x\frac{-\cos nx}{n} + \int \pi\frac{\cos nx}{n}\,dx + x^2\frac{\cos nx}{n} - 2\int x\frac{\cos nx}{n}\,dx\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\frac{-\pi x\cos nx}{n} + \frac{\pi\sin nx}{n^2} + \frac{x^2\cos nx}{n} - \frac{2x\sin nx}{n^2} + 2\int\frac{\sin nx}{n^2}\,dx\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\frac{-\pi x\cos nx}{n} + \frac{\pi\sin nx}{n^2} + \frac{x^2\cos nx}{n} - \frac{2x\sin nx}{n^2} - \frac{2\cos nx}{n^3}\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\frac{-\pi^2(-1)^n}{n} + \frac{\pi^2(-1)^n}{n} - \frac{2(-1)^n}{n^3} + \frac{2}{n^3}\right]$$
$$= \frac{4}{\pi n^3}[1 - (-1)^n] = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{8}{n^3\pi}, & \text{if } n \text{ is odd.} \end{cases}$$

Hence, the half-range Fourier sine series of f is

$$\frac{8}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\sin(2n-1)x.$$

Now, by using Parseval's identity (for $f \in L^2(0,\pi)$)

$$\frac{2}{\pi}\int_0^{\pi} |f(x)|^2\,dx = \sum_{n=1}^{\infty} |b_n|^2$$
$$\Rightarrow \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)^2\,dx = \sum_{n=1}^{\infty} \frac{64}{\pi^2}\cdot\frac{1}{(2n-1)^6}$$
$$\Rightarrow \frac{2}{\pi}\int_0^{\pi} (\pi^2x^2 - 2\pi x^3 + x^4)\,dx = \frac{64}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$$

$$\Rightarrow \frac{2}{\pi}\left[\pi^2\frac{x^3}{3} - 2\pi\frac{x^4}{4} + \frac{x^5}{5}\right]_0^{\pi} \cdot \frac{\pi^2}{64} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}$$

$$\Rightarrow \frac{\pi}{32}\left[\frac{\pi^5}{3} - \frac{\pi^5}{2} + \frac{\pi^5}{5}\right] = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}$$

$$\Rightarrow \frac{\pi^6}{32}\left[\frac{10-15+6}{30}\right] = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}.$$

Hence, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^6} = \frac{\pi^6}{960}$. □

Example 1.2.40. Compute the half-range Fourier cosine series of the function

$$f(x) = \begin{cases} \frac{2k}{\pi}x, & 0 < x < \frac{\pi}{2} \\ \frac{2k}{\pi}(\pi - x), & \frac{\pi}{2} < x < \pi. \end{cases}$$

Solution. Here,

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \\ &= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \frac{2k}{x}x\,dx + \int_{\frac{\pi}{2}}^{\pi}\frac{2k}{x}(\pi - x)\,dx\right] \\ &= \frac{4k}{\pi^2}\left\{\left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} + \left[\pi x - \frac{x^2}{2}\right]_{\frac{pi}{2}}^{\pi}\right\} \\ &= \frac{4k}{\pi^2}\left[\frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right] = \frac{4k}{\pi^2}\frac{2\pi^2}{8} = k. \end{aligned}$$

For $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \\ &= \frac{2}{\pi}\left[\frac{2k}{\pi}\int_0^{\frac{\pi}{2}} x\cos nx\,dx + \frac{2k}{\pi}\int_{\frac{\pi}{2}}^{\pi}(\pi - x)\cos nx\,dx\right] \\ &= \frac{4k}{\pi^2}\left\{\left[x\frac{\sin nx}{n} - \int\frac{\sin nx}{n}\,dx\right]_0^{\frac{\pi}{2}} + \left[(\pi - x)\frac{\sin nx}{n} + \int\frac{\sin nx}{n}\,dx\right]_{\frac{\pi}{2}}^{\pi}\right\} \\ &= \frac{4k}{\pi^2}\left\{\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\frac{\pi}{2}} + \left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{\frac{\pi}{2}}^{\pi}\right\} \\ &= \frac{4k}{\pi^2}\left[\frac{\pi}{2}\frac{\sin n\pi/2}{n} + \frac{\cos n\pi/2}{n^2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2} - \frac{\pi}{2}\frac{\sin n\pi/2}{n} + \frac{\cos n\pi/2}{n^2}\right] \\ &= \frac{4k}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - (1 + (-1)^n)\right] \\ &= \begin{cases} 0, & \text{if } n \text{ is odd} \\ -\frac{16k}{n^2\pi^2}, & n = 2, 6, 10, \ldots, \text{ i.e. } n = 4m - 2 \\ 0, & n = 4, 8, 12, \ldots, \text{ i.e. } n = 4m \end{cases} \end{aligned}$$

Therefore half-range Fourier cosine series of f is

$$\frac{k}{2} - \frac{16k}{\pi^2}\left[\frac{1}{2^2}\cos 2x + \frac{1}{6^2}\cos 6x + \frac{1}{10^2}\cos 10x + \cdots\right]$$

$$= \frac{k}{2} - \frac{16k}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(4n-2)x}{(4n-2)^2}$$
$$\text{or } \frac{k}{2} - \frac{4k}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(4n-2)x}{(2n-1)^2}$$

□

Ex

1. Let $f(x) = 1 + x$, $-\pi < x \leq \pi$ Use the Parseval's formula to evaluate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
2. Let $f(x) = \begin{cases} 0, & -\pi < x \leq 0 \\ \sin x, & 0 < x \leq \pi. \end{cases}$ Use the Parseval's formula to evaluate the sum of the series $\frac{1}{3^2} + \frac{1}{15^2} + \frac{1}{35^2} + \cdots$.
3. Let $f(x) = 1$, $0 < x < \pi$. Use the half range Fourier sine series to evaluate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.
4. Let $f(x) = x(\pi - x)$, $0 < x < \pi$. Use the half range Fourier sine or cosine series to evaluate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

1.3 Periodic Functions on Other Intervals

Let $f : (-L, L] \to \mathbb{C}$ be a map. Extend f as a $2L$-periodic function on $\mathbb{R}$. We note the following orthogonality relations.

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \quad (m, n \in \mathbb{N}), \tag{1.17}$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \quad (m, n \in \mathbb{N} \cup \{0\}), \tag{1.18}$$

and

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \quad (m, n \in \mathbb{N} \cup \{0\}). \tag{1.19}$$

This orthogonality relations lead to the following natural definition of Fourier series of $2L$ periodic functions.

Definition 1.3.1. Let $L > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be a $2L$ periodic function with $\int_{-L}^{L} |f(x)|\, dx < \infty$. The series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \tag{1.20}$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx \quad (n \in \mathbb{N}\cup\{0\}) \tag{1.21}$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx \quad (n \in \mathbb{N}), \tag{1.22}$$

is called the *Fourier series* of f, and the scalars a_n's and b_n's are called the *Fourier coefficients* of f.

Theorem 1.3.2. *Let $L > 0$ and $f : \mathbb{R} \to \mathbb{C}$ be a $2L$ periodic function. If f satisfies the Dirichlet conditions on $[-L, L]$, then the Fourier series of at $x \in \mathbb{R}$ converges to $\frac{f(x^+)+f(x^-)}{2}$.*

Definition 1.3.3. Let $f \in L^1(0, L)$. Then the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \tag{1.23}$$

where

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx \quad (n \in \mathbb{N}\cup\{0\}), \tag{1.24}$$

is called the *half range Fourier cosine series* of f.

Definition 1.3.4. Let $f \in L^1(0, L)$. Then the series

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \tag{1.25}$$

where

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx \quad (n \in \mathbb{N}), \tag{1.26}$$

is called the *half range Fourier sine series* of f.

Example 1.3.5. Compute the Fourier series of a periodic function $f(x) = x^2$, $-L < x \leq L$. Use the series to compute the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

Solution. Since, the function f is even, $b_n = 0$ for all $n \geq 1$. Now,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx = \frac{1}{L}\int_{-L}^{L} x^2\, dx = \frac{2}{L}\left[\frac{x^3}{3}\right]_0^L = \frac{2L^2}{3}.$$

For $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx = \frac{1}{L}\int_{-L}^{L} x^2\cos\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{L}\int_{0}^{L} x^2\cos\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{L}\left[x^2\frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} - 2\int x\frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\,dx\right]_0^L \\ &= \frac{2}{n\pi}\left[x^2\sin\left(\frac{n\pi x}{L}\right) - 2x\frac{-\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} + 2\int\frac{-\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\,dx\right]_0^L \\ &= \frac{4L}{n\pi}\left[L\frac{(-1)^n}{n\pi}\right] = \frac{4L^2(-1)^n}{n^2\pi^2}. \end{aligned}$$

Therefore the Fourier series of f is $\frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{L}\right)$. Here, the function f is continuous on $\mathbb{R}$ and it satisfied Dirichlet condition on any interval of length $2L$. Therefore, by Dirichlet theorem

$$f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{L}\right).$$

Taking $x = 0$ in the above equation, we get

$$\begin{aligned} 0 = f(0) &= \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \\ \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} &= -\frac{L^2}{3}\frac{\pi^2}{4L^2} = -\frac{\pi^2}{12}. \end{aligned}$$

Also, taking $x = L$ in the above equation, we get

$$\begin{aligned} L^2 = f(L) &= \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \\ \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} &= \left(L^2 - \frac{L^2}{3}\right)\frac{\pi^2}{4L^2} = \frac{\pi^2}{6}. \end{aligned}$$

□

Example 1.3.6. Compute the Fourier series of the function $f(x) = x - x^2$, $-1 < x \leq 1$. Also compute $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Solution. Here, $L = 1$. Therefore,

$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1}(x - x^2)\,dx = -2\int_{0}^{1} x^2\,dx = -2\left[\frac{x^3}{3}\right]_0^1 = -\frac{2}{3}.$$

For $n \geq 1$,

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{1}(x - x^2)\cos(n\pi x)\,dx$$

$$
\begin{aligned}
&= -2\int_0^1 x^2\cos(n\pi x)\,dx \\
&= -2\left[x^2\frac{\sin(n\pi x)}{n\pi} - 2\int x\frac{\sin(n\pi x)}{n\pi}\,dx\right]_0^1 \\
&= \frac{-2}{n\pi}\left[x^2\sin(n\pi x) - 2x\frac{-\cos(n\pi x)}{n\pi} + 2\int\frac{-\cos(n\pi x)}{n\pi}\,dx\right]_0^1 \\
&= \frac{-4}{n\pi}\left[\frac{(-1)^n}{n\pi}\right] = \frac{-4(-1)^n}{n^2\pi^2}.
\end{aligned}
$$

Also for $n \geq 1$

$$
\begin{aligned}
b_n &= \int_{-1}^1 f(x)\sin(n\pi x)\,dx = \int_{-1}^1 (x-x^2)\sin(n\pi x)\,dx \\
&= 2\int_0^1 x\sin(n\pi x)\,dx \\
&= 2\left[x\frac{-\cos(n\pi x)}{n\pi} - \int\frac{-\cos(n\pi x)}{n\pi}\,dx\right]_0^1 \\
&= \frac{-2(-1)^n}{n\pi}.
\end{aligned}
$$

Therefore the Fourier series of f is

$$
-\frac{1}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi x).
$$

Note that the function f is continuous on $\mathbb{R}$ except on the set $\{(2n+1) : n \in \mathbb{Z}\}$. Now, $f((2n+1)^-) = f(1^-) = 0$ and $f((2n+1)^+) = f(1^+) = -2$. Also, the function f satisfies Dirichlet condition on any interval of length 2. Therefore by Dirichlet theorem, we have

$$
-\frac{1}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi x) = \begin{cases} f(x), & x \neq 2n+1 \\ -1, & x = 2n+1 \end{cases}
$$

By taking $x = 1$ in the above equation, we get

$$
\begin{aligned}
-\frac{1}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} &= -1 \\
\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{4}\left(1-\frac{1}{3}\right) &= \frac{\pi^2}{6}.
\end{aligned}
$$

□

Ex

1. Find the Fourier series of $f(x) = \begin{cases} -a, & -L < x \leq 0 \\ a, & 0 < x \leq L. \end{cases}$
2. Find the Fourier series of $f(x) = \begin{cases} -2, & -4 < x \leq -2 \\ x, & -2 < x \leq 2 \\ 2, & 2 < x \leq 4. \end{cases}$

3. Find the Fourier series of $f(x) = \begin{cases} 0, & -2 < x \le -1 \\ k, & -1 < x \le 1 \\ 0, & 1 < x \le 2. \end{cases}$
4. Find the Fourier series of $f(x) = x^2 - 2$, $-2 < x \le 2$.
5. Find the Fourier series of $f(x) = e^{-x}$, $-L < x \le L$.
6. Find the half range Fourier sine and cosine series of the following functions.
   (a) $f(x) = x$, $0 < x < 2$.
   (b) $f(x) = 2x - 1$, $0 < x < 1$.

1.3.1 Parseval's Identity

Theorem 1.3.7 (Parseval's identity)**.** *Let $f \in L^2[-L, L]$. Then*

$$\frac{1}{L}\int_{-L}^{L} |f(x)|^2\, dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2),$$

where $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$ *and* $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$

Proof. Same as proof in Parseval's identity in the previous case. Just replace π by L here. □

Example 1.3.8. For Example 1.3.6, compute the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ using Parseval's identity.

Solution. By Parseval's identity, we have

$$\begin{aligned}
&\frac{1}{L}\int_{-L}^{L} |f(x)|^2\, dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) \\
\Rightarrow\ &\int_{-1}^{1} (x - x^2)^2\, dx = \frac{2}{9} + \frac{16}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \\
\Rightarrow\ &2\int_{0}^{1} (x^2 + x^4)\, dx = \frac{2}{9} + \frac{16}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^2} + \frac{4}{\pi^2}\frac{\pi^2}{6} \\
\Rightarrow\ &2\left[\frac{1}{3} + \frac{1}{5}\right] = \frac{2+6}{9} + \frac{16}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^2} \\
\Rightarrow\ &\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^4}{90}.
\end{aligned}$$

□

Ex

1. Let $f(x) = \begin{cases} -1, & -L < x \le 0 \\ 1, & 0 < x \le L. \end{cases}$ Use the Parseval's formula to evaluate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

2. Let $f(x) = x^2$, $-\frac{1}{2} < x \le \frac{1}{2}$. Use the Parseval's formula to evaluate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.
3. Let $f(x) = |x|, -\frac{\pi}{2} < x \le \frac{\pi}{2}$. Use the Parseval's formula to evaluate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$.
4. Let $f,g \in L^2[-L,L]$. Let $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ be the Fourier series of f, and let $\frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{n\pi x}{L}\right)$ be the Fourier series of g. Then prove that
$$\int_{-L}^{L} f(x)\overline{g(x)}\, dx = L\left(\frac{a_0\overline{\alpha_0}}{2} + \sum_{n=1}^{\infty} a_n\overline{\alpha_n} + \sum_{n=1}^{\infty} b_n\overline{\beta_n}\right).$$
5. Let $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2-x), & 1 \le x < 2. \end{cases}$ Use half range Fourier cosine series to find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$.

1.4 Applications of Fourier series

We have already seen that the Fourier series is useful in finding the sum of some series. We now see its application in solving partial differential equations for example, wave equation, etc.

Example 1.4.1. A string is stretched along the x-axis, to which it is attached at $x = 0$ and $x = L$. Find the displacement $y(x,t)$ of the string in terms of x and t, at a time t if given that $y(x,0) = mx(L-x)$, $0 \le x \le L$.

Solution. The vibration of the string is governed (or given) by the wave equation
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}. \tag{1.27}$$
Since both the end points of the strings are fixed, we have $y(0,t) = y(L,t) = 0,\ t > 0$.
Since the string is at rest initially, we have $y_t(x,0) = 0,\ 0 \le x \le L$.
Initial position of the string is $y(x,0) = mx(L-x),\ 0 \le x \le L$. Hence the problem is to solve the wave equation (partial differential equation) given in equation (1.27) subject to the conditions $y(0,t) = y(L,t) = 0,\ t > 0,\ y_t(x,0) = 0,\ 0 \le x \le L$ and $y(x,0) = mx(L-x),\ 0 \le x \le L$. Let
$$y(x,t) = X(x)T(t).$$
Then equation (1.27) becomes
$$X\frac{d^T}{dt^2} = c^2 T \frac{d^2X}{dx^2}$$
$$\text{i.e.}\quad \frac{1}{c^2T}\frac{d^T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2}.$$
Since left hand side is a function of t only and right hand side is a function of x only, both the functions must be constant, say k i.e., $\frac{1}{c^2T}\frac{d^T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} = k.$

Case-I: Let $k>0$, i.e. $k=\lambda^2$ for some $\lambda>0$.
Hence $\dfrac{d^2T}{dt^2}-c^2\lambda^2T=0$ and $\dfrac{d^2X}{dx^2}-\lambda^2X=0$. Therefore $T(t)=c_1e^{c\lambda t}+c_2e^{-c\lambda t}$ and $X(x)=d_1e^{\lambda x}+d_2e^{-\lambda x}$. Therefore

$$y(x,t)=(c_1e^{c\lambda t}+c_2e^{-c\lambda t})(d_1e^{\lambda x}+d_2e^{-\lambda x}).$$

Since $y(0,t)=0$ for all t, $(d_1+d_2)(c_1e^{c\lambda t}+c_2e^{-c\lambda t})=0$ for all t.
If $d_1+d_2\neq 0$, then $c_1e^{c\lambda t}+c_2e^{-c\lambda t}=0$ for all t, which implies $c_1=c_2=0$ ($\because e^{c\lambda t}$ and $e^{-c\lambda t}$ are linearly independent). But then $y=0$, i.e. in this case we will end up with the zero solution. Therefore $d_1+d_2=0$ or $d_2=-d_1$. Then

$$y(x,t)=(c_1e^{\lambda ct}+c_2e^{-\lambda ct})(e^{\lambda x}-e^{-\lambda x}).$$

Since $y(L,t)=0$ for all t, $(c_1e^{c\lambda t}+c_2e^{-c\lambda t})(e^{\lambda L}-e^{-\lambda L})=0$, i.e. $(c_1e^{c\lambda t}+c_2e^{-c\lambda t})\sin h(\lambda L)=0$. Since $\sin h(\lambda L)\neq 0$, $c_1=c_2=0$, and hence we end with a trivial solution.

Case-II: $k=0$.
Then $\dfrac{d^2T}{dt^2}=0$ and $\dfrac{d^2X}{dx^2}=0$. Therefore $X(x)=c_1x+c_2$ and $T(t)=d_1t+d_2$. Therefore

$$y(x,t)=(c_1x+c_2)(d_1t+d_2).$$

Since $y(0,t)=0$, $c_2(d_1t+d_2)=0$. If $c_2\neq 0$, then $d_1=d_2=0$ because 1 and t are linearly independent. But then in this case we end up with the zero solution. Hence $c_2=0$. Therefore $y(x,t)=x(c_1t+c_2)$. Since $y(L,t)=0$, $L(c_1t+c_2)=0$ for all t. This implies $c_1=c_2=0$, which ends with trivial solution again.

Case-III: $k<0$. Then $k=-\lambda^2$, for some $\lambda>0$.
Therefore $\dfrac{d^2T}{dt^2}+c^2\lambda^2t=0$ and $\dfrac{d^2X}{dx^2}+\lambda^2x=0$. Therefore $X(x)=c_1\cos\lambda x+c_2\sin\lambda x$ and $T(t)=d_1\cos(\lambda ct)+d_2\sin(\lambda ct)$. Therefore

$$y(x,t)=(c_1\cos\lambda x+c_2\sin\lambda x)(d_1\cos(\lambda ct)+d_2\sin(\lambda ct)).$$

Since $y(0,t)=0$ for all t, we have $c_1(d_1\cos(\lambda ct)+d_2\sin(\lambda ct))=0$ for all t. If $c_1\neq 0$, then $d_1=d_2=0$ ($\because\cos(c\lambda t)$ and $\sin(c\lambda t)$ are linearly independent). Therefore $c_1=0$. So

$$y(x,t)=\sin(\lambda x)(c_1\cos(c\lambda t)+c_2\sin(c\lambda t)).$$

Since $y(L,t)=0$, $\sin(\lambda L)(c_1\cos(\lambda ct)+c_2\sin(\lambda ct))=0$ for all t. If $\sin(\lambda L)\neq 0$, then again $c_1=c_2=0$. Since we want a non-trivial solution, $\sin(\lambda L)=0$. This implies that $\lambda L=n\pi$ or $\lambda=\dfrac{n\pi}{L}$ for $n\in\mathbb{N}$. Therefore

$$y_n(x,t)=\sin\left(\frac{n\pi x}{L}\right)\left[c_1\cos\left(\frac{n\pi ct}{L}\right)+c_2\sin\left(\frac{n\pi ct}{L}\right)\right].$$

Therefore

$$y_t(x,t)=\frac{cn\pi}{L}\sin\left(\frac{n\pi x}{L}\right)\left[-c_1\sin\left(\frac{cn\pi t}{L}\right)+c_2\cos\left(\frac{cn\pi t}{L}\right)\right].$$

Since the string is at rest when $t=0$, we have $y_t(x,0)=0$ for all x. This will imply $c_2=0$. Hence for each $n\in\mathbb{N}$, we have a solution of the wave equation of the form

$$y_n(x,t)=b_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right).$$

satisfying the conditions $y(0,t)=y(L,t)=0$ for all t and $y_t(x,0)=0$ for all x. Hence by the principle of *super position*,

$$y(x,t)=\sum_{n=1}^{\infty}y_n(x,t)$$

is a solution of the wave equation satisfying the same conditions, i.e.,

$$y(x,t)=\sum_{n=1}^{\infty}b_n\cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right),\quad (0\le x\le L,\ t>0). \tag{1.28}$$

Now, $y(x,0)=mx(L-x)$. Therefore

$$y(x,0)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right).$$

This is the half range Fourier sine series of the function $y(x,0)=mx(L-x)$, $0<x<L$. Hence

$$\begin{aligned}
b_n&=\frac{2}{L}\int_0^L mx(L-x)\sin\left(\frac{n\pi x}{L}\right)\,dx\\
&=\frac{2}{L}\left[\int_0^L mLx\sin\left(\frac{n\pi x}{L}\right)\,dx-\int_0^L mx^2\sin\left(\frac{n\pi x}{L}\right)\,dx\right]\\
&=\frac{2m}{L}\left\{L\left[x\frac{-\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}+\frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\right]_0^L-\left[x^2\frac{-\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}+2\int x\frac{\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\right]_0^L\right\}\\
&=\frac{2m}{L}\left\{\left[Lx\frac{-\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}+L\frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\right]_0^L-\left[x^2\frac{-\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}+2x\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2}+2\frac{\cos\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^3}\right]_0^L\right\}\\
&=\begin{cases}0, & \text{if } n \text{ is even}\\ \frac{8mL^2}{n^3\pi^3}, & \text{if } n \text{ is odd.}\end{cases}
\end{aligned}$$

Hence

$$y(x,t)=\frac{8mL^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\cos\left(\frac{(2n-1)\pi ct}{L}\right)\sin\left(\frac{(2n-1)\pi x}{L}\right),\quad (0\le x\le L,\ t>0).$$

□

Example 1.4.2. A homogeneous rod of conducting material of length 100 cm has its ends kept at 0 temperature and the initial temperature is

$$u(x,0)=\begin{cases}x, & 0\le x\le 50\\ 100-x, & 50<x\le 100.\end{cases}$$

Find the temperature $u(x,t)$ at a point x at any time t.

Solution. We know that the temperature in the rod is governed by the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{1.29}$$

Since the ends of the rod are at 0 temperature, we have

$$u(0,t) = u(100,t) = 0, \ \forall\, t.$$

Hence the problem is to solve the heat equation in (1.29) subject to the conditions $u(0,t) = u(100,t) = 0$ for all t. Let the solution be of the form

$$u(x,t) = X(x)T(t).$$

Then equation (1.29) reduces to

$$X\frac{dT}{dt} = c^2 T \frac{d^X}{dx^2}$$
$$\text{i.e. } \frac{1}{c^2T}\frac{dT}{dt} = X\frac{d^2X}{dx^2}.$$

Since left hand side is a function of t only and right hand side is a function of x only, both the functions must be constant, say k.

Case-I: Let $k > 0$. Then $k = \lambda^2$ for some $\lambda > 0$.
Then $\frac{dT}{dt} - \lambda^2 c^2 T = 0$ and $\frac{d^2X}{dx^2} = 0$. Therefore $T(t) = c_1 e^{c^2\lambda t}$ and $X(x) = a_1 e^{\lambda x} + be^{-\lambda x}$ and so

$$u(x,t) = e^{c^2\lambda^2 t}(ae^{\lambda x} + be^{-\lambda x}).$$

Since $u(0,t) = 0$ for all t, $e^{c^2\lambda^2 t}(a+b) = 0$ for all t, i.e., $b = -a$. Hence

$$u(x,t) = ae^{c^2\lambda^2 t}(e^{\lambda x} - e^{-\lambda x}) = a_1 e^{c^2\lambda^2 t} \sin h(\lambda x).$$

Now, $u(100,t) = 0$ for all t. Therefore $a_1 e^{c^2\lambda^2 t} \sin h(100\lambda) = 0$ for all t. Since $\sin h(100\lambda) \neq 0$ and $e^{c^2\lambda^2 t} > 0$, $a_1 = 0$ and we end with the trivial solution. Therefore $k > 0$ will not work.

Case-II: $k = 0$.
Then $\frac{dT}{dt} = 0 \Rightarrow T(t) = c_1$ and $\frac{d^2X}{dx^2} = 0 \Rightarrow X(x) = a_1 x + b$. Therefore, we write

$$u(x,t) = ax + b.$$

Since $u(0,t) = 0$ for all t, $b = 0$. Since $u(100,t) = 0$ for all t, $a = 0$, which again ends with the trivial solution.

Case-III: $k < 0$. Let $k = -\lambda^2$ for some $\lambda > 0$.
Then $\frac{dT}{dt} + \lambda^2 c^2 T = 0$ and $\frac{d^2X}{dx^2} + \lambda^2 X = 0$. Therefore $u(x,t) = e^{-c^2\lambda^2 t}(a\cos(\lambda x) + b\sin(\lambda x))$. Since $u(0,t) = 0$ for all t, $a = 0$. Therefore $u(x,t) = be^{-c^2\lambda^2 t}\sin(\lambda x)$. Since we want a non-trivial solution, $b \neq 0$. Since $u(100,t) = 0$ for all t, $be^{-c^2\lambda^2 t}\sin(100\lambda) = 0$ for all t. Therefore $\sin(100\lambda) = 0$ and so $\lambda = \frac{n\pi}{100}$ for $n \in \mathbb{N}$. Hence for each $n \in \mathbb{N}$,

$$u_n(x,t) = b_n e^{-\frac{c^2 n^2 \pi^2}{100^2}t} \sin\left(\frac{n\pi x}{100}\right)$$

is a solution of the heat equation satisfying $u(0,t) = u(100,t) = 0$ for all t. Therefore by the principle of *super position*

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{c^2n^2\pi^2}{100^2}t} \sin\left(\frac{n\pi x}{100}\right)$$

is also a solution of the same satisfying the same conditions. So,

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{100}\right)$$

which is half-range Fourier sine series of the function $u(x,0)$. It is given that $u(x,0) = \begin{cases} x, & 0 \le x \le 50 \\ 100 - x, & 50 < x \le 100 \end{cases}$. Therefore

$$\begin{aligned}
b_n &= \frac{2}{100}\int_0^{100} u(x,0)\sin\left(\frac{n\pi x}{100}\right)\,dx \qquad (n \in \mathbb{N}) \\
&= \frac{1}{50}\left[\int_0^{50} x\sin\left(\frac{n\pi x}{100}\right)\,dx + \int_{50}^{100}(100-x)\sin\left(\frac{n\pi x}{100}\right)\,dx\right] \\
&= \frac{1}{50}\left\{\left[x\frac{-\cos\left(\frac{n\pi x}{100}\right)}{\frac{n\pi}{100}} + \int \frac{\cos\left(\frac{n\pi x}{100}\right)}{\frac{n\pi}{100}}\,dx\right]_0^{50} + \left[(100-x)\frac{-\cos\left(\frac{n\pi x}{100}\right)}{\frac{n\pi}{100}} - \int \frac{\cos\left(\frac{n\pi x}{100}\right)}{\frac{n\pi}{100}}\,dx\right]_{50}^{100}\right\} \\
&= \frac{100}{50n\pi}\left\{\left[-x\cos\left(\frac{n\pi x}{100}\right) + \frac{\sin\left(\frac{n\pi x}{100}\right)}{\frac{n\pi}{100}}\right]_0^{50} + \left[(x-100)\cos\left(\frac{n\pi x}{100}\right) - \frac{\sin\left(\frac{n\pi x}{100}\right)}{\frac{n\pi}{100}}\right]_{50}^{100}\right\} \\
&= \frac{2}{n\pi}\left\{\left[-50\cos\left(\frac{n\pi}{2}\right) + \frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{100}}\right] + \left[50\cos\left(\frac{n\pi}{2}\right) + \frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{100}}\right]\right\} \\
&= \frac{2}{n\pi}\left[\frac{200}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right] = \frac{400}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \\
&= \begin{cases} 0, & n \text{ is even} \\ \frac{400}{n^2\pi^2}, & n = 1,5,9,\ldots, \text{ i.e. } n = 4k+1 \\ -\frac{400}{n^2\pi^2}, & n = 3,7,11,\ldots, \text{ i.e. } n = 4k+3. \end{cases}
\end{aligned}$$

Therefore, the solution is

$$u(x,t) = \frac{400}{n^2\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left(\frac{(2n-1)\pi x}{100}\right)e^{\frac{-(2n-1)^2c^2\pi^2t}{100^2}}.$$

□

Example 1.4.3 (The Interior Dirichlet problem for a circle). Solve $\nabla^2 u = 0$, $0 \le r < a$, $0 \le \theta \le 2\pi$, subject to the conditions $u(a,\theta) = f(\theta)$, $0 \le \theta \le 2\pi$, where f is a continuous function.

Solution. The Laplace equation in polar coordinate is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{1.30}$$

Let $u(r,\theta)=R(r)\Theta(\theta)$. Then equation (1.30) reduces to

$$\begin{aligned}\Theta\frac{d^2R}{dr^2}+\frac{\Theta}{r}\frac{dR}{dr}+\frac{R}{r^2}\frac{d^2\Theta}{d\theta^2} &=0\\ \frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}\right) &= -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}.\end{aligned}$$

Since left hand side is a function of r only and right hand side is a function of θ only, both the functions are constant, say k.

Case-I: $k<0$. Let $k=-\lambda^2$ for some $\lambda>0$.
Then

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+\lambda^2R=0 \text{ and } \frac{d^2\Theta}{d\theta^2}-\lambda^2\Theta=0. \tag{1.31}$$

Let $z=\log r$ i.e., $r=e^z$. Then $\frac{1}{r}=e^{-z}$. Therefore

$$\frac{dR}{dr}=\frac{dR}{dz}\frac{dz}{dr}=\frac{dR}{dz}\frac{1}{r}=\frac{dR}{dz}e^{-z}.$$

Also

$$\begin{aligned}\frac{d^2R}{dr^2} &= \frac{d}{dr}\left(\frac{dR}{dr}\right)=\frac{d}{dz}\left(\frac{dR}{dr}\right)\frac{dz}{dr}\\ &= \frac{d}{dz}\left(\frac{dR}{dz}e^{-z}\right)\frac{1}{r}\\ &= \left(e^{-z}\frac{d^2R}{dz^2}-\frac{dR}{dz}e^{-z}\right)\frac{1}{r}\\ &= \frac{1}{r^2}\left(\frac{d^2R}{dr^2}-\frac{dR}{dr}\right).\end{aligned}$$

Thus, equation (1.31) becomes

$$\frac{d^R}{dz^2}-\frac{dR}{dz}+\frac{dR}{dz}+\lambda^2R=0\Rightarrow\frac{d^R}{dz^2}+\lambda^2R=0.$$

Therefore, $R(z)=c_1\cos(\lambda z)+c_2\sin(\lambda z)$ i.e., $R(r)=c_1\cos(\lambda\log r)+c_2\sin(\lambda\log r)$ and $\Theta(\theta)=d_1e^{\lambda\theta}+d_2e^{-\lambda\theta}$. Therefore,

$$u(r,\theta)=(c_1\cos(\lambda\log r)+c_2\sin(\lambda\log r))(d_1e^{\lambda\theta}+d_2e^{-\lambda\theta}).$$

The above function is continuous at origin, only when $c_1=c_2=0$. But then u is zero function and we end with trivial solution. Thus, $k<0$ does not yield any non-trivial solution.

Case-II: $k=0$. Then

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}=0 \text{ and } \frac{d^\Theta}{d\theta^2}=0.$$

By the same procedure as above, we get $\dfrac{d^2R}{dz^2}=0$ and therefore $R(z)=az+b$ i.e., $R(r)=a\log r+b$ and $\Theta(\theta)=c\theta+d$. Therefore,

$$u(r,\theta)=(a\log r+b)(c\theta+d).$$

Since u is continuous at origin, we must have $a = 0$. Therefore $u(r,\theta) = c\theta + d$. Now,

$$\begin{aligned} & u(r,\theta+2\pi) = u(r,\theta) \\ \Rightarrow\ & c(\theta+2\pi)+d = c\theta+d \\ \Rightarrow\ & c = 0. \end{aligned}$$

Therefore $u(r,\theta) = d$, i.e. u is a constant function.

Case-III: Let $k > 0$. Then $k = \lambda^2$ for some $\lambda > 0$.
Then

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda^2 R = 0 \text{ and } \frac{d^2\Theta}{d\theta^2} + \lambda^2\Theta = 0.$$

Then by the same procedure as above, we get

$$\frac{d^2R}{dz^2} - \lambda^2 R = 0 \Rightarrow R(z) = c_1 e^{\lambda z} + c_2 e^{-\lambda z}.$$

Therefore $R(r) = c_1 r^{\lambda} + c_2 r^{-\lambda}$ and $\Theta(\theta) = d_1\cos(\lambda\theta) + d_2\sin(\lambda\theta)$. Therefore

$$u(r,\theta) = (c_1 r^{\lambda} + c_2 r^{-\lambda})(d_1\cos(\lambda\theta) + d_2\sin(\lambda\theta)).$$

Since u is continuous at origin, we have $c_2 = 0$. Therefore

$$u(r,\theta) = r^{\lambda}(d_1\cos(\lambda\theta) + d_2\sin(\lambda\theta)).$$

Now,

$$\begin{aligned} & u(r,\theta+2\pi) = u(r,\theta) \\ \Rightarrow\ & d_1\cos(\lambda(\theta+2\pi)) + d_2\sin(\lambda(\theta+2\pi)) = d_1\cos(\lambda\theta) + d_2\sin(\lambda\theta),\ \forall\theta \\ \Rightarrow\ & 2n\pi = 2\lambda\pi \text{ or } \lambda = n,\ n\in\mathbb{N} \\ \Rightarrow\ & u(r,\theta) = r^n(c_1\cos(n\theta) + c_2\sin(n\theta)). \end{aligned}$$

Hence, for each $n \in \mathbb{N}$,

$$u_n(r,\theta) = r^n(A_n\cos(n\theta) + B_n\sin(n\theta))$$

is a solution of the Laplace equation $\nabla^2 u = 0$, i.e. equation given by (1.30). Since the constant function is also a solution (by Case-II) of the same, by principle of superposition,

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n(A_n\cos n\theta + B_n\sin n\theta) \tag{1.32}$$

is a solution of the Laplace equation. Since $u(a,\theta) = f(\theta)$,

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n(A_n\cos n\theta + B_n\sin n\theta)$$

which is Fourier series of the function f. Therefore

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$A_n = \frac{1}{a^n\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta, \quad n \geq 1$$
$$B_n = \frac{1}{a^n\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta, \quad n \geq 1.$$

Substituting these values in equation (1.32), we have

$$\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\, d\tau + \sum_{n=1}^{\infty}\frac{r^n}{a^n\pi}\left(\int_{-\pi}^{\pi} f(\tau)(\cos n\tau\cos n\theta + \sin n\tau\sin n\theta)\, d\tau\right)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\, d\tau + \sum_{n=1}^{\infty}\frac{r^n}{a^n\pi}\int_{-\pi}^{\pi} f(tau)\cos\big(n(\tau-\theta)\big)\, d\tau\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\, d\tau + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau)\left(\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\frac{e^{in(\tau-\theta)}+e^{-in(\tau-\theta)}}{2}\right) d\tau\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\, d\tau + \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\left(\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left(e^{i(\tau-\theta)}\right)^n + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left(e^{-i(\tau-\theta)}\right)^n\right) d\tau\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\, d\tau + \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\left(\frac{\frac{r}{a}e^{i(\tau-\theta)}}{1-\frac{r}{a}e^{i(\tau-\theta)}} + \frac{\frac{r}{a}e^{-i(\tau-\theta)}}{1-\frac{r}{a}e^{-i(\tau-\theta)}}\right) d\tau. \qquad (1.33)
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{\frac{r}{a}e^{i(\tau-\theta)}}{1-\frac{r}{a}e^{i(\tau-\theta)}} + \frac{\frac{r}{a}e^{-i(\tau-\theta)}}{1-\frac{r}{a}e^{-i(\tau-\theta)}} &= \frac{\frac{r}{a}e^{i(\tau-\theta)}\left(1-\frac{r}{a}e^{-i(\tau-\theta)}\right) + \frac{r}{a}e^{-i(\tau-\theta)}\left(1-\frac{r}{a}e^{i(\tau-\theta)}\right)}{1-\frac{r}{a}e^{i(\tau-\theta)} - \frac{r}{a}e^{-i(\tau-\theta)} + \frac{r^2}{a^2}}\\
&= \frac{\frac{r}{a}\left[e^{i(\tau-\theta)} - \frac{r}{a} + r^{-i(\tau-\theta)} - \frac{r}{a}\right]}{1-2\frac{r}{a}\cos(\tau-\theta)+\frac{r^2}{a^2}}\\
&= \frac{2\frac{r}{a}\left[\cos(\tau-\theta)-\frac{r}{a}\right]}{1-2\frac{r}{a}\cos(\tau-\theta)+\frac{r^2}{a^2}}
\end{aligned}$$

Substituting this simplification in equation (1.33), we get

$$\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\left(1+\frac{2\frac{r}{a}\left(\cos(\tau-\theta)-\frac{r}{a}\right)}{1-2\frac{r}{a}\cos(\tau-\theta)+\frac{r^2}{a^2}}\right) d\tau\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\frac{1-\frac{r^2}{a^2}}{1-2\frac{r}{a}\cos(\tau-\theta)+\frac{r^2}{a^2}}\, d\tau.
\end{aligned}$$

Therefore

$$\boxed{u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\frac{a^2-r^2}{a^2-2ra\cos(\tau-\theta)+r^2}d\tau.}$$

It is called the *Poisson Integral Formula*. □

Example 1.4.4 (Exterior Dirichlet problem for a circle). Solve $\nabla^2 u = 0$, $r > a$, $0 \leq \theta \leq 2\pi$, subject to the conditions $u(a,\theta) = f(\theta)$, $0 \leq \theta \leq 2\pi$, where f is a continuous function, and u is bounded as $r \to \infty$.

Solution. The Laplace equation in polar coordinate is

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0. \tag{1.34}$$

Let $u(r,\theta)=R(r)\Theta(\theta)$. Then the above equation will become

$$\begin{aligned} 0 &= \Theta\frac{d^2R}{dr^2}+\frac{\Theta}{r}\frac{dR}{dr}+\frac{R}{r^2}\frac{d^2\Theta}{d\theta^2} \\ &= \frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}+\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}. \end{aligned}$$

Therefore $\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}=-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}$. Since LHS is function of r only and RHS is a function of θ only, both the functions are constant, say λ.

If $\lambda=0$, then $u(r,\theta)=(a\log r+b)(c\theta+d)$. Since u is 2π periodic in θ variable, $c=0$. Hence $u(r,\theta)=A\log r+B$. Since u is bounded as $r\to\infty$, $A=0$. Hence u is a constant function.

Let $\lambda>0$. Then $\lambda=k^2$ for some $k>0$. Then

$$u(r,\theta)=(ar^k+br^{-k})(c\cos k\theta+d\sin k\theta).$$

Since u is bounded as $r\to\infty$, $a=0$. Hence $u(r,\theta)=r^{-k}(A\cos k\theta+b\sin k\theta)$. Since u is 2π periodic in the variable θ, we have $k\in\mathbb{N}$. Hence for each $n\in\mathbb{N}$, $u_n(r,\theta)=r^{-n}(a_n\cos n\theta+b_n\sin n\theta)$ is a solution of $\nabla^2=0$. A constant function is also a solution of the same. Hence by the *principle of super position*

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}r^{-n}(a_n\cos n\theta+b_n\sin n\theta)$$

is a solution of the same. Since $u(a,\theta)=f(\theta)$, we have

$$f(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a^{-n}(a_n\cos n\theta+b_n\sin n\theta).$$

This is a Fourier series of f in $[-\pi,\pi]$. Therefore

$$a_n=\frac{a^n}{\pi}\int_{-\pi}^{\pi}f(\tau)\cos n\tau d\tau \quad \text{and } b_n=\frac{a^n}{\pi}\int_{-\pi}^{\pi}f(\tau)\sin n\tau d\tau.$$

Therefore

$$u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\tau)\left[1+\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\cos n(\tau-\theta)\right]d\tau$$

By simplifying it we get

$$u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\tau)\frac{r^2-a^2}{r^2-2ra\cos(\tau-\theta)+a^2}d\tau.$$

It is called the *Poisson Integral Formula*. □

Example 1.4.5 (The Dirichlet problem for a rectangle). Solve $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0,\ 0<x<a,\ 0<y<b$, subject to the conditions $u(0,y)=u(a,y)=0$ for all y, $u(x,b)=0$ for all x and $u(x,0)=f(x)$ for all x.

Solution. Let $u(x,y)=X(x)Y(y)$ be a solution of above equation. Then $Y\dfrac{d^2X}{dx^2}+X\dfrac{d^2Y}{dy^2}=0$, i.e.,

$$\frac{1}{X}\frac{d^2X}{dx^2}=-\frac{1}{Y}\frac{d^2Y}{dy^2}. \tag{1.35}$$

Since left hand side is a function of x only and right hand side is a function of y only, both the functions have to be constant, say k.

Case I: Let $k>0$, i.e., $k=\lambda^2$ for some $\lambda>0$.
Then the above equations will become $\dfrac{d^2X}{dx^2}-\lambda^2X=0$ and $\dfrac{d^2Y}{dy^2}+\lambda^2Y=0$. Solving these two equations, we have $X(x)=c_1\cosh(\lambda x)+c_2\sinh(\lambda x)$ and $Y(y)=d_1\cos(\lambda y)+d_2\sin(\lambda y)$. Hence

$$u(x,y)=(c_1\cosh(\lambda x)+c_2\sinh(\lambda x))(d_1\cos(\lambda y)+d_2\sin(\lambda y)).$$

Since $u(0,y)=0$ for all y, $c_1(d_1\cos(\lambda y)+d_2\sin(\lambda y))=0$ for all y. If $c_1\neq 0$, then $d_1=d_2=0$ as $\cos(\lambda y)$ and $\sin(\lambda y)$ are linearly independent. Thus $c_1=0$. This implies that

$$u(x,y)=\sinh(\lambda x)(d_1\cos(\lambda y)+d_2\sin(\lambda y)).$$

Now $u(a,y)=0$ for all y gives $\sinh(\lambda a)(d_1\cos(\lambda y)+d_2\sin(\lambda y))=0$ for all y. Since $\sinh(\lambda a)>0$ (as $\lambda a>0$) and $\cos(\lambda y)$ and $\sin(\lambda y)$ are linearly independent, we get $d_1=d_2=0$. Thus $u=0$. Therefore this case is not possible.

Case II: Let $k=0$.
Then $X(x)=c_1x+c_2$ and $Y(y)=d_1y+d_2$. Then

$$u(x,y)=(c_1x+c_2)(d_1y+d_2).$$

Using $u(0,y)=0$ for all y, we have $c_2(d_1y+d_2)=0$ for all y. If $c_2\neq 0$, then $d_1=d_2=0$ as the functions 1 and y are linearly independent. Thus $c_2=0$. Therefore

$$u(x,y)=x(d_1y+d_2).$$

As $u(a,y)=0$ for all y, $a(d_1y+d_2)=0$ for all y. Therefore $d_1=d_2=0$ and hence $u=0$. So, this case is also not possible.

Case III: Let $k<0$, i.e., $k=-\lambda^2$ for some $\lambda>0$.
Solving $\dfrac{d^2X}{dx^2}+\lambda^2X=0$ and $\dfrac{d^2Y}{dy^2}-\lambda^2Y=0$ we have $X(x)=c_1\cos(\lambda x)+c_2\sin(\lambda x)$ and $Y(y)=d_1\cosh(\lambda y)+d_2\sinh(\lambda y)$. Therefore

$$u(x,y)=(c_1\cos(\lambda x)+c_2\sin(\lambda x))(d_1\cosh(\lambda y)+d_2\sinh(\lambda y)).$$

Since $u(0,y)=0$ for all y,

$$c_1(d_1\cosh(\lambda y)+d_2\sinh(\lambda y))=0 \quad \text{for all } y.$$

Therefore, we get $c_1 = 0$. Hence

$$u(x,y) = \sin(\lambda x)(d_1\cosh(\lambda y) + d_2\sinh(\lambda y)).$$

Now $u(a,y) = 0$ for all y gives

$$\sin(\lambda a)(d_1\cosh(\lambda y) + d_2\sinh(\lambda y)) = 0 \quad \text{for all } y.$$

If $\sin(\lambda a) \neq 0$, then $d_1 = d_2 = 0$ which will imply that $u = 0$. Since we want non-trivial solution, $\sin(\lambda a) = 0$. Since $\lambda > 0$, $\lambda = \frac{n\pi}{a}$, $n \in \mathbb{N}$. Therefore

$$u(x,y) = \sin\left(\frac{n\pi x}{a}\right)\left(d_1\cosh\left(\frac{n\pi y}{a}\right) + d_2\sinh\left(\frac{n\pi y}{a}\right)\right).$$

Now since $u(x,b) = 0$ for all x,

$$\sin\left(\frac{n\pi x}{a}\right)\left(d_1\cosh\left(\frac{n\pi b}{a}\right) + d_2\sinh\left(\frac{n\pi b}{a}\right)\right) = 0 \quad \text{for all } x.$$

There is x such that $\sin\left(\frac{n\pi x}{a}\right) \neq 0$. Therefore

$$d_1\cosh\left(\frac{n\pi b}{a}\right) + d_2\sinh\left(\frac{n\pi b}{a}\right) = 0 \Rightarrow d_2 = -d_1\frac{\cosh\left(\frac{n\pi b}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}.$$

Substituting this value in the solution, we get

$$\begin{aligned}
u(x,y) &= d\sin\left(\frac{n\pi x}{a}\right)\left[\cosh\left(\frac{n\pi y}{a}\right) - \frac{\cosh\left(\frac{n\pi b}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\sin h\left(\frac{n\pi y}{a}\right)\right] \\
&= d\frac{1}{\sinh\left(\frac{n\pi b}{a}\right)}\sin\left(\frac{n\pi x}{a}\right)\left[\cosh\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{n\pi b}{a}\right) - \cosh\left(\frac{n\pi b}{a}\right)\sin h\left(\frac{n\pi y}{a}\right)\right] \\
&= d\frac{\sin\left(\frac{n\pi x}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\sinh\left(\frac{(b-y)n\pi}{a}\right).
\end{aligned}$$

Thus for each $n \in \mathbb{N}$,

$$u_n(x,y) = d\frac{\sin\left(\frac{n\pi x}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\sinh\left(\frac{(b-y)n\pi}{a}\right)$$

is a solution of the Laplace equation satisfying conditions $u(0,y) = u(a,y) = 0$ for all y and $u(x,b) = 0$ for all x. Therefore by principle of superposition,

$$\begin{aligned}
u(x,y) &= \sum_{n=1}^{\infty} u_n(x,y) \\
&= \sum_{n=1}^{\infty} d_n\frac{1}{\sinh\left(\frac{n\pi b}{a}\right)}\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{(b-y)n\pi}{a}\right)
\end{aligned}$$

is also a solution of the Laplace equation satisfying the same conditions. Since $u(x,0) = f(x)$ for all x,

$$f(x) = \sum_{n=1}^{\infty} d_n\sin\left(\frac{n\pi x}{a}\right).$$

This is the half range Fourier sine series of f with x in the interval $(0,a)$. Therefore

$$d_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)$$

i.e.,

$$u(x,y) = \sum_{n=1}^{\infty} d_n \frac{\sin\left(\frac{n\pi x}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} \sinh\left(\frac{(b-y)n\pi}{a}\right)$$

is the solution of the given Laplace equation, where $d_n = \dfrac{2}{a}\displaystyle\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)$. □

Example 1.4.6 (The Neumann problem for a rectangle). Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $0 < x < a$, $0 < y < b$, subject to the conditions $u_x(0,y) = u_x(a,y) = 0$, $u_y(x,0) = 0$ and $u_y(x,b) = f(x)$.

Example 1.4.7. A thin annulus occupies the region $0 < a \le r \le b$, $0 \le \theta \le 2\pi$. The faces are insulated. Along the inner edge the temperature is maintained at $0°$, while along the outer edge the temperature is held at $K\cos(\frac{\theta}{2})$, where K is constant. Determine the temperature distribution in the annulus.

Here we have to solve the Laplace equation $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$, $a < r < b$, $\theta \in \mathbb{R}$ subject to conditions $u(r,\theta) = u(r,\theta + 2\pi)$ for all θ, $u(a,\theta) = 0$ and $u(b,\theta) = K\cos(\frac{\theta}{2})$ for all θ.

Remark 1.4.8. 1. (Riemann Lebesgue Lemma) Let $f \in L^1[-\pi,\pi]$.

   (a) Let $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ be the Fourier series of f. Then $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$.

   (b) Let $\sum_{n\in\mathbb{Z}} c_n e^{inx}$ be the Fourier series of f, then $c_n \to 0$ as $|n| \to \infty$.

2. (a) Let $(a_0, a_1, \dots)$ and $(b_1, b_2, \dots)$ be two complex sequences such that both (a_n) and (b_n) tends to 0 as $n \to \infty$. Then there may *not* exist $f \in L^1[-\pi,\pi]$ such that the Fourier coefficients of f are $a_n{}'s$ and $b_n{}'s$.

   (b) Let (c_n) be a complex sequence such that $c_n \to 0$ as $n \to \infty$. Then there may *not* exist $f \in L^1[-\pi,\pi]$ whose fourier coefficients are $c_n{}'s$.

3. (Identity Theorem) Let $f,g \in L^1[-\pi,\pi]$. If f and g have the same Fourier series, then $f = g$ a.e. on $[-\pi,\pi]$.

CHAPTER 2

FOURIER TRANSFORM

2.1 Fourier Transform

2.1.1 Definition and Examples

Let $L^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} : f \text{ is measurable}, \int_{\mathbb{R}} |f|\, dm < \infty\}$, where the integral in the set is the Lebesgue integral.

Definition 2.1.1. Let $f \in L^1(\mathbb{R})$. Then the *Fourier transform* $F[f]$ of f is defined as

$$F[f][s] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-isx}\, dx \quad (s \in \mathbb{R}). \tag{2.1}$$

If $f \in L^1(\mathbb{R})$, then $\left|\int_{-\infty}^{\infty} f(x)e^{-isx}dx\right| \le \int_{-\infty}^{\infty} |f(x)|dx < \infty$. Thus $F[f](s)$ is a complex number for every $s \in \mathbb{R}$ and hence $F[f]$ is a function from $\mathbb{R}$ to $\mathbb{C}$.

Example 2.1.2. Let $a > 0$. Compute the Fourier transform of $f = \chi_{[-a,a]}$, i.e.

$$f(x) = \begin{cases} 1, & -a \le x \le a \\ 0, & \text{otherwise} \end{cases}.$$

Solution. Here $f(x) = 1$ if $|x| \le a$ and $f(x) = 0$ if $|x| > a$. The function f is an even function. Let $s \in \mathbb{R}$ and $s \neq 0$. Then

$$\begin{aligned} F[f](s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-isx}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1\, e^{-isx}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-isx}}{-is}\right]_{-a}^{a}, \qquad s \neq 0 \end{aligned}$$

$$= \frac{1}{-is\sqrt{2\pi}}[e^{-ias} - e^{ias}] = \sqrt{\frac{2}{\pi}}\frac{\sin sa}{s}.$$

For $s = 0$,

$$F[f](s) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} 1\, dx = \sqrt{\frac{2}{\pi}}a.$$

Therefore

$$F[f](s) = \begin{cases} \sqrt{\frac{2}{\pi}}\frac{\sin(sa)}{s}, & s \neq 0 \\ \sqrt{\frac{2}{\pi}}, a & s = 0 \end{cases}$$

□

Example 2.1.3. Find the Fourier transform of $f(x) = e^{-a|x|}$, where $a > 0$.

Solution. Here f is an even function. Let $0 \neq s \in \mathbb{R}$. Then

$$\begin{aligned} F[f](s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}\, dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\cos sx\, dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\sin sx\, dx \\ &= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} e^{-ax}\cos(sx)dx \\ &= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{s^2+a^2}(-a\cos(sx) + s\sin(bs))\right]_0^\infty \end{aligned}$$

We know that if $a, b \in \mathbb{R}$, then $|a\cos x + b\sin x| \leq \sqrt{a^2+b^2}$ for every $x \in \mathbb{R}$. Therefore $\left|\frac{e^{-ax}}{s^2+a^2}(-a\cos(sx) + s\sin(bs))\right| \leq \frac{e^{-ax}}{s^2+a^2}\sqrt{s^2+a^2} = \frac{e^{-ax}}{\sqrt{s^2+a^2}} \leq \frac{e^{-ax}}{a}$. Since $\lim_{x\to\infty} e^{-ax} = 0$, it follows that $\frac{e^{-ax}}{s^2+a^2}(-a\cos(sx) + s\sin(bs)) \to 0$ as $x \to \infty$. Hence, we have

$$F[f](s) = \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}, \quad \forall\, s \in \mathbb{R}.$$

□

Example 2.1.4. Let $a > 0$. Compute the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| \leq a \\ 0, & \text{otherwise} \end{cases}$.

Solution. Given function is an even function. Let $s \neq 0$. Then

$$\begin{aligned} F[f](s) &= \sqrt{\frac{2}{\pi}}\int_{0}^{a}(a^2 - x^2)\cos(sx)dx \\ &= \sqrt{\frac{2}{\pi}}\left[a^2\frac{\sin sx}{s} - x^2\frac{\sin sx}{s} + 2\int x\frac{\sin sx}{s}dx\right]_0^a \\ &= \sqrt{\frac{2}{\pi}}\left[a^2\frac{\sin sx}{s} - x^2\frac{\sin sx}{s} - 2x\frac{\cos sx}{s^2} + 2\frac{\sin sx}{s^3}\right]_0^a \\ &= \sqrt{\frac{2}{\pi}}\left[a^2\frac{\sin sa}{s} - a^2\frac{\sin sa}{s} - 2a\frac{\cos sa}{s^2} + 2\frac{\sin sa}{s^3}\right] \end{aligned}$$

$$= \sqrt{\frac{2}{\pi}} 2 \left(\frac{-as\cos sa + \sin sa}{s^3} \right), \quad s \neq 0.$$

For $s = 0$, we have

$$F[f](0) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)\, dx = \sqrt{\frac{2}{\pi}} \left[a^2 x - \frac{x^3}{3} \right]_0^a = \sqrt{\frac{2}{\pi}} \frac{2a^3}{3}.$$

Thus,

$$F[f](s) = \begin{cases} 2\sqrt{\frac{2}{\pi}} \left(\frac{-as\cos sa}{s^3} + \frac{\sin sa}{s^3} \right), & s \neq 0 \\ \frac{2}{3}\sqrt{\frac{2}{\pi}} a^3, & s = 0 \end{cases}.$$

□

Example 2.1.5. Evaluate $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx$.

Solution. We have $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx = 2\int_0^{\infty} e^{-x^2}\, dx$. Let $I = \displaystyle\int_0^{\infty} e^{-x^2} dx$. Then

$$I^2 = \left(\int_0^{\infty} e^{-x^2}\, dx \right)^2 = \left(\int_0^{\infty} e^{-x^2}\, dx \right) \left(\int_0^{\infty} e^{-y^2}\, dy \right) = \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)}\, dxdy.$$

Take $x = r\cos\theta$ and $y = r\sin\theta$. Then $dxdy = \frac{\partial(x,y)}{\partial(r,\theta)}\, dr\, d\theta = r\, dr\, d\theta$. Since $x \geq 0$ and $y \geq 0$ (as the integral are from 0 to ∞), we have $r \geq 0$ and θ will vary from 0 to $\frac{\pi}{2}$. Hence the above equation will be

$$I^2 = \int_0^{\infty} \int_0^{\frac{\pi}{2}} e^{-r^2}\, r\, dr\, d\theta = \left(\int_0^{\frac{\pi}{2}} d\theta \right) \left(\int_0^{\infty} re^{-r^2} dr \right) = \frac{\pi}{2} \int_0^{\infty} re^{-r^2} dr.$$

Take $r^2 = u$. Then $2r\, dr = du \Rightarrow rdr = \frac{du}{2}$. Since in the above integral r goes from 0 to ∞, u also ranges from 0 to ∞. Thus

$$I^2 = \frac{\pi}{2} \int_0^{\infty} e^{-u} \frac{du}{2} = \frac{\pi}{4} [-e^{-u}]_0^{\infty} = \frac{\pi}{4}.$$

This gives $I = \dfrac{\sqrt{\pi}}{2}$ and hence $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$. □

Example 2.1.6. If $a > 0$, then compute the Fourier transform of the function $f(x) = e^{-ax^2}$.

Solution. The function f is even. Let $s \neq 0$. Then

$$\begin{aligned} F[f](s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} \cos sx\, dx \qquad (\because e^{-ax^2} \sin sx \text{ is odd}) \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax^2} \cos sx\, dx \end{aligned}$$

Take $I(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax^2}\cos sx dx$. Then

$$\begin{aligned}\frac{dI}{ds} &= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax^2}(-\sin sx)x\,dx\\ &= -\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax^2}x\sin sx\,dx\\ &= -\sqrt{\frac{2}{\pi}}\left[\sin sx\int xe^{-ax^2}\,dx-\int\left(s\cos sx\int xe^{-ax^2}\,dx\right)\,dx\right]_0^\infty\\ &= -\sqrt{\frac{2}{\pi}}\left[-\frac{\sin sx}{2a}e^{-ax^2}+\int\frac{s\cos sx}{2a}e^{-ax^2}\,dx\right]_0^\infty\\ &= -\frac{s}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax^2}\cos sx\,dx=-\frac{s}{2a}I.\end{aligned}$$

For $\int xe^{-ax^2}\,dx$, take $x^2=t$. Then $x\,dx=\frac{dt}{2}$. Therefore

$$\begin{aligned}&\int xe^{-ax^2}\,dx\\ &=\int e^{-at}\,\frac{dt}{2}\\ &=\frac{1}{2}\frac{e^{-at}}{-a}\\ &=-\frac{1}{2a}e^{-ax^2}.\end{aligned}$$

Therefore

$$\frac{dI}{ds}=-\frac{s}{2a}I\Rightarrow\frac{dI}{I}=-\frac{s}{2a}ds\Rightarrow\log I=-\frac{s^2}{4a}+\log c.$$

Therefore

$$I(s)=ce^{-\frac{s^2}{4a}}.$$

$$\text{Now, } c=I(0)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax^2}\,dx=\sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{a}}\frac{1}{2}=\frac{1}{\sqrt{2a}}.$$

For $\int_0^\infty e^{-ax^2}\,dx$, take $\sqrt{a}x=t$. Then $\int_0^\infty e^{-t^2}\,\frac{dt}{\sqrt{a}}=\frac{1}{2}\sqrt{\frac{\pi}{a}}$.

Hence, $I(s)=\frac{1}{\sqrt{2a}}e^{-s^2/4a}$, i.e.

$$\boxed{F[f](s)=F\left[e^{-ax^2}\right](s)=\frac{1}{\sqrt{2a}}e^{-\frac{s^2}{4a}}.}$$

□

Corollary 2.1.7. *Taking $a=2$ in the above example, we get*

$$F[e^{-\frac{x^2}{2}}](s)=e^{-\frac{s^2}{2}}.$$

Thus, example of a function whose Fourier transform is itself is $f(x)=e^{-\frac{x^2}{2}}$, i.e. $F[f]=f$.

2.1.2 Properties of Fourier Transform

Let $a\in\mathbb{R}$ and $f\in L^1(\mathbb{R})$. Define

$$T_af(x)=f(x-a).$$

Let $a>0$ and $f\in L^1(\mathbb{R})$. Define

$$M_af(x)=f(ax).$$

The following are some of the properties satisfied by Fourier Transform.

1. The Fourier transform is a linear map, i.e.,

$$F[\alpha f+\beta g]=\alpha F[f]+\beta F[g], \quad \forall\, f,g\in L^1(\mathbb{R}),\ \forall\, \alpha,\beta\in\mathbb{C}.$$

Proof. For all $s\in\mathbb{R}$, we have

$$\begin{aligned}
F[\alpha f+\beta g](s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\alpha f+\beta g)(x)e^{-isx}dx\\
&= \frac{1}{\sqrt{2\pi}}\alpha\int_{-\infty}^{\infty}f(x)e^{-isx}dx+\frac{1}{\sqrt{2\pi}}\beta\int_{-\infty}^{\infty}g(x)e^{-isx}dx\\
&= \alpha F[f](s)+\beta F[g](s)\\
&= (\alpha F[f]+\beta F[g])(s).
\end{aligned}$$

Therefore $F[\alpha f+\beta g]=\alpha F[f]+\beta F[g]$, i.e., Fourier transform is a linear map. □

2. Let $a\in\mathbb{R}$ and $f\in L^1(\mathbb{R})$. Then

$$\begin{aligned}
F[T_a f] &= e^{-ias}F[f](s), \quad s\in\mathbb{R}\\
F[M_a f] &= \frac{1}{a}M_{\frac{1}{a}}F[f](s), \quad (a>0)
\end{aligned}$$

Proof. For all $s\in\mathbb{R}$,

$$\begin{aligned}
F[T_a f](s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}T_a f(x)e^{-isx}dx\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x-a)e^{-isx}dx\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-is(t+a)}dt \qquad (\text{Taking } x-a=t)\\
&= e^{-isa}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-ist}dt\\
&= e^{-isa}F[f](s).
\end{aligned}$$

Therefore $F[T_a f](s)=e^{-isa}F[f](s)$.
Also, for $s\in\mathbb{R}$,

$$\begin{aligned}
F[M_a f](s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}M_a f(x)e^{-isx}dx\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(ax)e^{-isx}dx\\
&= \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\frac{s}{a}t}dt \qquad (\text{Taking } ax=t)\\
&= \frac{1}{a}F[f]\left(\frac{s}{a}\right)\\
&= \frac{1}{a}M_{\frac{1}{a}}F[f](s).
\end{aligned}$$

Therefore $F[M_a f]=\frac{1}{a}M_{\frac{1}{a}}F[f]$. □

3. If $x^n f(x) \in L^1(\mathbb{R})$, then

$$F[x^n f(x)](s) = i^n \frac{d^n}{ds^n} F[f](s).$$

Proof. Here,

$$\begin{aligned}
\frac{d^n}{ds^n} F[f](s) &= \frac{d^n}{ds^n}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}dx\right] \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)(-i)^n x^n e^{-isx}dx \\
&= (-i)^n \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)x^n e^{-isx}dx \\
&= (-i)^n F[f(x)x^n](s).
\end{aligned}$$

Therefore

$$F[x^n f(x)](s) = i^n \frac{d^n}{ds^n} F[f](s) \qquad \left(\because \frac{1}{-i} = i\right).$$

□

Example 2.1.8. Compute the Fourier transform of $x^2 e^{-ax^2}$.

Solution. By Property 3 above, we have

$$\begin{aligned}
F[x^2 e^{-ax^2}](s) &= i^2 \frac{d^2}{ds^2} F\left[e^{-ax^2}\right](s) \\
&= i^2 \frac{d^2}{ds^2}\left[\frac{1}{\sqrt{2a}} e^{-\frac{s^2}{4a}}\right] && \text{(by Example 2.1.6)} \\
&= (-1)\frac{1}{\sqrt{2a}}\frac{d}{ds}\left[e^{-\frac{s^2}{4a}}\frac{-2s}{4a}\right] \\
&= \frac{1}{2a\sqrt{2a}}\left[e^{-\frac{s^2}{4a}}(1) + s e^{-\frac{s^2}{4a}}\frac{-2s}{4a}\right] \\
&= \frac{1}{2a\sqrt{2a}}\left[e^{-\frac{s^2}{4a}} - \frac{s^2}{2a} e^{-\frac{s^2}{4a}}\right] \\
&= \frac{e^{-\frac{s^2}{4a}}}{2a\sqrt{2a}} - \frac{s^2}{4a^2}\frac{e^{-\frac{s^2}{4a}}}{\sqrt{2a}}.
\end{aligned}$$

□

3. If f is n-times differentiable function and $f^{(r)}(x) \to 0$ as $|x| \to \infty$ for $r = 0, 1, \ldots, n-1$, then

$$F[f^{(n)}](s) = (is)^n F[f](s).$$

Proof. Here

$$F[f^{(n)}](s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^{(n)}(x)e^{-isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[e^{-isx} f^{(n-1)}(x) + \int f^{(n-1)}(x) ise^{-isx} dx \right]_{-\infty}^{\infty}$$
$$= is \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^{(n-1)}(x) e^{-isx}\, dx$$
$$= isF[f^{(n-1)}](s)$$

Repeating this process n-times, we get the required result. Hence,

$$F[f^{(n)}](s) = (is)^n F[f](s).$$

□

Example 2.1.9. Compute the Fourier transform of $xe^{-\frac{x^2}{4}}$.

Solution. By Example 2.1.6, we know that $F\left[e^{-\frac{x^2}{4}}\right](s) = \sqrt{2}e^{-s^2}$ (taking $a = \frac{1}{4}$). Also, if $f(x) = e^{-\frac{x^2}{4}}$, then $f'(x) = -\frac{x}{2}e^{-\frac{x^2}{4}}$. Since $f(x) \to 0$ as $|x| \to \infty$, by Property 3 above and linearity of Fourier transform, we have $F[f'(x)](s) = (is)F[f](s)$, i.e.

$$F\left[xe^{-\frac{x^2}{4}}\right](s) = -2F\left[-\frac{x}{2}e^{-\frac{x^2}{4}}\right](s)$$
$$= -2(is)F\left[e^{-\frac{x^2}{4}}\right](s)$$
$$= -2(is)\left(\sqrt{2}e^{-s^2}\right) = -2i\sqrt{2}se^{-s^2}.$$

□

2.2 Fourier Integral Representation

2.2.1 Inverse Fourier Transform

Definition 2.2.1 (Inverse Fourier Transform)**.** Let $f \in L^1(\mathbb{R})$. Then the *inverse Fourier transform* of f, $F^{-1}[f]$, is

$$F^{-1}[f](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{isx} ds.$$

Given $f \in L^1(\mathbb{R})$, we can find its Fourier transform. If we are given a function g, does there exists a function f whose Fourier transform is g? The following theorem answers this question.

Theorem 2.2.2 (Fourier Inversion Theorem)**.** *Let $f \in L^1(\mathbb{R})$, and let the Fourier transform $F[f]$ of f be in $L^1(\mathbb{R})$. Then*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f](s) e^{isx} ds, \quad \textit{a.e. on } \mathbb{R}. \tag{2.2}$$

It follows from the Fourier inversion theorem that if $f \in L^1(\mathbb{R})$ and if $F[f] \in L^1(\mathbb{R})$, then $f(x) = F^{-1}[F[f]](x)$, a.e. on $\mathbb{R}$. Therefore,

$$\begin{aligned} f(x) &= F^{-1}[F[f]](x) \quad \text{a.e. } x \quad (x \in \mathbb{R}) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f](s) e^{isx}\, ds \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ist}\, dt \right) e^{isx}\, ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-is(t-x)}\, ds\, dt \quad \text{a.e. } x \quad (x \in \mathbb{R}). \end{aligned}$$

Therefore,

$$\boxed{f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-is(t-x)}\, ds\, dt \quad \text{a.e. } x \quad (x \in \mathbb{R})}.$$

The above equation is called *Fourier integral representation* of f.

Theorem 2.2.3 (Fourier Integral Theorem)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be a map. If $f \in L^1(\mathbb{R})$ and if satisfies the Dirichlet conditions on every compact interval of $\mathbb{R}$, then*

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{-is(t-x)} ds dt = \frac{f(x^+) + f(x^-)}{2}.$$

Definition 2.2.4. Let $f \in L^1[0,\infty)$. Then the *Fourier cosine transform* $F_c[f]$ of f is defined as

$$F_c[f][s] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\, dx \quad (s \in \mathbb{R}). \tag{2.3}$$

Definition 2.2.5. Let $f \in L^1[0,\infty)$. Then the *Fourier sine transform* $F_s[f]$ of f is defined as

$$F_s[f][s] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx \quad (s \in \mathbb{R}). \tag{2.4}$$

Remark 2.2.6. We have the following observations:

1. If f is an even integrable function, then $F[f] = F_c[f]$.
2. If f is an odd integrable function, then $F[f] = -iF_s[f]$.

Ex Prove the following properties of Fourier transform:

1. $F[f(x)\cos ax](s) = \frac{1}{2}[F[f](s+a) + F[f](s-a)]$.
2. $F\left[\int_a^x f(t)dt\right](s) = -i\frac{F[f](s)}{s}$.
3. $F_s[\alpha f + \beta g] = \alpha F_s[f] + \beta F_s[g]$.
4. $F_s[M_a f] = \frac{1}{a} F_s[M_{\frac{1}{a}} f]$.
5. $F_c[M_a f] = \frac{1}{a} F_c[M_{\frac{1}{a}} f]$.
6. $F_s[f(x)\sin ax](s) = \frac{1}{2}[F_c[f](s-a) - F_c[f](s+a)]$.

7. $F_c[f(x)\sin ax](s) = \frac{1}{2}[F_s[f](s+a) - F_c[f](s-a)]$.
8. $F_s[f(x)\cos ax](s) = \frac{1}{2}[F_s[f](s+a) + F_s[f](s+a)]$.

Example 2.2.7. Compute the Fourier transform of $e^{-|x|}$.

Solution. Since the given function is even, its Fourier transform is same as its Fourier cosine transform. Therefore

$$\begin{aligned} F[f](s) = F_c[f](s) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos sx\,dx \\ &= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+s^2}(-\cos sx + s\sin sx)\right]_0^\infty \\ &= \sqrt{\frac{2}{\pi}}\left[0 - \left(\frac{-1}{1+s^2}\right)\right]. \end{aligned}$$

Hence,

$$F[f](s) = F_c[f](s) = \frac{1}{1+s^2}\sqrt{\frac{2}{\pi}}.$$

□

Example 2.2.8. Compute the Fourier transform of $f(x) = \begin{cases} e^{-x}, & x > 0 \\ -e^{x}, & x \le 0 \end{cases}$.

Solution. Since the given function is odd, its Fourier transform is related to its Fourier sine transform and given by $F[f](s) = -iF_s[f]$. Therefore

$$\begin{aligned} F[f](s) = -iF_s[f](s) &= -i\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx \\ &= -i\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin sx\,dx \\ &= -i\sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+s^2}(-\sin sx - s\cos sx)\right]_0^\infty \\ &= -i\sqrt{\frac{2}{\pi}}\left[0 - \left(\frac{-1}{1+s^2}(-s)\right)\right] = \frac{-is}{1+s^2}\sqrt{\frac{2}{\pi}}. \end{aligned}$$

Hence,

$$F[f](s) = -iF_s[f](s) = -\frac{is}{1+s^2}\sqrt{\frac{2}{\pi}}.$$

□

Remark 2.2.9. Note that in computation of the Fourier sine transform, one can take any half interval i.e. either $[0,\infty)$ or $(-\infty,0]$. For instance, in the above example we have

$$F[f](s) = -iF_s[f](s) = -i\sqrt{\frac{2}{\pi}}\int_{-\infty}^0 f(x)\sin sx\,dx$$

$$= -i\sqrt{\frac{2}{\pi}}\int_{-\infty}^{0} -e^{x}\sin sx\,dx$$
$$= i\sqrt{\frac{2}{\pi}}\left[\frac{e^{x}}{1+s^{2}}(\sin sx - s\cos sx)\right]_{0}^{\infty}$$
$$= i\sqrt{\frac{2}{\pi}}\left[\frac{1}{1+s^{2}}(-s)\right] = \frac{-is}{1+s^{2}}\sqrt{\frac{2}{\pi}}.$$

Example 2.2.10. Let $f \in L^1(0,\infty)$ be twice differentiable. If $f^{(r)}(x) \to 0$ as $x \to \infty$, $r = 0,1$, then

1. $F_s[f^{(2)}](s) = s\sqrt{\frac{2}{\pi}}f(0) - s^2F_s[f](s)$ and
2. $F_c[f^{(2)}](s) = \sqrt{\frac{2}{\pi}}f^{(1)}(0) - s^2F_c[f](s)$.

Solution. 1. By the definition of the Fourier sine transform of f,

$$\begin{aligned}
F_s[f^{(2)}](s) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f^{(2)}(x)\sin sxdx \\
&= \sqrt{\frac{2}{\pi}}\left(\left[f^{(1)}(x)\sin sx\right]_0^\infty - \int_0^\infty f^{(1)}(x)s\cos sxdx\right) \\
&= -s\sqrt{\frac{2}{\pi}}\int_0^\infty f^{(1)}(x)\cos sxdx \\
&= -s\sqrt{\frac{2}{\pi}}\left([f(x)\cos sx]_0^\infty + \int_0^\infty f(x)s\sin sxdx\right) \\
&= s\sqrt{\frac{2}{\pi}}f(0) - s^2F_s[f](s).
\end{aligned}$$

2. By the definition of the Fourier cosine transform of f,

$$\begin{aligned}
F_s[f^{(2)}](s) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f^{(2)}(x)\cos sxdx \\
&= \sqrt{\frac{2}{\pi}}\left(\left[f^{(1)}(x)\cos sx\right]_0^\infty + \int_0^\infty f^{(1)}(x)s\sin sxdx\right) \\
&= \sqrt{\frac{2}{\pi}}\left(f^{(1)}(0) + s\int_0^\infty f^{(1)}(x)\sin sxdx\right) \\
&= \sqrt{\frac{2}{\pi}}\left(f^{(1)}(0) + s\,[f(x)\sin sx]_0^\infty - s\int_0^\infty f(x)s\cos sxdx\right) \\
&= \sqrt{\frac{2}{\pi}}f^{(1)}(0) - s^2F_c[f](s).
\end{aligned}$$

□

Examples 2.2.11. Let $a > 0$. Show that

1. $F[e^{-a|x|}](s) = F_c[e^{-ax}](s) = \sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}$.

2. $F_s[e^{-ax}](s) = \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}$.
3. $F[\chi_{[-a,a]}](s) = \sqrt{\frac{2}{\pi}} \frac{\sin sa}{s}$.

Definition 2.2.12. Let $f \in L^1(0,\infty)$.

1. The *inverse Fourier cosine transform*, $F_c^{-1}[f]$, is defined as

$$F_c^{-1}[f](x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(s) \cos sx dx.$$

2. The *inverse Fourier sine transform*, $F_s^{-1}[f]$, is defined as

$$F_s^{-1}[f](x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(s) \sin sx dx.$$

If $f \in L^1(0,\infty)$, then it follows by Fourier cosine integral representation and Fourier sine integral representation of f that

$$f(x) = F_c^{-1}\left[F_c[f]\right](x) \text{ and } f(x) = F_s^{-1}\left[F_s[f]\right](x), \;\; a.e.$$

2.2.2 Fourier Cosine and Fourier Sine Integral Representation

Let $f \in L^1(\mathbb{R})$ be an even function such that $F[f] \in L^1(\mathbb{R})$. Then by the Fourier integral representation of f, we have

$$\begin{aligned} f(x) =& \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(t) e^{-is(t-x)} \, ds \, dt \\ =& \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(t)[\cos(s(t-x)) - i\sin(s(t-x))] \, ds \, dt \\ =& \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \cos(s(t-x)) \, ds \, dt - i\frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \sin(s(t-x)) \, ds \, dt. \end{aligned}$$

Now,

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \cos(s(t-x)) \, ds \, dt \\ =& \frac{1}{2\pi} \int_{-\infty}^\infty \left(\int_{-\infty}^\infty f(t) \cos(st) \, dt \right) \cos(sx) \, ds + \frac{1}{2\pi} \int_{-\infty}^\infty \left(\int_{-\infty}^\infty f(t) \sin(st) \, dt \right) \sin(sx) \, ds \end{aligned}$$

Since $\sin st$ is an odd function of t and f is an even function of t, we have $\int_{-\infty}^\infty f(t) \sin st \, dt = 0$ and hence the second term in the above equation is zero. Therefore

$$\frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \cos(s(t-x)) \, ds \, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cos(st)\,dt\right)\cos(sx)\,ds$$
$$= \frac{2}{2\pi}\int_{-\infty}^{\infty}\left(\int_{0}^{\infty} f(t)\cos(st)\,dt\right)\cos(sx)\,ds \qquad (\because f(t)\cos st \text{ is an even function of } t)$$
$$= \frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty} f(t)\cos(st)\cos(sx)\,ds\,dt \qquad (\because \cos st\cos sx \text{ is an even function of } s)$$

Now,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\sin(s(t-x))\,ds\,dt$$
$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\sin(st)\,dt\right)\cos(sx)\,ds - \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cos(st)\,dt\right)\sin(sx)\,ds$$

Since $f(t)\sin st$ is an odd function of t, the first term in the above equation is zero. Also, since $\int_{-\infty}^{\infty} f(t)\cos st\,dt$ is an even function of s and $\sin sx$ is an odd function of s, the second integral in the above equation is zero. Therefore,

$$\boxed{f(x) = \frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty} f(t)\cos(st)\cos(sx)\,ds\,dt \quad \text{a.e. on } \mathbb{R}}. \tag{2.5}$$

Equation (2.5) is called the *Fourier cosine integral representation* of f.

Similarly, if $f \in L^1(\mathbb{R})$ is an odd function and if $F[f] \in L^1(\mathbb{R})$, then

$$\boxed{f(x) = \frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty} f(t)\sin(st)\sin(sx)\,ds\,dt \quad \text{a.e. on } \mathbb{R}}. \tag{2.6}$$

Equation (2.6) is called the *Fourier sine integral representation* of f.

If $f \in L^1(0,\infty)$, then it follows by Fourier cosine integral representation and Fourier sine integral representation of f that

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_c[f](s)\cos sx\,ds = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_s[f](s)\sin sx\,ds, \;\; a.e}.$$

Example 2.2.13. Obtain the Fourier transform and the Fourier integral representation of $f = \chi_{[-1,1]}$. Hence evaluate the integrals $\int_{0}^{\infty}\frac{\sin x\cos(\lambda x)}{x}\,dx$ and $\int_{0}^{\infty}\frac{\sin x}{x}\,dx$.

Solution. Here $f = \chi_{[-1,1]}$, i.e. $f(x) = \begin{cases} 1, & x \in [-1,1] \\ 0, & \text{otherwise} \end{cases}$. Then for $s \in \mathbb{R}$, $s \neq 0$

$$F[f](s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-isx}\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-isx}}{-is}\right]_{-1}^{1}$$

$$= \frac{-1}{is\sqrt{2\pi}}[e^{-is} - e^{is}] = \sqrt{\frac{2}{\pi}}\frac{\sin s}{s}, \qquad s \neq 0$$

For $s = 0$,

$$F[f](0) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} 1\, dx = \sqrt{\frac{2}{\pi}}.$$

Since f is an even function, the Fourier integral representation of f is the Fourier cosine integral representation of f. Hence for a.e. x

$$\begin{aligned} f(x) &= \frac{2}{\pi}\int_0^\infty \int_0^\infty f(t)\cos(st)\cos(sx)\, ds\, dt \\ &= \frac{2}{\pi}\int_0^\infty \cos(sx)\left(\int_0^1 \cos(st)\, dt\right) ds \\ &= \frac{2}{\pi}\int_0^\infty \left[\frac{\sin st}{s}\right]_0^1 \cos sx\, ds \\ &= \frac{2}{\pi}\int_0^\infty \frac{\sin s \cos sx}{s}\, ds \quad \text{ a.e. on } \mathbb{R}. \end{aligned}$$

By the Fourier integral theorem, we have

$$\frac{2}{\pi}\int_0^\infty \frac{\sin s \cos sx}{s}\, ds = \frac{f(x^+) + f(x^-)}{2}.$$

The function f is discontinuous at ± 1 only and the average of left and right limit of f at these points is $\frac{1}{2}$. Hence,

$$\frac{2}{\pi}\int_0^\infty \frac{\sin s \cos sx}{s}\, ds = \begin{cases} f(x), & x \neq \pm 1 \\ \frac{1}{2}, & x = \pm 1 \end{cases}$$

$$\text{i.e. } \frac{2}{\pi}\int_0^\infty \frac{\sin s \cos sx}{s}\, ds = \begin{cases} 1, & x \in (-1,1) \\ \frac{1}{2}, & x = \pm 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore \int_0^\infty \frac{\sin x \cos \lambda x}{x}\, dx = \begin{cases} \frac{\pi}{2}, & |\lambda| < 1 \\ \frac{\pi}{4}, & |\lambda| = 1 \\ 0, & |\lambda| > 1. \end{cases}$$

In particular, taking $x = 0$, we get $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$. □

Example 2.2.14. Compute the Fourier sine transform of $f(x) = e^{-\beta x}$, $x > 0$. Also show that $\displaystyle\frac{\pi}{2}e^{-\beta u} = \int_0^\infty \frac{x \sin ux}{\beta^2 + x^2}\, dx \quad (u > 0)$.

Solution. Fourier sine transform of f is given by

$$F_s[f](s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\beta x} \sin sx \, dx$$
$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-\beta x}}{s^2+\beta^2} (-\beta \sin sx - s\cos sx) \right]_0^\infty$$
$$= \sqrt{\frac{2}{\pi}} \frac{s}{s^2+\beta^2}.$$

Since f is continuous, it follows by Fourier integral theorem that, for $x > 0$

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \sin st \sin sx \, ds \, dt$$
$$\therefore \frac{\pi}{2} e^{-\beta x} = \int_0^\infty \left(\int_0^\infty e^{-\beta t} \sin st \, dt \right) \sin sx \, ds$$
$$= \int_0^\infty \left[\frac{e^{-\beta x}}{s^2+\beta^2} (-\beta \sin sx - s\cos sx) \right]_0^\infty \sin sx \, ds$$
$$= \int_0^\infty \frac{s \sin sx}{\beta^2+s^2} \, ds.$$

Therefore

$$\int_0^\infty \frac{x \, \sin ux}{\beta^2 + x^2} \, dx = \frac{\pi}{2} e^{-\beta u}.$$

□

Example 2.2.15. Find the Fourier transform of $f(x) = \chi_{[-1,1]}(1-x^2)$ and hence evaluate the integral $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) \, dx$.

Solution. Since f is continuous, by Fourier integral theorem, the Fourier integral representation of f is f. Since f is an even function, the Fourier integral representation of f is the Fourier cosine integral representation of f. Therefore

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos st \cos sx \, ds \, dt$$
$$= \frac{2}{\pi} \int_0^\infty \left(\int_0^1 (1-t^2) \cos st \, dt \right) \cos sx \, ds$$
$$= \frac{2}{\pi} \int_0^\infty \left\{ \left[\frac{\sin st}{s} \right]_0^1 - \left[t^2 \frac{\sin st}{s} - 2t \frac{-\cos st}{s^2} + 2\frac{-\sin st}{s^3} \right]_0^1 \right\} \cos sx \, ds$$
$$= \frac{2}{\pi} \int_0^\infty \left\{ \frac{\sin s}{s} - \left[\frac{\sin s}{s} + 2\frac{\cos s}{s^2} - 2\frac{\sin s}{s^3} \right] \right\} \cos sx \, ds$$
$$= -\frac{4}{\pi} \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3} \right) \cos sx \, ds.$$

Now,

$$\frac{3}{4} = f\left(\frac{1}{2}\right) = -\frac{4}{\pi} \int_0^\infty \frac{s\sin s - \sin s}{s^3} \cos\left(\frac{s}{2}\right) \, ds.$$

Hence,

$$\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) \, dx = -\frac{3\pi}{16}.$$

□

2.3 Parseval's identity

Theorem 2.3.1 (Parseval's Identity for Fourier transform)**.** *Let $f,g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and let $F[f], F[g] \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then*

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} F[f](x)\overline{F[g](x)}dx.$$

Proof. We see that

$$\begin{aligned}
\int_{-\infty}^{\infty} F[f](x)\overline{F[g](x)}dx &= \int_{-\infty}^{\infty} F[f](x)\overline{\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(s)e^{-isx}ds\right)}dx \\
&= \int_{-\infty}^{\infty} F[f](x)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{g(s)}e^{isx}ds\right)dx \\
&= \int_{-\infty}^{\infty} \overline{g(s)}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f](x)e^{isx}dx\right)ds \\
&= \int_{-\infty}^{\infty} \overline{g(s)}f(s)ds. \qquad \left(\because f(t) = F^{-1}\left[F[f]\right](t)\right)
\end{aligned}$$

This proves the result. □

Corollary 2.3.2. *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $F[f] \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then*

$$\int_{-\infty}^{\infty} |f(x)|^2dx = \int_{-\infty}^{\infty} |F[f](x)|^2dx.$$

Theorem 2.3.3 (Parseval's Identity)**.** *Let $f,g \in L^1(0,\infty) \cap L^2(0,\infty)$, and let $F_c[f]$, $F_s[f]$, $F_c[g]$, $F_s[g]$ be in $L^1(0,\infty) \cap L^2(0,\infty)$. Then*

$$\int_0^{\infty} f(x)\overline{g(x)}\,dx = \int_0^{\infty} F_c[f](x)\overline{F_c[g](x)}\,dx = \int_0^{\infty} F_s[f](x)\overline{F_s[g](x)}\,dx.$$

Proof. Let $f,g \in L^2(0,\infty)$. Extend f and g as even functions on $\mathbb{R}$. Then by the Parseval's identity

$$\int_{\mathbb{R}} f(x)\overline{g(x)}dx = \int_{\mathbb{R}} F[f](s)\overline{F[g](s)}ds.$$

But $F[f](s) = F_c[f](s)$ and $F[g](s) = F_c[g](s)$. Since left hand side of the above equation is an even function and both $F_c[f]$ and $F_c[g]$ are even functions s, we have

$$\int_0^{\infty} f(x)\overline{g(x)}dx = \int_0^{\infty} F_c[f](s)\overline{F_c[g](s)}ds.$$

Now, extend f and g as odd functions on $\mathbb{R}$. Then by the Parseval's identity

$$\int_{\mathbb{R}} f(x)\overline{g(x)}dx = \int_{\mathbb{R}} F[f](s)\overline{F[g](s)}ds.$$

But $F[f](s) = -iF_s[f](s)$ and $F[g](s) = -iF_s[g](s)$. Since left hand side of the above equation is an even function and both $F_c[f]$ and $F_c[g]$ are odd functions s, we have

$$\begin{aligned}\int_0^\infty f(x)\overline{g(x)}dx &= \int_0^\infty -iF_s[f](s)\overline{(-iF_s[g](s))}ds \\ &= \int_0^\infty F_s[f](s)\overline{F_s[g](s)}ds.\end{aligned}$$

□

Example 2.3.4. Evaluate $\displaystyle\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx,\ a,b>0$

Solution. Let $f(x) = e^{-ax}$, $x > 0$ and $g(x) = e^{-bx}$, $x > 0$. Then

$$F_s[e^{-ax}](s) = \sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2} \text{ and } F_s[e^{-bx}](s) = \sqrt{\frac{2}{\pi}}\frac{s}{s^2+b^2}.$$

By using Parseval's identity, we have

$$\int_0^\infty e^{-ax}e^{-bx}\,dx = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+a^2}\frac{s}{s^2+b^2}\,ds.$$

Therefore

$$\begin{aligned}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\,ds &= \frac{\pi}{2}\int_0^\infty e^{-(a+b)x}\,dx \\ &= \frac{\pi}{2}\left[\frac{e^{-(a+b)}}{-(a+b)}\right]_0^\infty = \frac{\pi}{2}\frac{1}{a+b}.\end{aligned}$$

Hence, $\displaystyle\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{2(a+b)}.$ □

Example 2.3.5. Evaluate $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)},\ a,b>0$

Solution. Let $f(x) = e^{-ax}$, $x > 0$ and $g(x) = e^{-bx}$, $x > 0$. Then

$$F_c[e^{-ax}](s) = \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2} \text{ and } F_c[e^{-bx}](s) = \sqrt{\frac{2}{\pi}}\frac{b}{s^2+b^2}.$$

By using Parseval's identity, we have

$$\int_0^\infty e^{-ax}e^{-bx}\,dx = \frac{2}{\pi}\int_0^\infty \frac{a}{s^2+a^2}\frac{b}{s^2+b^2}\,ds.$$

Therefore

$$\begin{aligned}\int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}\,ds &= \frac{\pi}{2ab}\int_0^\infty e^{-(a+b)x}\,dx \\ &= \frac{\pi}{2ab}\left[\frac{e^{-(a+b)}}{-(a+b)}\right]_0^\infty = \frac{\pi}{2ab}\frac{1}{(a+b)}.\end{aligned}$$

Hence, $\displaystyle\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{2ab(a+b)}.$ □

Example 2.3.6. Evaluate $\int_{-\infty}^{\infty} \frac{\sin at}{t(t^2+b^2)} dt$, where $a, b > 0$.

Solution. Let $f = \chi_{[-a,a]}$, i.e. $f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$ and $g(x) = e^{-b|x|}$, $x > 0$. Then

$$F_c[f](s) = \sqrt{\frac{2}{\pi}} \frac{\sin as}{s} \quad \text{and} \quad F_c[g](s) = \sqrt{\frac{2}{\pi}} \frac{b}{s^2+b^2}.$$

By Parseval's identity

$$\begin{aligned}
\int_{-a}^{a} 1\, e^{-b|x|}\, dx &= \frac{2b}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s(s^2+b^2)}\, ds \\
\Rightarrow 2\int_{0}^{a} e^{-bx}\, dx &= \frac{2b}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s(s^2+b^2)}\, ds \\
\Rightarrow \int_{-\infty}^{\infty} \frac{\sin as}{s(s^2+b^2)}\, ds &= \frac{\pi}{b} \int_{0}^{a} e^{-bx}\, dx \\
&= \frac{\pi}{b} \left[\frac{e^{-bx}}{-b} \right]_0^a = \frac{\pi}{b^2}(-e^{-ba}+1).
\end{aligned}$$

Hence, $\int_{-\infty}^{\infty} \frac{\sin at}{t(t^2+b^2)}\, dt = \frac{\pi}{b^2}(1-e^{-ba})$. □

Example 2.3.7. Evaluate $\int_{0}^{\infty} \frac{\sin at}{t(t^2+b^2)} dt$, $a, b > 0$.

Solution. Same as above. Answer is $\frac{\pi}{2b^2}(1-e^{-ba})$. □

Example 2.3.8. Evaluate $\int_{-\infty}^{\infty} \left(\frac{\sin at}{t}\right)^2 dt$, $a > 0$.

Solution. Let $f(x) = \chi_{[-a,a]}(x)$. Then $F[f](s) = \sqrt{\frac{2}{\pi}} \frac{\sin as}{s}$. Using Parseval's identity, we get

$$\begin{aligned}
\int_{-a}^{a} 1\, dx &= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds \\
\Rightarrow \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds &= \frac{\pi}{2} [x]_{-a}^{a} = a\pi.
\end{aligned}$$

Hence, $\int_{-\infty}^{\infty} \left(\frac{\sin at}{t}\right)^2 dt = a\pi$. □

Example 2.3.9. Evaluate $\int_{0}^{\infty} \left(\frac{\sin at}{t}\right)^2 dt$, $a > 0$.

Solution. Same as above. Answer is $\frac{a\pi}{2}$. □

Example 2.3.10. Evaluate $\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt$ and $\int_{0}^{\infty} \frac{\sin t}{t}\, dt$.

Solution. Same as above. Answer is $\frac{\pi}{2}$ for both the integrals. □

Ex

1. Find the Fourier transform of the following functions.
   (a) $f(x) = \begin{cases} x, & |x| \leq a \\ 0, & |x| > a, \end{cases}$ where $a > 0$.
   (b) $f(x) = \begin{cases} x^2, & |x| \leq a \\ 0, & |x| > a, \end{cases}$ where $a > 0$.
   (c) $f(x) = \begin{cases} a - |x|, & |x| \leq a \\ 0, & |x| > a, \end{cases}$ where $a > 0$.
2. Find the Fourier sine transform of the following functions.
   (a) $f(x) = \frac{x}{1+x^2}$.
   (b) $f(x) = e^{-ax}$, where $a > 0$.
   (c) $f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \\ 0, & x > 2. \end{cases}$
   (d) $f(x) = \chi_{[0,a]}$, where $a > 0$.
   (e) $f(x) = \frac{e^{-ax}}{x}$, where $a > 0$.
   (f) $f(x) = \frac{1}{x}$.
   (g) $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x \geq a \end{cases}$
3. Find the Fourier cosine transform of the following functions.
   (a) $f(x) = e^{-ax}$, where $a > 0$.
   (b) $f(x) = \chi_{[0,a]}$, where $a > 0$.
   (c) $f(x) = \frac{1}{1+x^2}$.
   (d) $f(x) = \begin{cases} x, & 0 \leq x \leq \frac{1}{2} \\ 1 - x, & \frac{1}{2} < x \leq 1 \\ 0, & x > 1. \end{cases}$
   (e) $f(x) = \begin{cases} \cos x, & 0 < x \leq a \\ 0, & x > a. \end{cases}$
4. Find the Fourier cosine integral representation and the Fourier sine integral representation of $f(x) = e^{-\beta x}$. Hence evaluate

   (a) $\int_0^\infty \frac{\cos ux}{u^2+\beta^2} du$; (b) $\int_0^\infty \frac{u \sin ux}{u^2+\beta^2} du$.

5. Let $f = \chi_{[0,\pi]}$. Then find the Fourier sine integral representation of f. Hence evaluate $\int_0^\infty \frac{1-\cos \pi\lambda}{\lambda} \sin \lambda x d\lambda$.
6. Find the Fourier integral representation of $f(x) = \chi_{[-a,a]}(x)(a - |x|)$, where $a > 0$. Hence evaluate $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$.
7. Find f if $\int_0^\infty f(x) \sin sx dx = \begin{cases} 1 & 0 < s < 1 \\ 2 & 1 < s < 2 \\ 0 & s > 2 \end{cases}$.

2.4 Convolution Product

Let $f(x) = \chi_{(0,1)}(x)\frac{1}{\sqrt{x}}$, $x \in \mathbb{R}$. Then $f \in L^1(\mathbb{R})$. But $f^2(x) = \chi_{(0,1)}(x)\frac{1}{x}$ is not in $L^1(\mathbb{R})$. Hence the product of two L^1-functions may not be $L^1(\mathbb{R})$. The following operation (called convolution) on $L^1(\mathbb{R})$ makes $L^1(\mathbb{R})$ an algebra. With this convolution product $L^1(\mathbb{R})$ is a commutative algebra (in fact commutative Banach algebra).

Definition 2.4.1. Let $f, g \in L^1(\mathbb{R})$. Then the convolution product $*$ of f and g is defined as

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)g(x-y)\,dy \quad (x \in \mathbb{R}). \tag{2.7}$$

We first note that $f*g \in L^1(\mathbb{R})$, this will follow by Fubini-Tonelli theorem.

Example 2.4.2. If $f, g \in L^1(\mathbb{R})$, then $f*g \in L^1(\mathbb{R})$.

Solution. Since $f, g \in L^1(\mathbb{R})$,

$$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty \text{ and } \int_{-\infty}^{\infty} |g(x)|\,dx < \infty.$$

We have to prove that $\int_{-\infty}^{\infty} |(f*g)(x)|\,dx < \infty$.

$$\begin{aligned}
\int_{-\infty}^{\infty} |(f*g)(x)|dx &= \int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)g(x-y)\,dy \right| dx \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y)||g(x-y)|\,dx\,dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(y)| \left(\int_{-\infty}^{\infty} |g(x-y)|\,dx \right) dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(y)| \left(\int_{-\infty}^{\infty} |g(t)|\,dt \right) dy \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} |g(t)|\,dt \right) \left(\int_{-\infty}^{\infty} |f(y)|\,dy \right) < \infty.
\end{aligned}$$

Hence $f*g \in L^1(\mathbb{R})$. □

Ex Let $f, g, h \in L^1(\mathbb{R})$. Then show that the convolution product satisfies following properties:

(a) $f*g = g*f$, i.e. convolution product is commutative.
(b) $(f*g)*h = f*(g*h)$, i.e. convolution is associative.
(c) $f*(g+h) = (f*g)+(f*h)$, i.e. convolution is distributive over addition.

Theorem 2.4.3 (Convolution Theorem)**.** *Let $f, g \in L^1(\mathbb{R})$. Then*

$$F[f*g] = F[f]F[g]. \tag{2.8}$$

Dr. P. A. Dabhi lightatinfinite@gmail.com
Dr. Jay Mehta jay_mehta@spuvvn.edu

Proof. Let $s \in \mathbb{R}$. Then

$$\begin{aligned}
F[f*g](s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f*g)(x)e^{-isx}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)g(x-y)dy\right)e^{-isx}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-y)e^{-is(x-y)}dx\right)e^{-isy}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)F[g](s)e^{-isy}dy \\
&= F[g](s)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{-isy}dy \\
&= F[f](s)F[g](s) = (F[f](s)F[g])(s).
\end{aligned}$$

Hence, $F[f*g] = F[f]F[g]$.
In the proof we used Fubini–Tonelli theorem (where?). □

Example 2.4.4. Find f if $\displaystyle\int_{-\infty}^{\infty} f(t)e^{-|x-t|}dt = \phi(x)$.

Solution. Let $g(x) = e^{-|x|}$. Then $F[g](s) = \sqrt{\frac{2}{\pi}}\frac{1}{1+s^2}$. Now

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-|x-t|}dt = \frac{1}{\sqrt{2\pi}}\phi(x),$$

i.e. $(f*g)(x) = \frac{1}{\sqrt{2\pi}}\phi(x)$. Let $F[\phi]$ be the Fourier transform of ϕ. Applying the Fourier transform on both the sides of the above equation and applying the convolution theorem, we get

$$\begin{aligned}
F[f*g](s) &= \frac{1}{\sqrt{2\pi}}F[\phi](s) \\
\text{i.e. } F[f](s)F[g](s) &= \frac{1}{\sqrt{2\pi}}F[\phi](s) \\
\text{i.e. } F[f](s)\sqrt{\frac{2}{\pi}}\frac{1}{1+s^2} &= \frac{1}{\sqrt{2\pi}}F[\phi](s)
\end{aligned}$$

Therefore,

$$\begin{aligned}
F[f](s) &= \frac{1}{2}(1+s^2)F[\phi](s) \\
&= \frac{1}{2}\left[F[\phi](s) + s^2F[\phi](s)\right] \\
&= \frac{1}{2}\left[F[\phi](s) - (is)^2F[\phi](s)\right] \\
&= \frac{1}{2}\left[F[\phi](s) - F[\phi''](s)\right] \qquad \left(\because F[f^{(n)}](s) = (is)^nF[f](s)\right)
\end{aligned}$$

Applying the inverse transform on both sides, we get

$$f(x) = \frac{1}{2}(\phi(x) - \phi''(x)).$$

□

Example 2.4.5. Let $f \in L^1(\mathbb{R})$ be even. Then $F[f] = F^{-1}[f] = F_c[f]$.

Solution. Let $s \in \mathbb{R}$. Then

$$\begin{aligned}
F^{-1}[f](s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos sx\, dx + i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \sin sx\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos sx\, dx && (\because f \text{ is even}) \\
&= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos sx\, dx && (\text{which is } F_c[f]) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx}\, dx \\
&= F[f](s).
\end{aligned}$$

□

Example 2.4.6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{1+x^2}$. Then evaluate $f * f$.

Solution. Let $g(x) = e^{-|x|}$. Then $F[g](s) = \sqrt{\frac{2}{\pi}} \frac{1}{1+s^2} = \sqrt{\frac{2}{\pi}} f(s)$. Since g is an even function, $F[g] = F^{-1}[g]$. Therefore

$$F^{-1}[g] = F[g] = \sqrt{\frac{2}{\pi}} f.$$

So, $F[f] = \sqrt{\frac{\pi}{2}} g$. Then by Convolution theorem,

$$F[f * f] = F[f]F[f] = \frac{\pi}{2} g^2.$$

Applying inverse Fourier transform, we get

$$\begin{aligned}
(f * f)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\pi}{2} e^{-2|s|} e^{isx}\, ds \\
&= \sqrt{\frac{\pi}{2}} \int_{0}^{\infty} e^{-2s} \cos sx\, ds \\
&= \sqrt{\frac{\pi}{2}} \left[\frac{e^{-2s}}{4+x^2} (-2\cos sx + x \sin sx) \right]_0^{\infty} \\
&= \sqrt{\frac{\pi}{2}} \frac{2}{4+x^2} = \frac{\sqrt{2\pi}}{x^2+4}.
\end{aligned}$$

□

Example 2.4.7. Find f if $\int_0^{\infty} f(s) \cos sx ds = e^{-ax}$, where $a > 0$.

Here

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(s) \cos sx ds = \sqrt{\frac{2}{\pi}} e^{-ax}.$$

Applying the inverse Fourier cosine transform, we get

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} e^{-as} \cos sx ds = \frac{2}{\pi} \frac{a}{a^2+x^2} = \frac{2a}{\pi(a^2+x^2)}.$$

2.5 Applications of Fourier Transform

Example 2.5.1. Solve $y'' - y = e^{-\alpha|x|}$, where $\alpha > 0$, $\alpha \neq 1$, subject to the conditions $y(x), y'(x) \to 0$ as $|x| \to \infty$.

Solution. Let Y be the Fourier transform of y. Applying the Fourier transform to the equation $y'' - y = e^{-\alpha|x|}$, we get

$$F[y''](s) - F[y](s) = F\left[e^{-\alpha|x|}\right](s).$$

Now, $F[y''](s) = (is)^2 F[y](s) = -s^2[y](s)$. Therefore,

$$-s^2 Y - Y = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + s^2}.$$

Therefore

$$Y = -\sqrt{\frac{2}{\pi}} \frac{1}{1+s^2} \frac{\alpha}{\alpha^2 + s^2} = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 - 1} \left[\frac{1}{\alpha^2 + s^2} - \frac{1}{1+s^2}\right].$$

Applying the inverse Fourier transform, we get

$$\begin{aligned}
y &= \frac{\alpha}{\alpha^2 - 1} \sqrt{\frac{2}{\pi}} \left\{ F^{-1}\left[\frac{1}{s^2 + \alpha^2}\right] - F^{-1}\left[\frac{1}{s^2 + 1}\right] \right\} \\
&= \frac{\alpha}{\alpha^2 - 1} \sqrt{\frac{2}{\pi}} \left\{ \sqrt{\frac{\pi}{2}} \frac{1}{\alpha} e^{-\alpha|x|} - \sqrt{\frac{\pi}{2}} e^{-|x|} \right\} \\
&= -\frac{\alpha}{\alpha^2 - 1} \left(e^{-|x|} - \frac{1}{\alpha} e^{-\alpha|x|} \right) = \frac{e^{-\alpha|x|} - \alpha e^{-|x|}}{\alpha^2 - 1}.
\end{aligned}$$

□

Definition 2.5.2. Let $u(x,t)$ be a complex valued function of two variables. Then the *Fourier transform* of $u(x,t)$ in the first variable, i.e. with respect to x is defined as

$$F[u](s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-isx} dx.$$

The *Fourier sine transform* of $u(x,t)$ with respect to variable x is defined as

$$F_s[u](s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} u(x,t) \sin sx dx.$$

The *Fourier cosine transform* of $u(x,t)$ with respect to variable x is defined as

$$F_c[u](s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} u(x,t) \cos sx dx.$$

Example 2.5.3. Let $u(x,t)$ be a function of two variables and let $u(x,t)$ and $u_x(x,t)$ both tend to 0 as $|x| \to \infty$ for all t. Then $F[u_{xx}](s) = -s^2 F[u](s) = i^2 s^2 F[u](s)$.

Solution. Here,

$$\begin{aligned}
F[u_{xx}](s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_{xx}(x,t)e^{-ist}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\left[u_x(x,t)e^{-isx} + is\int e^{-ist}u_x(x,t)\right]_{-\infty}^{\infty} \\
&= \frac{1}{\sqrt{2\pi}}\left[u_x(x,t)e^{-isx} + isu(x,t)e^{-isx} - s^2\int e^{-isx}u(x,t)\,dx\right]_{-\infty}^{\infty} \\
&= -s^2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-isx}\,dx \\
&= -s^2F[u](s).
\end{aligned}$$

Similarly, note that, $F[u_x](s) = isF[u](s)$. □

Theorem 2.5.4. *Let $u(x,t)$ be a function of two variables and let $u(x,t)$ and $u_x(x,t)$ both tend to 0 as $|x| \to \infty$. Then*

1. $F_c[u_x](s) = -\sqrt{\frac{2}{\pi}}u(0,t) + sF_s[u](s)$.
2. $F_c[u_{xx}](s) = -\sqrt{\frac{2}{\pi}}\,u_x(0,t) - s^2F_c[u](s)$.
3. $F_s[u_x](s) = -sF_c[u](s)$.
4. $F_s[u_{xx}](s) = \sqrt{\frac{2}{\pi}}\,su(0,t) - s^2F_s[u](s)$.

Proof. 1.

$$\begin{aligned}
F_c[u_x](s) &= \sqrt{\frac{2}{\pi}}\int_0^{\infty} u_x(x,t)\cos sx\,dx \\
&= \sqrt{\frac{2}{\pi}}\left[u(x,t)\cos sx + s\int \sin sx\,u(x,t)\,dx\right]_0^{\infty} \\
&= \sqrt{\frac{2}{\pi}}(-u(0,t)) + s\sqrt{\frac{2}{\pi}}\int_0^{\infty} u(x,t)\sin sx\,dx \\
&= = \sqrt{\frac{2}{\pi}}u(0,t) + sF_s[u](s).
\end{aligned}$$

2.

$$\begin{aligned}
F_c[u_{xx}](s) &= \sqrt{\frac{2}{\pi}}\int_0^{\infty} u_{xx}(x,t)\cos sx\,dx \\
&= \sqrt{\frac{2}{\pi}}\left[u_x(x,t)\cos sx + s\int \sin sx\,u_x(x,t)\,dx\right]_0^{\infty} \\
&= \sqrt{\frac{2}{\pi}}\left[u_x(x,t)\cos sx + s\sin sx\,u(x,t) - s^2\int \cos sx\,u(x,t)\,dx\right]_0^{\infty} \\
&= -\sqrt{\frac{2}{\pi}}u_x(0,t) - \sqrt{\frac{2}{\pi}}s^2\int_0^{\infty} u(x,t)\cos sx\,dx \\
&= -\sqrt{\frac{2}{\pi}}u_x(0,t) - s^2F_c[u](s).
\end{aligned}$$

Dr. P. A. Dabhi lightatinfinite@gmail.com
Dr. Jay Mehta jay_mehta@spuvvn.edu

3.

$$\begin{aligned}
F_s[u_x](s) &= \sqrt{\frac{2}{\pi}} \int_0^\infty u_x(x,t) \sin sx \, dx \\
&= \sqrt{\frac{2}{\pi}} \left[u(x,t) \sin sx - s \int \cos sx u(x,t) \, dx \right]_0^\infty \\
&= -s\sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \cos sx \, dx \\
&= -sF_c[u](s).
\end{aligned}$$

4.

$$\begin{aligned}
F_s[u_{xx}](s) &= \sqrt{\frac{2}{\pi}} \int_0^\infty u_{xx}(x,t) \sin sx \, dx \\
&= \sqrt{\frac{2}{\pi}} \left[u_x(x,t) \sin sx - s \int \cos sx \, u_x(x,t) \, dx \right]_0^\infty \\
&= \sqrt{\frac{2}{\pi}} \left[u_x(x,t) \sin sx - s \cos sx \, u(x,t) + s^2 \int \sin sx \, u(x,t) \, dx \right]_0^\infty \\
&= \sqrt{\frac{2}{\pi}} su(0,t) - s^2 \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \sin sx \, dx \\
&= \sqrt{\frac{2}{\pi}} su(0,t) - s^2 F_s[u](s).
\end{aligned}$$

□

Theorem 2.5.5. *Let $u(x,t)$ be a function of two variables. Suppose that $\frac{\partial^r u}{\partial x^r} \in L^1(\mathbb{R})$ and it is tending to 0 as $|x| \to \infty$ for $r = 0, 1, \ldots, n-1$. Then*

1. $F[\frac{\partial^n u}{\partial x^n}](s) = (is)^n F[u](s)$.
2. $F[\frac{\partial^n u}{\partial t^n}](s) = \frac{\partial^n}{\partial t^n} F[u](s)$.
3. $F_s\left[\frac{\partial^n u}{\partial t^n}\right](s) = \frac{\partial^n}{\partial t^n} F_s[u](s)$, $n \in \mathbb{N}$.
4. $F_c\left[\frac{\partial^n u}{\partial t^n}\right](s) = \frac{\partial^n}{\partial t^n} F_c[u](s)$, $n \in \mathbb{N}$.

Proof. 1. Same as proof of Property 3.

2. Let $s \in \mathbb{R}$. Then

$$\begin{aligned}
F\left[\frac{\partial^n u}{\partial t^n}\right](s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{\partial^n u}{\partial t^n}(x,t) e^{-isx} \, dx \\
&= \frac{\partial^n}{\partial t^n} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty u(x,t) e^{-isx} \, dx \right) \\
&= \frac{\partial^n}{\partial t^n} F[u](s).
\end{aligned}$$

3. Let $s \in \mathbb{R}$. Then

$$F_s\left[\frac{\partial^n u}{\partial t^n}\right](s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^n u}{\partial t^n}(x,t) \sin sx \, dx$$

$$= \frac{\partial^n}{\partial t^n}\left(\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\sin sx\,dx\right)$$
$$= \frac{\partial^n}{\partial t^n}F_s[u](s).$$

4. Let $s \in \mathbb{R}$. Then

$$F_c\left[\frac{\partial^n u}{\partial t^n}\right](s) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^n u}{\partial t^n}(x,t)\cos sx\,dx$$
$$= \frac{\partial^n}{\partial t^n}\left(\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\cos sx\,dx\right)$$
$$= \frac{\partial^n}{\partial t^n}F_c[u](s).$$

□

Example 2.5.6. Obtain the solution of the free vibration of semi-infinite string governed by the partial differential equation $u_{tt} = c^2u_{xx}$, $0 < x < \infty$, $t > 0$ subject to $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

Solution: Since one end point of the string is fixed, $u(0,t) = 0, t > 0$. Hence we have to solve the PDE $u_{tt} = c^2u_{xx}$, $0 < x < \infty$, $t > 0$ subject to $u(x,0) = f(x)$, $u_t(x,0) = g(x)$ and $u(0,t) = 0, t > 0$.

Let $U(s,t)$ be the Fourier sine transform of $u(x,t)$ and let F and G be the Fourier sine transforms of f and g respectively. Applying the Fourier sine transform to the equation $u_{tt} = c^2u_{xx}$, we get

$$\frac{\partial^2}{\partial t^2}U(s,t) = c^2\left[\sqrt{\frac{2}{\pi}}su(0,t) - s^2U(s,t)\right]$$
$$= -c^2s^2U(s,t) \qquad (\because u(0,t) = 0).$$

By fixing s in the above equation, we may write it as $\frac{d^2U}{dt^2} + c^2s^2U = 0$. This implies that

$$U(s,t) = A(s)\cos(cst) + B(s)\sin(cst),$$

where $A(s)$ and $B(s)$ are arbitrary constants. Since $u(x,0) = f(x)$, $U(s,0) = F(s)$. Therefore $A(s) = F(s)$. Now since $u_t(x,0) = g(x)$, $\frac{dU}{dt}(s,0) = G(s)$. Hence $B(s) = \frac{1}{cs}G(s)$. This implies

$$U(s,t) = F(s)\cos(cst) + \frac{1}{cs}G(s)\sin(cst).$$

Since s was fixed arbitrarily, the above equation holds for every s and t. Applying the inverse Fourier sine transform, we obtain

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(s)\cos(cst)\sin sxdx + \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{cs}G(s)\sin(cst)\sin sxdx.$$

Now,

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}}\int_0^\infty F(s)\cos(cst)\sin sx\,ds\\
&=\sqrt{\frac{2}{\pi}}\int_0^\infty \left[\frac{1}{2}(\sin s(x+ct)+\sin s(x-ct))\right]F(s)\,ds\\
&=\frac{1}{\sqrt{2\pi}}\int_0^\infty F(s)\sin s(x+ct)\,ds+\frac{1}{\sqrt{2\pi}}\int_0^\infty F(s)\sin s(x-ct)\,ds\\
&=\frac{1}{2}\left[F_s^{-1}[F_s[f]](x+ct)+F_s^{-1}[F_s[f]](x-ct)\right] && (\because F=F_s[f])\\
&=\frac{1}{2}\left[f(x+ct)+f(x-ct)\right] && (\because f=F_s^{-1}[F_s[f]]).
\end{aligned}$$

Now,

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{G(s)}{cs}\sin sx\sin(cst)\,ds\\
&=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{G(s)}{cs}\left[\cos s(x-ct)-\cos s(x+ct)\right]\,ds\\
&=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{G(s)}{cs}\left(\int_{x-ct}^{x+ct}\sin su\,du\right)\,ds\\
&=\frac{1}{2c}\int_{x-ct}^{x+ct}\left(\sqrt{\frac{2}{\pi}}\int_0^\infty G(s)\sin su\,ds\right)\,du\\
&=\frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du.
\end{aligned}$$

Therefore

$$u(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(u)du.$$

This is called *D'Alembert's solution* which describes the vibrations of an infinite string.

Example 2.5.7. Solve $u_{xx}+u_{yy}=0$ ($x\in\mathbb{R}$, $y>0$) subject to the conditions $u(x,0)=f(x)$ ($x\in\mathbb{R}$), u is bounded as $y\to\infty$, both u and $\partial u/\partial x\to 0$ as $|x|\to\infty$.

Solution. Let U be the Fourier transform of u in the first variable, and let F be the Fourier transform of f. Applying the Fourier transform to the equation $u_{xx}+u_{yy}=0$, we have

$$\begin{aligned}
&F[u_{xx}](s)+F[u_{yy}](s)=0\\
\Rightarrow\ &(is)^2U(s,t)+\frac{\partial^2 U}{\partial y^2}(s,t)=-s^2U(s,t)+\frac{\partial^2 U}{\partial y^2}(s,t)=0.
\end{aligned}$$

Fixing s, the above equation reduces to $\frac{d^2U}{dy^2}+s^2U=0$. Therefore $U(s,y)=A(s)e^{sy}+B(s)e^{-sy}$. Since s is arbitrary, above equation holds for every s and every y, i.e.

$$U(s,y)=A(s)e^{sy}+B(s)e^{-sy}\ \text{ for all } s \text{ and } t.$$

Since u is bounded as $y \to \infty$, $U(s,y)$ is bounded as $y \to \infty$. If $s > 0$, then $U(s,t)$ has to be of the form $B(s)e^{-sy}$, i.e. $A(s) = 0$; and if $s < 0$, then $U(s,y)$ has to be of the form $A(s)e^{sy}$, i.e. $B(s) = 0$. Thus

$$U(s,y) = \begin{cases} B(s)e^{-sy}, & s > 0 \\ A(s)e^{sy}, & s < 0 \end{cases} = C(s)e^{-|s|y}.$$

Since $u(x,0) = f(x)$, $U(s,0) = F(s)$. Therefore $F(s) = U(s,0) = C(s)$. Hence

$$U(s,y) = F(s)e^{-|s|y}.$$

Applying the inverse Fourier transform to this equation, we obtain

$$\begin{aligned}
u(x,y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)e^{-|s|y}e^{isx}ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)e^{-isu}du \right) e^{-|s|y}e^{isx}ds \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left(\int_{-\infty}^{\infty} e^{-|s|y}e^{is(x-u)}ds \right) du \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left(\int_{-\infty}^{\infty} e^{-|s|y}[\cos s(u-x) + i\sin s(u-x)]ds \right) du \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left(\int_{-\infty}^{\infty} e^{-|s|y}\cos s(u-x)ds \right) du \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left(2\int_{0}^{\infty} e^{-sy}\cos s(u-x)ds \right) du \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \left[\frac{e^{-sy}}{y^2+(u-x)^2} \left(-y\cos s(u-x) + (u-x)\sin s(u-x)\right) \right]_0^{\infty} du \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \left[\frac{y}{y^2+(u-x)^2} \right] du \\
&= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(u)}{y^2+(u-x)^2}\, du
\end{aligned}$$

Therefore

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(u)}{y^2+(u-x)^2}\, du.$$

□

Example 2.5.8. Solve $u_t = ku_{xx}$ $(x \in \mathbb{R},\ t > 0)$ subject to the condition $u(x,0) = f(x)$ $(x \in \mathbb{R})$ and both $u, u_x \to 0$ as $|x| \to \infty$.

Solution. Let U be the Fourier transform of u in the first variable and F be the Fourier transform of f. Applying the Fourier transform to the equation $u_t = ku_{xx}$, we get

$$F[u_t](s) = kF[u_{xx}](s) \Rightarrow \frac{\partial}{\partial t}U(s,t) = -ks^2U(s,t).$$

Fixing s in the above equation will become $\frac{dU}{dt} + ks^2U = 0$. Therefore

$$U(s,t) = A(s)e^{-ks^2t}, \text{ where } A(s) \text{ is an arbitrary constant.}$$

Since $u(x,0) = f(x)$, $U(s,0) = F(s)$. Therefore $A(s) = F(s)$. Since s is arbitrary, $U(s,t) = F(s)e^{-ks^2t}$ for all s and t. Applying the inverse Fourier transform to this equation, we obtain

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-ks^2t}e^{isx}ds \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-isu}du\right)e^{-ks^2t}e^{isx}ds \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left(\int_{-\infty}^{\infty} e^{-ks^2t}(\cos s(x-u) + i\sin s(x-u))ds\right)du \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\left(\int_{0}^{\infty} e^{-ks^2t}\cos s(x-u)ds\right)du \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\left(\frac{1}{2}\sqrt{\frac{\pi}{kt}}\exp\left(-\frac{(x-u)^2}{4kt}\right)\right)du \qquad (\because F[e^{-kts^2}](x-u)) \\
&= \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} f(u)\exp\left(-\frac{(x-u)^2}{4kt}\right)du
\end{aligned}$$

Take $\frac{u-x}{2\sqrt{kt}} = \theta$. Then $\frac{du}{2\sqrt{kt}} = d\theta \Rightarrow du = 2\sqrt{kt}d\theta$. Now, $u \to \infty \Rightarrow \theta \to \infty$ and $u \to -\infty \Rightarrow \theta \to -\infty$. Therefore, we have

$$\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} f(x+2\theta\sqrt{kt})e^{-\theta^2}2\sqrt{kt}\,d\theta \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2\theta\sqrt{kt})e^{-\theta^2}\,d\theta.
\end{aligned}$$

□

Example 2.5.9. Solve $u_t = ku_{xx}$ $(x > 0,\ t > 0)$ subject to the conditions $u(x,0) = 0$ and $u_x(0,t) = -a$ and both $u, u_x \to 0$ as $x \to \infty$.

Solution. Let U be the Fourier cosine transform of u in the first variable. Applying the Fourier cosine transform to the equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, we get $F_c[u_t](s) = kF_c[u_{xx}](s)$. Therefore,

$$\begin{aligned}
\frac{\partial}{\partial t}U(s,t) &= -k\sqrt{\frac{2}{\pi}}u_x(0,t) - ks^2U(s,t) \\
\Rightarrow \frac{\partial}{\partial t}U(s,t) &= ka\sqrt{\frac{2}{\pi}} - ks^2U(s,t).
\end{aligned}$$

Fixing s, the above equation reduces to $\frac{dU}{dt} + ks^2U = ka\sqrt{\frac{2}{\pi}}$. Therefore,

$$\text{Complementary Function (C.F.)} = A(s)e^{-ks^2t} \quad \text{and}$$

$$\begin{aligned}
\text{Particular Integral (P.I.)} &= \frac{1}{D+ks^2}\,ka\sqrt{\frac{2}{\pi}} \\
&= ka\sqrt{\frac{2}{\pi}}\frac{1}{ks^2\left(1+\frac{D}{ks^2}\right)}\cdot 1 = \frac{a}{s^2}\sqrt{\frac{2}{\pi}}.
\end{aligned}$$

Therefore, $U(s,t) =$ C.F + P.I. $= A(s)e^{-ks^2t} + \frac{a}{s^2}\sqrt{\frac{2}{\pi}}$. Now since $u(x,0) = 0$, $U(s,0) = 0$. This implies

$$A(s) = -\frac{a}{s^2}\sqrt{\frac{2}{\pi}}.$$

Therefore, $U(s,t) = \frac{a}{s^2}\sqrt{\frac{2}{\pi}}\left(1 - e^{-ks^2t}\right)$. Applying the Fourier cosine transform, we get

$$u(x,t) = \frac{2a}{\pi}\int_0^\infty \frac{1 - e^{-ks^2t}}{s^2}\cos sx\, ds.$$

□

Example 2.5.10. Solve $u_{tt} = c^2u_{xx}$ $(x \in \mathbb{R},\ t > 0)$ subject to the conditions $u(x,0) = f(x)$ and $u_t(x,0) = 0$ for all $x \in \mathbb{R}$ and both $u, u_x \to 0$ as $|x| \to \infty$.

Solution. Let U be the Fourier transform of u in the first variable and let F be the Fourier transform of f. Applying Fourier transform to $u_{tt} = c^2u_{xx}$, we get

$$F[u_{tt}](s) = c^2F[u_{xx}](s)$$
$$\therefore \frac{\partial^2}{\partial t^2}U(s,t) = c^2\left(-s^2U(s,t)\right).$$

Fix s. Then we may write above equation as $\frac{d^2U}{dt^2} + c^2s^2U = 0$. Therefore

$$U(s,t) = A(s)\cos(cst) + B(s)\sin(cst),$$

where $A(s)$ and $B(s)$ are arbitrary constants. Since s was fixed arbitrarily, the above solution holds for all s and for all t. Also,

$$U_t(s,t) = -cs\,A(s)\sin(cst) + cs\,B(s)\cos(cst).$$

Since $u(x,0) = f(x)$, $U(s,0) = F(s)$. Therefore $A(s) = F(s)$. Again since $u_t(x,0) = 0$, $U_t(s,0) = 0$. This implies $cs\,B(s) = 0$ and so $B(s) = 0$. Therefore the required solution takes the form

$$U(s,t) = F(s)\cos(cst).$$

Applying inverse Fourier transform, we get

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\cos(cst)e^{isx}\,ds \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-isu}\,du\right)\frac{e^{isct} + e^{-isct}}{2}e^{isx}\,ds \\
&= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{-is(u-ct-x)}\,du\,ds + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{-is(u-ct-x)}\,du\,ds\right] \\
&= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right].
\end{aligned}$$

This is called *D'Alembert's solution.* □

Definition 2.5.11. The *error function* $\text{erf}(x)$ is defined as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\sigma^2} d\sigma.$$

We note that if $a > 0$ and $b \in \mathbb{R}$, then

$$\int_0^\infty e^{-ax^2} \cos bx dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \exp\left(-\frac{b^2}{4a}\right).$$

Example 2.5.12. Solve $u_t = ku_{xx}$ $(x > 0,\ t > 0)$ subject to the conditions $u(0,t) = 0$ $(t > 0)$ and $u(x,0) = f(x)$ $(x > 0)$.

Solution. Let U be the Fourier sine transform of u in the first variable and F be the Fourier sine transform of f. Applying the Fourier sine transform to the equation $u_t = ku_{xx}$, we get

$$\begin{aligned} F_s[u_t](s) &= k\, F_s[u_{xx}](s) \\ \therefore \frac{\partial}{\partial t} U(s,t) &= k \left[\sqrt{\frac{2}{\pi}} s u(0,t) - s^2 U(s,t) \right] \\ \therefore \frac{\partial}{\partial t} U(s,t) &= -ks^2 U(s,t). \end{aligned}$$

Fixing s the above equation will become $\dfrac{dU}{dt} + ks^2 U = 0$. Therefore

$$U(s,t) = c(s)e^{-ks^2 t}, \text{ where } c(s) \text{ is an arbitrary constant.}$$

Since $u(x,0) = f(x)$, $U(s,0) = F(s)$. Therefore $c(s) = F(s)$. Since s is arbitrary, we have $U(s,t) = F(s)e^{-ks^2 t}$ for all s and t. Applying the inverse Fourier sine transform to this equation, we obtain

$$\begin{aligned} u(x,t) &= \sqrt{\frac{2}{\pi}} \int_0^\infty F(s)e^{-ks^2 t} \sin sx\, ds \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\sqrt{\frac{2}{\pi}} \int_0^\infty f(u) \sin su\, du \right) e^{-ks^2 t} \sin sx\, ds \\ &= \frac{1}{\pi} \int_0^\infty f(u) \left(\int_0^\infty e^{-ks^2 t} 2 \sin su \sin sx\, ds \right) du \\ &= \frac{1}{\pi} \int_0^\infty f(u) \left(\int_0^\infty e^{-ks^2 t} \cos s(u-x)\, ds - \int_0^\infty e^{-ks^2 t} \cos s(u+x)\, ds \right) du \\ &= \frac{1}{\pi} \int_0^\infty f(u) \left[\frac{1}{2}\sqrt{\frac{\pi}{kt}} \exp\left(-\frac{(u-x)^2}{4kt}\right) - \frac{1}{2}\sqrt{\frac{\pi}{kt}} \exp\left(-\frac{(u+x)^2}{4kt}\right) \right] du \\ &= \frac{1}{2\sqrt{\pi kt}} \left[\int_0^\infty f(u) \exp\left(-\frac{(u-x)^2}{4kt}\right) du - \int_0^\infty f(u) \exp\left(-\frac{(u+x)^2}{4kt}\right) du \right] \\ &= \frac{1}{\sqrt{\pi}} \left[\int_{-\frac{x}{2\sqrt{kt}}}^\infty f(x + 2\theta\sqrt{kt}) e^{-\theta^2} d\theta - \int_{\frac{x}{2\sqrt{kt}}}^\infty f(-x + 2\theta\sqrt{kt}) e^{-\theta^2} d\theta \right]. \end{aligned}$$

If $f(x) = 1$, then it follows that

$$u(x,t) = \operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right).$$

□

Example 2.5.13. Solve $u_{xx} + u_{yy} = 0$, $(x \in \mathbb{R},\ y > 0)$ subject to the conditions $u_y(x,0) = f(x)$ $(x \in \mathbb{R})$, u is bounded as $y \to \infty$, and both $u, u_x \to 0$ as $|x| \to \infty$.

Solution. Let $\phi(x,y) = u_y(x,y)$. Then $\phi_{xx} + \phi_{yy} = u_{xxy} + u_{yyy} = \frac{\partial}{\partial y}(u_{xx} + u_{yy}) = 0$. Also, $\phi(x,0) = u_y(x,0) = f(x)$. Then by Example 2.5.7

$$\phi(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(u)}{y^2 + (u-x)^2}\, du$$

$$\text{i.e. } u_y(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(u)}{y^2 + (u-x)^2}\, du.$$

But then

$$\begin{aligned} u(x,y) &= \int \phi(x,y)\, dy \\ &= \int \left(\frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(u)}{y^2 + (u-x)^2}\, du\right) dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left(\int \frac{2y}{y^2 + (u-x)^2}\, dy\right) du \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)(\ln\left(y^2 + (u-x)^2\right) + c)\, du, \end{aligned}$$

where c is a constant. □

CHAPTER 3

LAPLACE TRANSFORM

3.1 Definitions and Examples

Definition 3.1.1. Let $f:(0,\infty)\to\mathbb{C}$ be a map. Then the *Laplace transform*, $L[f]$, of f is defined as

$$L[f](s)=\int_0^\infty f(t)e^{-st}dt.$$

The domain of the Laplace transform $L[f]$ of f is the set of all those $s\in\mathbb{C}$ such that $\int_0^\infty f(t)e^{-st}dt$ exists.

Example 3.1.2. Compute the Laplace transform of the following:

1. 1.

Solution. $L[1](s)=\int_0^\infty 1\cdot e^{-st}\,dt=\left[\frac{e^{-st}}{-s}\right]_0^\infty=\frac{1}{s}\qquad (\because e^{-st}\to 0 \text{ as } t\to\infty).$

Thus, $\boxed{L[1](s)=\frac{1}{s}}$, for all $s\in\mathbb{C}$ and $\operatorname{Re}s>0$. □

2. t

Solution.
$$L[t](s)=\int_0^\infty te^{-st}\,dt$$
$$=\left[t\left(\frac{e^{-st}}{-s}\right)-\frac{e^{-st}}{s^2}\right]_0^\infty=\frac{1}{s^2}.$$

$\boxed{L[t](s)=\frac{1}{s^2}}$, for all $s\in\mathbb{C}$ such that $\operatorname{Re}s>0$. □

3. t^n

Solution. $L[t^n](s) = \int_0^\infty t^n e^{-st}\, dt.$
Take $st = \theta$, then $s\,dt = d\theta \Rightarrow t = \frac{\theta}{s}$ and $dt = \frac{d\theta}{s}$. Then we have,

$$\begin{aligned} L[t^n](s) &= \int_0^\infty \frac{\theta^n}{s^n} e^{-\theta}\, \frac{d\theta}{s} \\ &= \frac{1}{s^{n+1}} \int_0^\infty \theta^n e^{-\theta}\, d\theta \\ &= \frac{1}{s^{n+1}} \Gamma(n+1) = \frac{n!}{s^{n+1}}. \qquad (\because \Gamma(z) = \int_0^\infty x^z e^{-z}\, dx \text{ and } \Gamma(n) = (n-1)!) \end{aligned}$$

$\boxed{L[t^n](s) = \frac{n!}{s^{n+1}}}$, for all $s \in \mathbb{C}$ such that $\operatorname{Re} s > 0$. □

4. e^{at}

Solution.
$$\begin{aligned} L[e^{at}](s) &= \int_0^\infty e^{at} e^{-st}\, dt \\ &= \int_0^\infty e^{-t(s-a)}\, dt \\ &= \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a} \end{aligned}$$

$\boxed{L[e^{at}](s) = \frac{1}{s-a}}$, for all $s \in \mathbb{C}$ such that $\operatorname{Re}(s-a) > 0$. □

5. $\cos(at)$

Solution.
$$\begin{aligned} L[\cos at](s) &= \int_0^\infty \cos(at)\, e^{-st}\, dt \\ &= \left[\frac{e^{-st}}{s^2+a^2}\,(-s\cos at + a\sin at)\right]_0^\infty = \frac{s}{s^2+a^2}. \end{aligned}$$

$\boxed{L[\cos(at)](s) = \frac{s}{s^2+a^2}}$. □

6. $\sin(at)$

Solution.
$$\begin{aligned} L[\cos at](s) &= \int_0^\infty \sin(at)\, e^{-st}\, dt \\ &= \left[\frac{e^{-st}}{s^2+a^2}\,(-s\sin at - a\cos at)\right]_0^\infty = \frac{a}{s^2+a^2}. \end{aligned}$$

$\boxed{L[\sin(at)](s) = \frac{a}{s^2+a^2}}$. □

7. $\cosh(at)$

Solution.
$$\begin{aligned} L[\cosh(at)](s) &= \int_0^\infty \cosh(at)e^{-st}\,dt \\ &= \int_0^\infty \left(\frac{e^{at}+e^{-at}}{2}\right)e^{-st}\,dt \\ &= \frac{1}{2}\int_0^\infty \left(e^{-(s-a)t}+e^{-(s+a)t}\right)dt \\ &= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)}+\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty \\ &= \frac{1}{2}\left[\frac{1}{s-a}+\frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right]. \end{aligned}$$

$$\boxed{L[\cosh(at)](s) = \frac{s}{s^2-a^2}}.$$ □

8. $\sinh(at)$

Solution.
$$\begin{aligned} L[\sinh(at)](s) &= \int_0^\infty \sinh(at)e^{-st}\,dt \\ &= \int_0^\infty \left(\frac{e^{at}-e^{-at}}{2}\right)e^{-st}\,dt \\ &= \frac{1}{2}\int_0^\infty \left(e^{-(s-a)t}-e^{-(s+a)t}\right)dt \\ &= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)}-\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty \\ &= \frac{1}{2}\left[\frac{1}{s-a}-\frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]. \end{aligned}$$

$$\boxed{L[\sinh(at)](s) = \frac{a}{s^2-a^2}}.$$ □

From Examples 7 and 8 above, it can be observed that Laplace transform will be linear. Indeed it is. We have the following properties of Laplace transform

3.2 Properties of Laplace transform

1. Laplace transform is linear, i.e. if $f,g:(0,\infty)\to\mathbb{C}$ and $\alpha,\beta\in\mathbb{C}$, then
$$L[\alpha f+\beta g] = \alpha L[f]+\beta L[g].$$

Proof.
$$\begin{aligned} L[\alpha f+\beta g](s) &= \int_0^\infty (\alpha f+\beta g)(t)e^{-st}\,dt \\ &= \alpha\int_0^\infty f(t)e^{-st}\,dt+\beta\int_0^\infty g(t)e^{-st}\,dt \\ &= \alpha L[f](s)+\beta L[g](s). \end{aligned}$$
Thus, $L[\alpha f+\beta g](s) = (\alpha L[f]+\beta L[g])(s)$ and since s is arbitrary, we have
$$L[\alpha f+\beta g] = \alpha L[f]+\beta L[g].$$

□

2. **(Shifting property):**
$$L[e^{at}f(t)](s) = L[f](s-a).$$

Proof.
$$\begin{aligned} L[e^{at}f(t)](s) &= \int_0^\infty f(t)e^{at}e^{-st}\,dt \\ &= \int_0^\infty f(t)e^{-(s-a)t}\,dt \\ &= L[f](s-a). \end{aligned}$$
□

Example 3.2.1. Compute the Laplace transforms of the following:

1. $e^{at}\cos bt$

Solution. We know that $L[\cos(bt)](s) = \frac{s}{s^2+b^2}$. Therefore
$$\begin{aligned} L[e^{at}\cos(bt)](s) &= L[\cos(bt)](s-a) \\ &= \frac{s-a}{(s-a)^2+b^2}. \end{aligned}$$
□

2. $e^{at}\sin bt$

Solution. We know that $L[\sin(bt)](s) = \frac{b}{s^2+b^2}$. Therefore
$$\begin{aligned} L[e^{at}\sin(bt)](s) &= L[\sin(bt)](s-a) \\ &= \frac{b}{(s-a)^2+b^2}. \end{aligned}$$
□

3. $e^{at}\cosh bt$

Solution. We know that $L[\cosh(bt)](s) = \frac{s}{s^2-b^2}$. Therefore
$$\begin{aligned} L[e^{at}\cosh(bt)](s) &= L[\cosh(bt)](s-a) \\ &= \frac{s-a}{(s-a)^2-b^2}. \end{aligned}$$
□

4. $e^{at}\sinh bt$

Solution. We know that $L[\sinh(bt)](s) = \frac{b}{s^2-b^2}$. Therefore
$$\begin{aligned} L[e^{at}\sinh(bt)](s) &= L[\sinh(bt)](s-a) \\ &= \frac{b}{(s-a)^2-b^2}. \end{aligned}$$
□

5. $e^{at}t^n$

Solution. We know that $L[t^n](s) = \frac{n!}{s^{n+1}}$. Therefore

$$L[e^{at}t^n](s) = L[t^n](s-a) = \frac{n!}{(s-a)^{n+1}}.$$

□

3. **(Multiplication by power of t):**

$$L[t^n f(t)](s) = (-1)^n \frac{d^n}{ds^n} L[f](s).$$

Proof.
$$\frac{d^n}{ds^n}L[f](s) = \frac{d^n}{ds^n}\left(\int_0^\infty f(t)e^{-st}dt\right) = \int_0^\infty f(t)(-t)^n e^{-st}dt = (-1)^n \int_0^\infty f(t)t^n e^{-st}dt = (-1)^n L[t^n f(t)](s).$$

Therefore

$$L[t^n f(t)](s) = (-1)^n \frac{d^n}{ds^n} L[f](s).$$

□

Example 3.2.2. Compute the Laplace transform of the following:

1. $t^2\cos(at)$

Solution. We know that $L[\cos(at)](s) = \frac{s}{s^2+a^2}$. Therefore

$$L[t^2\cos(at)](s) = (-1)^2 \frac{d^2}{ds^2} L[\cos(at)](s) = \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right).$$

□

2. $te^{2t}\sin(3t)$

Solution. We know that $L[e^{2t}\sin(3t)](s) = \frac{3}{(s-2)^2+3^2}$. Therefore

$$L[te^{2t}\sin(3t)](s) = (-1)^1 \frac{d}{ds} L[e^{2t}\sin(3t)](s) = -\frac{d}{ds}\left(\frac{3}{(s-2)^2+3^2}\right).$$

□

3. $\cos(at)\cosh(bt)$

Solution. We know that

$$\begin{aligned} L[\cos(at)\cosh(bt)](s) &= \frac{1}{2}\, L[e^{bt}\cos(at)](s) + \frac{1}{2}\, L[e^{-bt}\cos(at)](s) \\ &= \frac{1}{2}\,\frac{s-b}{(s-b)^2+a^2} + \frac{1}{2}\,\frac{s+b}{(s+b)^2+a^2}. \end{aligned}$$

□

4. $\cos(at)\sinh(bt)\,t^2$

Solution. We know that

$$\begin{aligned} L[\cos(at)\sinh(bt)](s) &= \frac{1}{2}\, L[e^{bt}\cos(at)](s) - \frac{1}{2}\, L[e^{-bt}\cos(at)](s) \\ &= \frac{1}{2}\,\frac{s-b}{(s-b)^2+a^2} - \frac{1}{2}\,\frac{s+b}{(s+b)^2+a^2}. \end{aligned}$$

Therefore

$$\begin{aligned} L[\cos(at)\sinh(bt)\,t^2](s) &= (-1)^2 \frac{d^2}{ds^2} L[\cos(at)\sinh(bt)](s) \\ &= (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{2}\,\frac{s-b}{(s-b)^2+a^2} - \frac{1}{2}\,\frac{s+b}{(s+b)^2+a^2}\right) \\ &= \frac{1}{2}\,\frac{d^2}{ds^2}\left(\frac{s-b}{(s-b)^2+a^2} - \frac{s+b}{(s+b)^2+a^2}\right). \end{aligned}$$

□

Remark 3.2.3. Let $s_0 \in \mathbb{R}$. Let $f : (0,\infty) \to \mathbb{C}$ be a map. If $\int_0^\infty |f(t)|e^{-s_0 t}\, dt < \infty$, then the Laplace transform of f exists for all $s \in \mathbb{C}$ with $\operatorname{Re} s > s_0$.

Proof. Let $s = s_1 + is_2 \in \mathbb{C}$ with $\operatorname{Re} s = s_1 > s_0$. Then for all $t \geq 0$,

$$|e^{-st}| = |e^{-(s_1+is_2)t}| = e^{-s_1 t} \leq e^{-s_0 t}.$$

Therefore

$$\int_0^\infty |f(t)||e^{-st}|\, dt \leq \int_0^\infty |f(t)|e^{-s_0 t}\, dt < \infty.$$

Hence, $L[f](s) = \int_0^\infty f(t)e^{-st}\, dt$ exists for all $s \in \mathbb{C}$ satisfying $\operatorname{Re} s > s_0$. □

4. **(Division by t):**

$$L\left[\frac{f(t)}{t}\right](s) = \int_s^\infty L[f](u)\, du.$$

Proof. We have

$$
\begin{aligned}
\int_s^\infty L[f](u)du &= \int_s^\infty \left(\int_0^\infty f(t)e^{-tu}\,dt\right)du \\
&= \int_0^\infty f(t)\left(\int_s^\infty e^{-tu}\,du\right)dt \\
&= \int_0^\infty f(t)\left[\frac{e^{-tu}}{-t}\right]_s^\infty dt \\
&= \int_0^\infty \frac{f(t)}{t}e^{-st}dt = L\left[\frac{f(t)}{t}\right](s).
\end{aligned}
$$

□

Example 3.2.4. Compute the Laplace transform of $\frac{1-\cos 2t}{t}$ and $\frac{\cos 2t - \cos 3t}{t}$.

Solution. We have, $L[1-\cos 2t](s) = \frac{1}{s} - \frac{s}{s^2+4}$. Now,

$$
\begin{aligned}
L\left[\frac{1-\cos 2t}{t}\right](s) &= \int_s^\infty L[f](u)\,du \\
&= \int_s^\infty \left(\frac{1}{u} - \frac{u}{u^2+4}\right)du \\
&= \left[\log u - \frac{1}{2}\log(u^2+4)\right]_s^\infty \\
&= \left[\log\left(\frac{u}{\sqrt{u^2+4}}\right)\right]_s^\infty \\
&= \log 1 - \log\left(\frac{s}{\sqrt{s^2+4}}\right) = \log\left(\frac{\sqrt{s^2+4}}{s}\right).
\end{aligned}
$$

Next, $L[\cos 2t - \cos 3t](s) = \frac{s}{s^2+4} - \frac{s}{s^2+9}$. Therefore

$$
\begin{aligned}
L\left[\frac{\cos 2t - \cos 3t}{t}\right](s) &= \int_s^\infty L[f](u)\,du \\
&= \int_s^\infty \left(\frac{u}{u^2+4} - \frac{u}{u^2+9}\right)du \\
&= \left[\frac{1}{2}\log(u^2+4) - \frac{1}{2}\log(u^2+9)\right]_s^\infty \\
&= \left[\log\left(\frac{\sqrt{u^2+4}}{\sqrt{u^2+9}}\right)\right]_s^\infty \\
&= \log 1 - \log\sqrt{\frac{s^2+4}{s^2+9}} = \log\left(\sqrt{\frac{s^2+9}{s^2+4}}\right).
\end{aligned}
$$

□

Definition 3.2.5 (Heaviside step function)**.** Let $a \geq 0$. Then the map H defined by

$$H_a(t) = \begin{cases} 0, & t \leq a \\ 1, & t > a \end{cases}$$

is called *Heaviside function* at a sometimes also called *Heaviside step function*. We denote it by $H(t-a)$, i.e. $H_a(t) = H(t-a)$.

Example 3.2.6. We compute below the Laplace transform of Heaviside function.

$$L[H(t-a)](s) = \int_0^\infty H(t-a)e^{-st}\,dt = \int_a^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-sa}}{s}.$$

Recall the translation operator given by $T_a f(x) = f(x-a)$ for $a \geq 0$. Then we have the following property:

5. **(Multiplication by Heaviside function $H(t-a)$):**
   Let $f : (0,\infty) \to \mathbb{C}$ be a map and $a \geq 0$. Then

$$L[f(t)H(t-a)](s) = e^{-sa}L[T_{-a}f](s).$$

Proof. $L[f(t)H(t-a)](s) = \int_0^\infty f(t)H(t-a)e^{-st}\,dt = \int_a^\infty f(t)e^{-st}\,dt.$
Take $t-a = u \Rightarrow t = a+u$ and $dt = du$. Also, if $t = a$ then $u = 0$, and if $t \to \infty$ then $u \to \infty$. Therefore

$$\begin{aligned} L[f(t)H(t-a)](s) &= \int_a^\infty f(t)e^{-st}dt \\ &= \int_0^\infty f(u+a)e^{-s(u+a)}du \\ &= e^{-sa}\int_0^\infty f(u+a)e^{-su}du \\ &= e^{-sa}L[f(t+a)](s) = e^{-sa}L[T_{-a}f](s). \end{aligned}$$

□

6. **(Laplace transform of a periodic function):**
   If $f : (0,\infty) \to \mathbb{C}$ is a periodic function with period T, then

$$L[f](s) = \frac{1}{(1-e^{-sT})}\int_0^T f(t)e^{-st}dt.$$

Proof. Since f is T-periodic function, $f(t+T) = f(t)$ for all t. Now,

$$\begin{aligned} L[f](s) &= \int_0^\infty f(t)e^{-st}\,dt \\ &= \int_0^T f(t)e^{-st}dt + \int_T^\infty f(t)e^{-st}\,dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^T f(t)e^{-st}dt + \int_T^\infty f(t+T)e^{-st}\,dt \\
&= \int_0^T f(t)e^{-st}dt + \int_0^\infty f(u)e^{-s(u-T)}\,du \\
&= \int_0^T f(t)e^{-st}dt + e^{-sT}\int_0^\infty f(u)e^{-su}\,du \\
&= \int_0^T f(t)e^{-st}dt + e^{-sT}L[f](s).
\end{aligned}$$

Therefore $L[f](s) = \dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T f(t)e^{-st}\,dt.$ □

Example 3.2.7. Use the above result to compute the Laplace transform of the functions $\cos t, \sin t, \cos 2t, \sin 2t$.

Solution. We know that, $\cos t$ is a 2π-periodic function, i.e. $T = 2\pi$. Therefore,

$$\begin{aligned}
L[\cos t](s) &= \frac{1}{1-e^{-s2\pi}}\int_0^{2\pi} \cos t e^{-st}\,dt \\
&= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-st}}{s^2+1}(-s\cos t+\sin t)\right]_0^{2\pi} \\
&= \frac{1}{1-e^{-2\pi s}}\left[\frac{-se^{-2\pi s}}{s^2+1}+\frac{s}{s^2+1}\right] \\
&= \frac{s}{(1-e^{-2\pi s})(s^2+1)}(1-e^{-2\pi s}) = \frac{s}{s^2+1}.
\end{aligned}$$

Similarly,

$$L[\sin t](s) = \frac{1}{s^2+1}.$$

□

Example 3.2.8. Let $f:(0,\infty)\to\mathbb{C}$ be a map and let $a>0$. Let $g(t) = f(t-a)H(t-a)$, i.e.

$$g(t) = \begin{cases} f(t-a), & t>a \\ 0, & t\le a. \end{cases}$$

Then $L[g](s) = e^{-sa}L[f](s)$.

Solution. We have

$$\begin{aligned}
L[g](s) &= \int_0^\infty g(t)e^{-st}\,dt \\
&= \int_a^\infty f(t-a)e^{-st}\,dt \\
&= \int_0^\infty f(\theta)e^{-s(\theta+a)}\,d\theta \\
&= e^{-sa}\int_0^\infty f(\theta)e^{-s\theta}\,d\theta = e^{-sa}L[f](s).
\end{aligned}$$

□

Remark 3.2.9. Notice that the above example can be solved using Property 5 also. Since $g(t) = f(t-a)H(t-a)$, by that property

$$L[g](s) = e^{-sa}L[T_{-a}f(t-a)](s) = e^{-sa}L[f((t-a)+a)](s) = e^{-sa}L[f](s).$$

Example 3.2.10. Compute the Laplace transform of $g(t) = (t-1)^2H(t-1)$.

Solution. Taking $f(t) = (t-1)^2$, by above example, we have

$$L[g](s) = e^{-s}L[t^2](s) = e^{-s}\frac{2}{s^3}.$$

□

Example 3.2.11. Compute the Laplace transform of $f(t) = \sin(t-\pi)H(t-\pi)$.

Solution. By Example 3.2.8, we have

$$L[f](s) = e^{-s\pi}L[\sin t](s) = \frac{e^{-s\pi}}{s^2+1}.$$

□

Example 3.2.12. Let $\alpha > 0$. Compute the Laplace transform of the function t^α.

Solution. $L[t^\alpha](s) = \int_0^\infty t^\alpha e^{-st}\,dt$. Take $st = u$. Then $dt = \frac{du}{s}$. Therefore

$$\begin{aligned} L[t^\alpha](s) &= \int_0^\infty \left(\frac{u}{s}\right)^\alpha e^{-u}\,\frac{du}{s} \\ &= \frac{1}{s^{\alpha+1}}\int_0^\infty u^\alpha e^{-u}\,du \\ &= \frac{1}{s^{\alpha+1}}\int_0^\infty u^{(\alpha+1)-1}e^{-u}\,du = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}. \end{aligned}$$

□

7. **(Differentiation property):**
Let f be n times differentiable. If $f^{(r)}(t)e^{-st} \to 0$ as $t \to \infty$ for all $r = 0,\ldots,n-1$ and s in the domain of $L[f]$, then

$$L[f^{(n)}](s) = s^nL[f](s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Proof. We prove the result using induction on n. For $n = 1$, we have

$$\begin{aligned} L[f^{(1)}](s) &= \int_0^\infty f^{(1)}(t)e^{-st}\,dt \\ &= \left[e^{-st}f(t) - \int(-s)e^{-st}f(t)\,dt\right]_0^\infty \end{aligned}$$

$$= -f(0) + s\int_0^\infty f(t)e^{-st}\,dt$$
$$= sL[f](s) - f(0).$$

Hence, the result is true for $n = 1$. Assume that the result is true for $n-1$, i.e.

$$L[f^{(n-1)}](s) = s^{n-1}L[f](s) - s^{n-1}f(0) - \cdots - sf^{(n-3)}(0) - f^{(n-2)}(0).$$

Now,

$$\begin{aligned}
L[f^{(n)}](s) &= \int_0^\infty f^{(n)}(t)e^{-st}\,dt \\
&= \left[e^{-st}f^{(n-1)}(t) - \int(-s)e^{-st}f^{(n-1)}(t)dt\right]_0^\infty \\
&= s\int_0^\infty f^{(n-1)}(t)e^{-st}dt - f^{(n-1)}(0) \\
&= s\,L[f^{(n-1)}](s) - f^{(n-1)}(0) \\
&= s^nL[f](s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).
\end{aligned}$$

□

Ex Find the Laplace transform of the following functions.

1. $t\cos at$
2. $te^{-t}\sin t$
3. $te^t\sin 2t$
4. $(1+t)^2e^{at}$
5. $\frac{1-e^t}{t}$
6. $J_1(t)$
7. $tJ_1(t)$
8. $f(t) = \begin{cases} t/\tau, & 0 < t < \tau \\ 1, & t \geq \tau \end{cases}$
9. $f(t) = \begin{cases} t, & 0 < t \leq 1 \\ 1-t, & 1 < t \leq 2 \\ 0, & t > 2 \end{cases}$

1. $f(t) = \begin{cases} t, & 0 < t \leq 1 \\ 0, & 1 < t \leq 2 \end{cases}$, $f(t+2) = f(t)$ for all $t > 0$.
2. $f(t) = \begin{cases} t, & 0 < t \leq 6 \\ 0, & 6 < t \leq 12 \end{cases}$, $f(t+12) = f(t)$ for all $t > 0$.
3. $f(t) = \begin{cases} 1, & 0 < t \leq b \\ -1, & b < t \leq 2b \end{cases}$, $f(t+2b) = f(t)$ for all $t > 0$.

3.3 Inverse Laplace transform

Definition 3.3.1. If F the Laplace transform of f, then f is called the *inverse Laplace transform* of F. We denote it by $L^{-1}[F] = f$.

Example 3.3.2. Compute the inverse Laplace transform of the following functions.

1. $\frac{1}{s^n}$.
Since $L\left[\frac{t^{n-1}}{(n-1)!}\right](s) = \frac{1}{(n-1)!}\frac{(n-1)!}{s^n}$,

$$L^{-1}\left[\frac{1}{s^n}\right](t) = \frac{t^{n-1}}{(n-1)!}.$$

2. $\frac{1}{(s-a)^n}$.
Since $L\left[e^{at}\frac{t^{n-1}}{(n-1)!}\right](s) = L\left[\frac{t^{n-1}}{(n-1)!}\right](s-a) = \frac{1}{(s-a)^n}$,

$$L^{-1}\left[\frac{1}{(s-a)^n}\right](t) = e^{at}\frac{t^{n-1}}{(n-1)!}.$$

3. $\frac{s}{s^2+a^2}$.
We know that $L[\cos at](s) = \frac{s}{s^2+a^2}$. Therefore

$$L^{-1}\left[\frac{s}{s^2+a^2}\right](t) = \cos at.$$

4. $\frac{1}{s^2+a^2}$.
We know that $L[\frac{1}{a}\sin at](s) = \frac{1}{a}\frac{a}{s^2+a^2} = \frac{1}{s^2+a^2}$. Therefore

$$L^{-1}\left[\frac{1}{s^2+a^2}\right](t) = \frac{\sin at}{a}.$$

5. $\frac{s}{s^2-a^2}$.
We know that $L[\cosh at](s) = \frac{s}{s^2-a^2}$. Therefore

$$L^{-1}\left[\frac{s}{s^2-a^2}\right](t) = \cosh at.$$

6. $\frac{1}{s^2-a^2}$.
We know that $L[\frac{1}{a}\sinh at](s) = \frac{1}{a}\frac{a}{s^2-a^2} = \frac{1}{s^2-a^2}$. Therefore

$$L^{-1}\left[\frac{1}{s^2-a^2}\right](t) = \frac{\sinh at}{a}.$$

7. $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right](t) = \frac{1}{b}e^{at}\sin bt.$
8. $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right](t) = e^{at}\cos bt.$

3.3.1 Properties of inverse Laplace transform

1. Inverse Laplace transform is linear, i.e.

$$L^{-1}[\alpha F+\beta G]=\alpha L^{-1}[F]+\beta L^{-1}[G].$$

Proof. Let f and g be the inverse Laplace transform of F and G respectively. Therefore $L[f]=F$ and $L[g]=G$. Now

$$L[\alpha f+\beta g]=\alpha L[f]+\beta L[g]=\alpha F+\beta G.$$

Therefore

$$L^{-1}[\alpha F+\beta G]=\alpha f+\beta g=\alpha L^{-1}[F]+\beta L^{-1}[G].$$

□

Example 3.3.3. Compute the inverse Laplace transform of $\frac{s}{s^4+s^2+1}$.

Solution. We have,

$$\begin{aligned}\frac{s}{s^4+s^2+1} &= \frac{s}{(s^2+1)^2-s^2}\\ &= \frac{s}{(s^2+1-s)(s^2+1+s)}\\ &= \frac{1}{2}\left[\frac{1}{s^2-s+1}-\frac{1}{s^2+s+1}\right]\\ &= \frac{1}{2}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}-\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]\\ &= \frac{1}{2}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}-\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right].\end{aligned}$$

Therefore

$$\begin{aligned}L^{-1}\left[\frac{s}{s^4+s^2+1}\right](t) &= \frac{1}{2}L^{-1}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]-\frac{1}{2}L^{-1}\left[\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]\\ &= \frac{1}{2}\frac{e^{\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}}-\frac{1}{2}\frac{e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}}\\ &= \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)2\left(\frac{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}{2}\right)\\ &= \frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\sinh\left(\frac{t}{2}\right).\end{aligned}$$

□

2. **(Shifting property):**
If $\overline{f}$ is the Laplace transform of f, then

$$L^{-1}[\overline{f}(s-a)](t) = e^{at}f(t) = e^{at}L^{-1}[\overline{f}](t).$$

Proof. We know that, $L[e^{at}f(t)](s) = L[f](s-a) = \overline{f}(s-a)$. Therefore

$$L^{-1}[\overline{f}(s-a)](t) = e^{at}f(t) = e^{at}L^{-1}[\overline{f}](t).$$

□

Example 3.3.4. Compute the inverse Laplace transform of

1. $f(t) = \dfrac{s-2}{(s-2)^2+25} + \dfrac{s+4}{(s+4)^2-9} + \dfrac{1}{(s+2)^2+16}$
2. $g(t) = \dfrac{1}{(s-4)^2} + \dfrac{5}{(s-2)^2+25} + \dfrac{s+3}{(s+3)^2+4}$

Solution. 1. Since $L^{-1}\left[\frac{s}{s^2+a^2}\right](t) = \cos at$, $L^{-1}\left[\frac{a}{s^2+a^2}\right](t) = \sin at$ and $L^{-1}[\overline{f}(s-a)](t) = e^{at}f(t)$, it follows that

$$L^{-1}[f](t) = e^{2t}\cos 5t + e^{-4t}\cosh 3t + \frac{1}{4}e^{-2t}\sin 4t.$$

2. $L^{-1}[g](t) = te^{4t} + e^{2t}\sin 5t + e^{-3t}\cos 2t.$

□

3. **(Second shifting property):**
If $\overline{f}$ is the Laplace transform of f and $a > 0$, then

$$L^{-1}[e^{-as}\overline{f}(s)](t) = f(t-a)H(t-a).$$

Proof. $L[f(t-a)H(t-a)](s) = \int_a^\infty f(t-a)e^{-st}dt = \int_0^\infty f(u)e^{-s(u+a)}du = e^{-sa}\overline{f}(s)$.
Therefore $L^{-1}[e^{-as}\overline{f}(s)](t) = f(t-a)H(t-a)$. □

Example 3.3.5. Compute the inverse Laplace transform of

1. $\dfrac{e^{-2s}}{s^2+1}$
2. $e^{-3s}\dfrac{1}{(s-1)^2+2}$
3. $\dfrac{e^{-2s}}{(s-4)^2}$

Solution. 1. Let $f(t) = \sin t$. Then $\overline{f}(s) = \dfrac{1}{s^2+1}$. Then

$$\begin{aligned} L^{-1}\left[e^{-2s}\frac{1}{s^2+1}\right](t) &= L^{-1}\left[e^{-2s}\overline{f}(s)\right](t) \\ &= f(t-2)H(t-2) \\ &= \sin(t-2)H(t-2). \end{aligned}$$

2. Let $f(t) = \frac{e^t}{\sqrt{2}}\sin(\sqrt{2}t)$. Then $\overline{f}(s) = \frac{1}{(s-1)^2+2}$. Then

$$\begin{aligned} L^{-1}\left[e^{-3s}\frac{1}{(s-1)^2+2}\right](t) &= L^{-1}\left[e^{-3s}\overline{f}(s)\right](t) \\ &= f(t-3)H(t-3) \\ &= \frac{e^{t-3}}{\sqrt{2}}\sin(\sqrt{2}(t-3))H(t-3). \end{aligned}$$

3. Let $f(t) = te^{4t}$. Then $\overline{f}(s) = \frac{1}{(s-4)^2}$.

$$L^{-1}\left[\frac{e^{-2s}}{(s-4)^2}\right](t) = L^{-1}[e^{-2s}\overline{f}(s)](t) = f(t-\pi)H(t-\pi) = (t-2)e^{4(t-2)}H(t-2).$$

□

4. **(Change of scale):**
Let $\overline{f}$ be the Laplace transform of f. If $a > 0$, then

$$L^{-1}[\overline{f}(as)](t) = \frac{1}{a}f\left(\frac{t}{a}\right).$$

Proof. We have

$$\begin{aligned} L\left[\frac{1}{a}f\left(\frac{t}{a}\right)\right](s) &= \int_0^\infty \frac{1}{a}f\left(\frac{t}{a}\right)e^{-st}dt \\ &= \frac{1}{a}\int_0^\infty f(u)e^{-sau}a\,du \qquad \left(\text{taking } \frac{t}{a} = u\right) \\ &= \int_0^\infty f(u)e^{-sau}\,du = \overline{f}(as). \end{aligned}$$

Therefore $L^{-1}[\overline{f}(as)](t) = \frac{1}{a}f\left(\frac{t}{a}\right)$. □

Example 3.3.6. If $L^{-1}\left[\frac{s^2-1}{(s^2+1)^2}\right](t) = t\cos t$, then find the inverse Laplace transform of $\frac{9s^2-1}{(9s^2+1)^2}$.

Solution. Let $f(t) = t\cos t$, then $\overline{f}(s) = \frac{s^2-1}{(s^2+1)^2}$. Now

$$\begin{aligned} L^{-1}\left[\frac{9s^2-1}{(9s^2+1)^2}\right](t) &= L^{-1}\left[\frac{(3s)^2-1}{((3s)^2+1)^2}\right] \\ &= L^{-1}[\overline{f}(3s)](t) \\ &= \frac{1}{3}f\left(\frac{t}{3}\right) = \frac{t}{9}\cos\left(\frac{t}{3}\right). \end{aligned}$$

□

Example 3.3.7. 1. If $f(t)=\sin t$, then find the inverse Laplace transform of $\dfrac{1}{4s^2+1}$.

2. If $f(t)=\cosh(2\sqrt{2}t)$, then find the inverse Laplace transform of $\dfrac{s}{s^2-8}$.

Solution. 1. Let $f(t)=\sin t$. Then $\overline{f}(s)=\dfrac{1}{s^2+1}$. Now,

$$\begin{aligned}L^{-1}\left[\frac{1}{4s^2+1}\right](t) &= L^{-1}\left[\frac{1}{(2s)^2+1}\right](t)\\ &= L^{-1}\left[\overline{f}(2s)\right](t)\\ &= \frac{1}{2}f\left(\frac{t}{2}\right)=\frac{1}{2}\sin\left(\frac{1}{2}\right).\end{aligned}$$

2. Let $f(t)=\cosh(2\sqrt{2}t)$. Then $\overline{f}(s)=\frac{s}{s^2-8}$. Now

$$\begin{aligned}L^{-1}\left[\frac{s}{s^2-8}\right](t) &= \frac{1}{\sqrt{2}}L^{-1}\left[\frac{\sqrt{2}s}{(\sqrt{2}s)^2-8}\right](t)\\ &= \frac{1}{\sqrt{2}}L^{-1}[\overline{f}(\sqrt{2}s)](t)\\ &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}f\left(\frac{t}{\sqrt{2}}\right)=\frac{1}{2}\cosh 2t.\end{aligned}$$

□

5. If $\overline{f}$ is the Laplace transform of f, then

$$L^{-1}[\overline{f}^{(n)}](t)=(-1)^n t^n f(t).$$

Proof. We know that $L[t^n f(t)](s)=(-1)^n\dfrac{d^n}{ds^n}L[f](s)=(-1)^n\overline{f}^{(n)}(s)$. Therefore, it follows that $L^{-1}[\overline{f}^{(n)}](t)=(-1)^n t^n f(t)$. □

Example 3.3.8. Compute the inverse Laplace transform of $\log\left(\dfrac{s^2+4}{s^2+9}\right)$.

Solution. Let $\overline{f}(s)=\log\left(\dfrac{s^2+4}{s^2+9}\right)$. Therefore, $\overline{f}'(s)=\dfrac{2s}{s^2+4}-\dfrac{2s}{s^2+9}$. Then $L^{-1}[\overline{f}'](t)=2\cos 2t-2\cos 3t$. But by above property, $L^{-1}[\overline{f}'](t)=-tf(t)$. Therefore

$$f(t)=-\frac{1}{t}L^{-1}[\overline{f}'](t)=\frac{2(\cos 3t-\cos 2t)}{t}.$$

□

Example 3.3.9. Compute the inverse Laplace transform of $\log\left(\dfrac{s^2+4}{s+9}\right)$.

Solution. Let $\overline{f}(s) = \log\left(\frac{s^2+4}{s+9}\right)$. Therefore, $\overline{f}'(s) = \frac{2s}{s^2+4} - \frac{2s}{s+9}$. Then $L^{-1}[\overline{f}'](t) = 2\cos 2t - e^{-9t}$. But by above property, $L^{-1}[\overline{f}'](t) = -tf(t)$. Therefore

$$f(t) = -\frac{1}{t}L^{-1}[\overline{f}'](t) = \frac{e^{-9t} - 2\cos 2t}{t}.$$

□

6. **(Division by s):**
   Let $\overline{f}$ be the Laplace transform of f, then

$$L^{-1}\left[\frac{\overline{f}(s)}{s}\right](t) = \int_0^t f(u)du.$$

Proof. Let $G(t) = \int_0^t f(u)du$, then $G(0) = 0$, and $G'(t) = f(t)$ for all t. Now

$$L[G'(t)](s) = sL[G(t)](s) - G(0) = sL\left[\int_0^t f(u)du\right](s).$$

Therefore

$$\frac{\overline{f}(s)}{s} = \frac{L[f](s)}{s} = L\left[\int_0^t f(u)du\right](s).$$

Hence

$$L^{-1}\left[\frac{\overline{f}(s)}{s}\right](t) = \int_0^t f(u)du.$$

□

Example 3.3.10. Compute the inverse Laplace transform of

1. $\frac{1}{s(s+1)^4}$
2. $\frac{1}{s(s^2+4)}$

Solution. 1. We have $L^{-1}\left[\frac{1}{(s+1)^4}\right](t) = e^{-t}L^{-1}\left[\frac{1}{s^4}\right](t) = e^{-t}\frac{t^3}{3!}$.

Let $f(t) = \frac{e^{-t}t^3}{6}$. Then $\overline{f}(s) = \frac{1}{(s+1)^4}$. Now,

$$\begin{aligned} L^{-1}\left[\frac{1}{s(s+1)^4}\right](t) &= L^{-1}\left[\frac{\overline{f}(s)}{s}\right](t) \\ &= \int_0^t \frac{e^{-u}u^3}{6}du \\ &= \frac{1}{6}\left[u^3\frac{e^{-u}}{-1} - 3u^2e^{-u} + 6u\frac{e^{-u}}{-1} - 6e^{-u}\right]_0^t \\ &= -\frac{1}{6}\left[t^3e^{-t} + 3t^2e^{-t} + 6te^{-t} + 6e^{-t} - 6\right] \\ &= -\frac{e^{-t}}{6}\left[t^3 + 3t^2 + 6t + 6 - 6e^t\right]. \end{aligned}$$

2. We have $L^{-1}\left[\frac{1}{s^2+4}\right](t) = \frac{1}{2}\sin 2t$. Let $f(t) = \frac{\sin 2t}{2}$. Then $\overline{f}(s) = \frac{1}{s^2+4}$. Now,

$$\begin{aligned} L^{-1}\left[\frac{1}{s(s^2+4)}\right](t) &= L^{-1}\left[\frac{\overline{f}(s)}{s}\right](t) \\ &= \int_0^t \frac{\sin 2u}{2}\, du \\ &= \frac{1}{2}\left[-\frac{\cos 2u}{2}\right]_0^t \\ &= \frac{1}{4}[-\cos 2t + 1] = \frac{1-\cos 2t}{4}. \end{aligned}$$

□

3.4 Convolution Product

Definition 3.4.1. Let $f, g : (0,\infty) \to \mathbb{C}$. Then the *convolution product* $*$ of f and g is defined as

$$(f*g)(t) = \int_0^t f(u)g(t-u)\, du \qquad (t>0).$$

Ex Show that

1. $f*g = g*f$, i.e. convolution is commutative.
2. $(f*g)*h = f*(g*h)$, i.e. convolution is associative.

Theorem 3.4.2 (Convolution Theorem)**.** *Let $\overline{f}$ and $\overline{g}$ be Laplace transforms of f and g respectively. Then*

$$L[f*g] = \overline{f}\overline{g} = L[f]L[g].$$

Corollary 3.4.3. *If $\overline{f}_i$ is the Laplace transform of f_i, $i = 1,2,\dots,n$, then*

$$L[f_1 * f_2 * \cdots * f_n] = L[f_1]L[f_2]\cdots L[f_n].$$

Corollary 3.4.4. *If $\overline{f}$ and $\overline{g}$ are Laplace transforms of f and g respectively, then*

$$L^{-1}[\overline{f}\overline{g}] = f*g.$$

Example 3.4.5. Compute the inverse Laplace transform of $\frac{1}{(s+a)(s+b)}$.

Solution. Let $f(t) = e^{-at}$ and $g(t) = e^{-bt}$. Then $\overline{f}(s) = \frac{1}{s+a}$ and $\overline{g}(s) = \frac{1}{s+b}$. Now,

$$L^{-1}\left[\frac{1}{(s+b)(s+b)}\right](t) = L^{-1}[\overline{f}\overline{g}](t)$$

$$
\begin{aligned}
&= (f*g)(t) \qquad \text{(by Convolution theorem)}\\
&= \int_0^t e^{-au}e^{-b(t-u)}\,du\\
&= e^{-bt}\int_0^t e^{(b-a)u}\,du\\
&= \frac{e^{-bt}}{b-a}\left[e^{(b-a)u}\right]_0^t\\
&= \frac{e^{-bt}}{b-a}\left[e^{(b-a)t}-1\right] = \frac{e^{-at}-e^{-bt}}{b-a}.
\end{aligned}
$$

□

Example 3.4.6. Compute the inverse Laplace transform of $\frac{1}{(s^2+a^2)(s^2+b^2)}$ by using convolution theorem.

Solution. Let $f(t) = \frac{1}{a}\sin at$ and $g(t) = \frac{1}{b}\sin bt$. Then $\overline{f}(s) = \frac{1}{s^2+a^2}$ and $\overline{g}(s) = \frac{1}{s^2+b^2}$. Now,

$$
\begin{aligned}
L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right](t) &= L^{-1}[\overline{f}\overline{g}](t) = (f*g)(t)\\
&= \frac{1}{ab}\int_0^t \sin au \sin b(t-u)\,du\\
&= -\frac{1}{2ab}\int_0^t -2\sin au\sin b(t-u)\,du\\
&= -\frac{1}{2ab}\int_0^t [\cos((a-b)u+bt)-\cos((a+b)u-bt)]\,du\\
&= -\frac{1}{2ab}\left[\frac{\sin((a-b)u+bt)}{a-b}-\frac{\sin((a+b)u-bt)}{a+b}\right]_0^t\\
&= -\frac{1}{2ab}\left[\frac{\sin at}{a-b}-\frac{\sin at}{a+b}-\frac{\sin bt}{a-b}+\frac{\sin(-bt)}{a+b}\right]\\
&= -\frac{1}{2ab}\left[\frac{-2b\sin at-2a\sin bt}{a^2-b^2}\right]\\
&= \frac{1}{ab}\left[\frac{b\sin at+a\sin bt}{a^2-b^2}\right].
\end{aligned}
$$

□

Example 3.4.7. Compute the inverse Laplace transform of $\frac{s}{(s^2+a^2)^2}$.

Solution. Let $f(t) = \cos at$ and $g(t) = \frac{1}{a}\sin at$. Then $\overline{f}(s) = \frac{s}{s^2+a^2}$ and $\overline{g}(t) = \frac{1}{s^2+a^2}$. Now,

$$
\begin{aligned}
L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right](t) &= L^{-1}[\overline{f}\overline{g}](t) = (f*g)(t)\\
&= \frac{1}{a}\int_0^t \cos au\sin a(t-u)\,du\\
&= \frac{1}{2a}\int_0^t [\sin(at)-\sin(2au-at)]\,du
\end{aligned}
$$

$$= \frac{1}{2a}\left[u\sin at + \frac{\cos(2au - at)}{2a}\right]_0^t$$
$$= \frac{1}{2a}\left[t\sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right]$$
$$= \frac{1}{2a} t\sin at.$$

□

Example 3.4.8. Compute the inverse Laplace transform of $\frac{1}{(s-1)(s^2+1)}$.

Solution. Let $f(t) = e^t$ and $g(t) = \sin t$. Then $\overline{f}(s) = \frac{1}{s-1}$ and $\overline{g}(s) = \frac{1}{s^2+1}$. Now,

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right](t) = L\left[\overline{f}\overline{g}\right](t) = (f*g)(t)$$
$$= \int_0^t e^u \sin(t-u)\,du$$
$$= \int_0^t e^{(t-u)} \sin u\,du \qquad (\because f*g = g*f)$$
$$= \int_0^t e^t e^{-u} \sin u\,du$$
$$= e^t\left[\frac{e^{-u}}{2}(-\sin u - \cos u)\right]_0^t$$
$$= e^t\left[\frac{e^{-t}}{2}(-\sin t - \cos t) - \frac{1}{2}(-1)\right]$$
$$= \frac{-\sin t - \cos t + e^t}{2}.$$

□

Example 3.4.9. Compute the inverse Laplace transform of $\frac{s+1}{(s^2+2s+2)^2}$.

Solution. We have $\frac{s+1}{(s^2+2s+2)^2} = \frac{s+1}{((s+1)^2+1^2}$.
We know that $L^{-1}\left[\frac{s}{(s^2+1)^2}\right](t) = \frac{t}{2}\sin t$. Let $h(t) = \frac{t}{2}\sin t$. Then

$$L^{-1}\left[\frac{s+1}{((s+1)^2+1)^2}\right](t) = L^{-1}\left[\overline{h}(s+1)\right](t)$$
$$= e^{-t}h(t) = \frac{e^{-t}}{2} t\sin t.$$

□

The above example is solved by direct method. Verify the solution by solving it using convolution theorem.

Example 3.4.10. Compute the inverse Laplace transform of $\frac{s}{(s^2+4)^3}$.

Solution. By Example 3.4.7 (taking $a=2$), we have

$$L^{-1}\left[\frac{s}{(s^2+4)^2}\right](t)=\frac{t}{4}\sin 2t.$$

Let $h(t)=\frac{t}{4}\sin 2t$ and $k(t)=\frac{1}{2}\sin 2t$. Then $\overline{h}(s)=\frac{s}{(s^2+4)^2}$ and $\overline{k}(s)=\frac{1}{s^2+4}$. Then

$$\begin{aligned}
L^{-1}\left[\frac{s}{(s^2+4)^3}\right](t) &= L^{-1}[\overline{hk}](t)=(h*k)(t)\\
&= \int_0^t \frac{u}{4}\sin 2u\,\frac{1}{2}\sin 2(t-u)\,du\\
&= -\frac{1}{16}\int_0^t [(u\cos 2t)-(u\cos(4u-2t))]\,du\\
&= -\frac{1}{16}\left[\frac{u^2}{2}\cos 2t-\left\{u\frac{\sin(4u-2t)}{4}-1\cdot\frac{-\cos(4u-2t)}{16}\right\}\right]_0^t\\
&= -\frac{1}{16}\left[\frac{t^2}{2}\cos 2t-t\frac{\sin 2t}{4}-\frac{\cos 2t}{16}+\frac{\cos 2t}{16}\right]\\
&= \frac{t\sin 2t-2t^2\cos 2t}{64}.
\end{aligned}$$

□

Solution. **[Alternate solution]**
Take $f(t)=\cos 2t$ and $g(t)=\frac{\sin 2t}{2}$. Then $\overline{f}(s)=\frac{s}{s^2+4}$ and $\overline{g}(s)=\frac{1}{s^2+4}$. Now

$$L^{-1}\left[\frac{s}{(s^2+4)^3}\right](t) = L^{-1}[\overline{f}\,\overline{g}\,\overline{g}](t)=(f*g*g)(t)=[f*(g*g)](t).$$

We shall find $g*g$. By definition

$$\begin{aligned}
(g*g)(t) &= \int_0^t g(u)g(t-u)du=\frac{1}{4}\int_0^t \sin 2u\sin(2t-2u)du\\
&= \frac{1}{8}\int_0^t [\cos(4u-2t)-\cos(2t)]du=\frac{1}{8}\left[\frac{\sin(4u-2t)}{4}-u\cos(2t)\right]_0^t\\
&= \frac{1}{8}\left[\frac{\sin 2t}{2}-t\cos 2t\right]
\end{aligned}$$

Now compute $f*(g*g)$ (Exercise). □

Other Miscellaneous Examples

Example 3.4.11. 1. Compute $L^{-1}\left[\frac{s}{s^4+4}\right]$.

$$L^{-1}\left[\frac{s}{s^4+4}\right] = L^{-1}\left[\frac{s}{(s^2-2s+2)(s^2+2s+2)}\right]$$

$$\begin{aligned}
&= \frac{1}{4}L^{-1}\left[\frac{1}{s^2-2s+2}-\frac{1}{s^2+2s+2}\right]\\
&= \frac{1}{4}L^{-1}\left[\frac{1}{s^2-2s+2}\right]-\frac{1}{4}L^{-1}\left[\frac{1}{s^2+2s+2}\right]\\
&= \frac{1}{4}L^{-1}\left[\frac{1}{(s-1)^2+1}\right]-\frac{1}{4}L^{-1}\left[\frac{1}{(s+1)^2+1}\right]\\
&= \frac{1}{4}e^t\sin t-\frac{1}{4}e^{-t}\sin t\\
&= \frac{1}{2}\sin t\sinh t.
\end{aligned}$$

2. Compute $L^{-1}\left[\frac{s(s^2+2)}{s^4+4}\right]$.

$$\begin{aligned}
L^{-1}\left[\frac{s(s^2+2)}{s^4+4}\right](t) &= L^{-1}\left[\frac{s(s^2+2)}{(s^2-2s+2)(s^2+2s+2)}\right](t)\\
&= \frac{1}{2}L^{-1}\left[\frac{s}{s^2-2s+2}+\frac{s}{s^2+2s+2}\right](t)\\
&= \frac{1}{2}L^{-1}\left[\frac{s}{s^2-2s+2}\right](t)-\frac{1}{2}L^{-1}\left[\frac{s}{s^2+2s+2}\right](t)\\
&= \frac{1}{2}L^{-1}\left[\frac{s-1}{(s-1)^2+1}+\frac{1}{(s-1)^2+1}\right](t)\\
&\quad -\frac{1}{2}L^{-1}\left[\frac{s+1}{(s+1)^2+1}-\frac{1}{(s+1)^2+1}\right](t)\\
&= \frac{1}{2}[e^t\cos t+e^t\sin t]-\frac{1}{2}[e^{-t}\cos t-e^{-t}\sin t]\\
&= \frac{e^t-e^{-t}}{2}\cos t+\frac{e^t+e^{-t}}{2}\sin t\\
&= \sinh t\cos t+\cosh t\sin t.
\end{aligned}$$

3. Find $L^{-1}\left[\ln\left(\frac{s+3}{s+2}\right)\right]$.
Let $\overline{f}(s)=\left[\ln\left(\frac{s+3}{s+2}\right)\right]$. Then $\overline{f}'(s)=\frac{1}{s+3}-\frac{1}{s+2}$. So, $L^{-1}[\overline{f}'(s)](t)=e^{-3t}-e^{-2t}$. Now $L^{-1}[\overline{f}^{(n)}](t)=(-1)^n t^n L^{-1}[\overline{f}](t)$. Therefore

$$e^{-3t}-e^{-2t}=(-1)tL^{-1}[\overline{f}](t)=(-1)tL^{-1}\left[\ln\left(\frac{s+3}{s+2}\right)\right].$$

Hence

$$L^{-1}\left[\ln\left(\frac{s+3}{s+2}\right)\right](t)=\frac{e^{-2t}-e^{-3t}}{t}.$$

4. Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$.
Let $f(t)=\sin at$. Then $\overline{f}(s)=\frac{a}{s^2+a^2}$. Therefore $\overline{f}'(s)=\frac{-2as}{(s^2+a^2)^2}$. Now

$$\begin{aligned}
L^{-1}\left[\frac{-2as}{(s^2+a^2)^2}\right](t) &= (-1)tL^{-1}[\overline{f}](t)\\
&= (-1)tf(t)=-t\sin at.
\end{aligned}$$

Therefore $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right](t)=\frac{1}{2a}t\sin at$.

5. Find $L^{-1}\left[\frac{1}{s(s+1)^2}\right]$.
   Let $f(t) = te^{-t}$. Then $\overline{f}(s) = \frac{1}{(s+1)^2}$. Now

$$\begin{aligned} L^{-1}\left[\frac{1}{s(s+1)^2}\right](t) &= \int_0^t f(u)du \\ &= \int_0^t ue^{-u}du \\ &= \left[u\frac{e^{-u}}{-1} + \int e^{-u}\right]_0^t \\ &= [-ue^{-u} - e^{-u}]_0^t \\ &= -te^{-t} - e^{-t} + 1 \\ &= 1 - e^{-t} - te^{-t}. \end{aligned}$$

Example 3.4.12. Evaluate the inverse Laplace Transform of following functions using Convolution Theorem.

1. $\frac{s}{(s+a)(s^2+1)}$
2. $\frac{1}{s^2(s+1)}$.

Solution. 1. Take $f(t) = e^{-at}$ and $g(t) = \cos t$. Then $\overline{f}(s) = \frac{1}{s+a}$ and $\overline{g}(s) = \frac{1}{s^2+1}$. Using Convolution Theorem, we get

$$\begin{aligned} L^{-1}\left[\frac{s}{(s+a)(s^2+1)}\right](t) &= L^{-1}[\overline{f}(s)\overline{g}(s)](t) = (f*g)(t) \\ &= \int_0^t e^{-a(t-u)}\cos u\,du \\ &= e^{-at}\left[\frac{e^{au}}{a^2+1}[a\cos u + \sin u]\right]_0^t \\ &= \frac{e^{-at}}{a^2+1}[e^{at}(a\cos t + \sin t) - a] \end{aligned}$$

2. Take $f(t) = t$ and $g(t) = e^{-t}$. Then $\overline{f}(s) = \frac{1}{s^2}$ and $\overline{g}(s) = \frac{1}{s+1}$. By Convolution Theorem

$$\begin{aligned} L^{-1}\left[\frac{1}{s^2(s+1)}\right](t) &= L^{-1}[\overline{f}(s)\overline{g}(s)](t) = (f*g)(t) \\ &= \int_0^t ue^{-(t-u)}\,du = e^{-t}\int_0^t ue^u\,du \\ &= e^t[ue^u - e^u]_0^t = e^t([te^t - e^t] - (-1)) \\ &= te^{2t} - e^{2t} + e^t \end{aligned}$$

□

Ex Compute the inverse Laplace transforms of following using Convolution Theorem.

lightatinfinite@gmail.com
jay_mehta@spuvvn.edu

1. $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$
2. $\frac{s^2}{(s^2-a^2)(s^2-b^2)}$
3. $\frac{1}{(s^2-a^2)(s^2-b^2)}$
4. $\frac{s}{(s^2-a^2)(s^2-b^2)}$
5. $\frac{1}{(s-a)\sqrt{s}}$

Ex

1. Show that $L^{-1}\left[\frac{e^{-\frac{1}{s}}}{s}\right](t) = J_0(2\sqrt{t})$.
2. Compute the inverse Laplace transforms of $\frac{(s+1)e^{-\pi s}}{s^2+s+1}$ and $\frac{s+1}{(s^2+2s+2)^2}$

3.5 Applications of Laplace transform

There are many applications of Laplace transform. Here, we shall see some of its applications to integral equations, ordinary and partial differential equations, and simultaneous differential equations.

3.5.1 Applications to Ordinary Differential Equations

We recall that $L[f^{(n)}](s) = s^n\overline{f}(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$ and $L[t^n f(t)](s) = (-1)^n \frac{d^n}{ds^n}\overline{f}(s)$.

Example 3.5.1. Solve $y'' + 9y = \cos 2t$ subject to $y(0) = 1 = y'(0)$.

Solution. Let Y be the Laplace transform of y. Applying Laplace transform to the given equation, we get

$$\begin{aligned} &L[y''](s) + 9L[y](s) = L[\cos 2t](s) \\ \Rightarrow\ & s^2L[y](s) - sy(0) - y'(0) + 9L[y](s) = \frac{s}{s^2+4} \\ \Rightarrow\ & s^2Y - s - 1 + 9Y = \frac{s}{s^2+4} \\ \Rightarrow\ & (s^2+9)Y = \frac{s}{s^2+4} + s + 1. \end{aligned}$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{s+1}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)} \\ &= \frac{s}{s^2+9} + \frac{1}{3}\frac{3}{s^2+9} + \frac{1}{5}\left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right]. \end{aligned}$$

Applying inverse Laplace transform, we get

$$\begin{aligned} y(t) &= \cos 3t + \frac{1}{3}\sin 3t + \frac{\cos 2t}{5} - \frac{\cos 3t}{5} \\ &= \frac{1}{3}\sin 3t + \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t. \end{aligned}$$

□

Example 3.5.2. Solve $2y'' + 5y' + 2y = e^{-2t}$ subject to $y'(0) = y(0) = 1$.

Solution. Let Y be the Laplace transform of y. Applying Laplace transform to the given equation, we get

$$\begin{aligned}
& 2L[y''](s) + 5L[y'](s) + 2L[y](s) = L[e^{-2t}](s) \\
\Rightarrow\ & 2(s^2Y(s) - sy(0) - y'(0)) + 5(sY(s) - y(0)) + 2Y(s) = \frac{1}{s+2} \\
\Rightarrow\ & (2s^2 + 5s + 2)Y(s) = 2s + 2 + 5 + \frac{1}{s+2} \\
\Rightarrow\ & (2s+1)(s+2)Y(s) = (2s+1) + 6 + \frac{1}{s+2} \\
\Rightarrow\ & Y(s) = \frac{1}{s+2} + \frac{6}{(2s+1)(s+2)} + \frac{1}{(2s+1)(s+2)^2} \\
\Rightarrow\ & Y(s) = \frac{1}{s+2} + \frac{12}{(2s+1)(2s+4)} + \frac{1}{2(s+\frac{1}{2})(s+2)^2} \\
\Rightarrow\ & Y(s) = \frac{1}{s+2} + \left\{\frac{12}{3}\left[\frac{1}{2s+1} - \frac{1}{2s+4}\right]\right\} + \frac{1}{2}\frac{1}{(s+\frac{1}{2})(s+2)^2} \\
\Rightarrow\ & Y(s) = \frac{1}{s+2} + \frac{2}{s+\frac{1}{2}} - \frac{2}{s+2} + \frac{1}{2}\frac{1}{(s+\frac{1}{2})(s+2)^2}
\end{aligned}$$

Applying inverse Laplace transform and using convolution theorem, we have

$$\begin{aligned}
y(t) &= e^{-2t} + 2e^{-\frac{t}{2}} - 2e^{-2t} + \frac{1}{2}\int_0^t ue^{-2u}e^{-\frac{1}{2}(t-u)}\,du \\
&= 2e^{-\frac{t}{2}} - e^{-2t} + \frac{1}{2}\int_0^t ue^{-\frac{3u}{2}-\frac{t}{2}}\,du \\
&= 2e^{-\frac{t}{2}} - e^{-2t} + \frac{1}{2}\left[u\left(\frac{e^{-\frac{3u}{2}-\frac{t}{2}}}{-\frac{3}{2}}\right) - \frac{e^{-\frac{3u}{2}-\frac{t}{2}}}{\frac{9}{4}}\right]_0^t \\
&= 2e^{-\frac{t}{2}} - e^{-2t} + \frac{1}{2}\left[\left(-2t\frac{e^{-2t}}{3} - 4\frac{e^{-2t}}{9}\right) + \frac{4}{9}e^{-\frac{t}{2}}\right] \\
&= \frac{18e^{-\frac{t}{2}} - 9e^{-2t} - 3te^{-2t} - 2e^{-2t} + 2e^{-\frac{t}{2}}}{9} \\
&= \frac{20e^{-\frac{t}{2}} - 3te^{-2t} - 11e^{-2t}}{9}.
\end{aligned}$$

□

Example 3.5.3. Solve $y^{(4)} - y = 1$ subject to $y(0) = y'(0) = y''(0) = y'''(0) = 0$.

Solution. Let Y be the Laplace transform of y. Applying Laplace transform to the given equation, we get

$$\begin{aligned}
& L[y^{(4)}](s) - L[y](s) = L[1](s) \\
\Rightarrow\ & s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - Y(s) = \frac{1}{s}
\end{aligned}$$

$$\Rightarrow Y(s) = \frac{1}{s(s^4-1)} = \frac{1}{2}\left[\frac{1}{s}\frac{1}{s^2-1} - \frac{1}{s}\frac{1}{s^2+1}\right].$$

Applying inverse Laplace transform, we get

$$\begin{aligned} y(t) &= \frac{1}{2}\int_0^t \sinh u\, du - \frac{1}{2}\int_0^t \sin u\, du \\ &= \frac{1}{2}[\cosh u + \cos u]_0^t = \frac{\cosh t + \cos t}{2} - 1. \end{aligned}$$

□

Example 3.5.4. Solve $ty'' + y' + 4ty = 0$ subject to $y(0) = 3$ and $y'(0) = 0$.

Solution. Let Y be the Laplace transform of y. Applying Laplace transform to the given equation, we get

$$\begin{aligned} & -\frac{d}{ds}L[y''](s) + L[y'](s) - 4\frac{d}{ds}L[y](s) = 0 \\ \Rightarrow\ & -\frac{d}{ds}[s^2Y(s) - sy(0) - y'(0)] + sY(s) - y(0) - 4\frac{dY}{ds} = 0 \\ \Rightarrow\ & -s^2\frac{dY}{ds} - 2sY(s) + y(0) + sY(s) - y(0) - 4\frac{dY}{ds} = 0 \\ \Rightarrow\ & -(s^2+4)\frac{dY}{ds} - sY = 0 \\ \Rightarrow\ & \frac{dY}{Y} + \frac{1}{2}\frac{2s}{s^2+4}ds = 0 \\ \Rightarrow\ & \log Y + \frac{1}{2}\log(s^2+4) = \log c \\ \Rightarrow\ & Y(s) = \frac{c}{\sqrt{s^2+4}}. \end{aligned}$$

Applying inverse Laplace transform, we get

$$y(t) = cL^{-1}\left[\frac{1}{\sqrt{s^2+4}}\right](t).$$

□

Example 3.5.5. Solve $y'' + 6y' + 9y = 6t^2e^{-3t}$ subject to $y(0) = 1$ and $y'(0) = 2$.

Solution. Let Y be the Laplace transform of y. Applying Laplace transform to the given equation, we get

$$\begin{aligned} & L[y''](s) + 6L[y'](s) + 9L[y](s) = 6L[t^2e^{-3t}](s) \\ \Rightarrow\ & s^2Y(s) - sy(0) - y'(0) + 6sY(s) - 6y(0) + 9Y(s) = 6\frac{2}{(s+3)^3} \\ \Rightarrow\ & (s^2+6s+9)Y(s) - s - 2 - 6 = \frac{12}{(s+3)^3} \\ \Rightarrow\ & (s+3)^2Y(s) = (s+3) + 5 + \frac{12}{(s+3)^3} \end{aligned}$$

$$\Rightarrow Y(s) = \frac{1}{s+3} + \frac{5}{(s+3)^2} + \frac{12}{(s+3)^5}.$$

Applying inverse Laplace transform, we get

$$y(t) = e^{-3t} + 5te^{-3t} + 12e^{-3t}\frac{t^4}{4!} = e^{-3t}\left[1 + 5t + \frac{t^4}{2}\right].$$

□

Example 3.5.6. Solve $y'' + 2y' + 5y = e^{-t}\sin t$ subject to $y(0) = 3$, $y'(0) = 1$.

Solution. Let Y be the Laplace transform of y. Applying Laplace transform to the given equation, we get

$$\begin{aligned}
& L[y''](s) + 2L[y'](s) + 5L[y](s) = L[e^{-t}\sin t](s) \\
\Rightarrow\ & s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + 5Y(s) = \frac{1}{(s+1)^2+1} \\
\Rightarrow\ & (s^2+2s+5)Y(s) - 3s - 1 - 6 = \frac{1}{(s+1)^2+1} \\
\Rightarrow\ & (s^2+2s+1+4)Y(s) = 3s + 7 + \frac{1}{(s+1)^2+1} \\
\Rightarrow\ & ((s+1)^2+4)Y(s) = 3(s+1) + 4 + \frac{1}{(s+1)^2+1} \\
\Rightarrow\ & Y(s) = 3\frac{(s+1)}{(s+1)^2+4} + \frac{4}{(s+1)^2+4} + \frac{1}{((s+1)^2+1)((s+1)^2+4)} \\
\Rightarrow\ & Y(s) = 3\frac{(s+1)}{(s+1)^2+4} + \frac{4}{(s+1)^2+4} + \frac{1}{3}\left[\frac{1}{(s+1)^2+1} - \frac{1}{(s+1)^2+4}\right].
\end{aligned}$$

Applying inverse Laplace transform, we get

$$\begin{aligned}
y(t) &= 3e^{-t}\cos 2t + 2e^{-t}\sin 2t + \frac{1}{3}e^{-t}\sin t - \frac{1}{6}e^{-t}\sin 2t \\
&= 3e^{-t}\cos 2t + \frac{11}{6}e^{-t}\sin 2t + \frac{1}{3}e^{-t}\sin t \\
&= \frac{e^{-t}}{6}\left[18\cos 2t + 11\sin 2t + 2\sin t\right].
\end{aligned}$$

□

Example 3.5.7. 1. Solve $y'' + 9y = \cos 2t$, $y(0) = 1$ and $y(\pi/2) = 1$ using Laplace transform methods.

Let Y be the Laplace transform of y. Applying Laplace transform to the equation $y'' + 9y = \cos 2t$ we get $s^2Y - sy(0) - y'(0) + 9Y = \frac{s}{s^2+4}$. Substituting $y(0) = 1$ and simplifying it we get

$$Y = \frac{s}{s^2+9} + y'(0)\frac{1}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}.$$

Applying inverse Laplace transform to the last equation we obtain

$$\begin{aligned} y(t) &= \cos 3t + \frac{y'(0)}{3}\sin 3t + \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t \\ &= \frac{4}{5}\cos 3t + \frac{y'(0)}{3}\sin 3t + \frac{1}{5}\cos 2t. \end{aligned}$$

Since $y(\pi/2) = 1$, we get $a = \frac{12}{5}$. Hence $y(t) = \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t$ is the required solution.

Ex Solve the following differential equations using Laplace transform methods.

1. $y'' + 4y = 9t$, $y(0) = 0$, $y'(0) = 7$.
2. $y'' + 6y' + 9y = 6t^2e^{-3t}$, $y(0) = y'(0) = 0$.
3. $(D^3 - D^2 + 4D - 4)y = 68e^t \sin t$, $y(0) = 1$, $y'(0) = -19$, $y''(0) = -37$.
4. $ty'' + 2y' + ty = \sin t$, $y(0) = 1$.
5. $ty'' + (1-2t)y' - 2y = 0$, $y(0) = 1$.
6. $y'' + 9y = 9u(t-3)$, $y(0) = y'(0) = 0$.

3.5.2 Applications to Integral Equations

We shall now solve some integral equations by using Laplace transform methods.

Example 3.5.8. Solve the integral equation $F(t) = 1 + 2\int_0^t F(t-u)e^{-2u}\,du$.

Solution. We want to find all F satisfying the above equation. Let $G(t) = e^{-2t}$. Then $\overline{G}(s) = \dfrac{1}{s+2}$. Then the given equation may be written as

$$F(t) = 1 + 2(F*G)(t).$$

Applying the Laplace transform to the above equation, we get

$$\overline{F}(s) = \frac{1}{s} + L[(F*G)](s) = \frac{1}{s} + 2\overline{F}(s)\overline{G}(s).$$

Therefore

$$\overline{F}(s) = \frac{1}{s} + 2\frac{\overline{F}(s)}{s+2} \Rightarrow \overline{F}(s) - 2\frac{\overline{F}(s)}{s+2} = \frac{1}{s}.$$

Therefore

$$\frac{s\overline{F}(s)}{s+2} = \frac{1}{s} \Rightarrow \overline{F}(s) = \frac{s+2}{s^2} = \frac{1}{s} + \frac{2}{s^2}.$$

Applying inverse Laplace transform, we get $F(t) = 1 + 2t$. □

Example 3.5.9. Solve $F(t) = 1 + 2\int_0^t F(t-u)\cos u\,du$.

OR

Solve $F(t) = 1 + 2\int_0^t F(t)\cos(t-u)\,du$.

Solution. Let $G(t) = \cos t$. Then $\overline{G}(s) = \dfrac{s}{s^2+1}$. Then the given equation may be written as

$$F(t) = 1 + 2(F * G)(t).$$

Applying Laplace transform on the above equation, we get

$$\begin{aligned}
& & \overline{F}(s) = 1 + 2\overline{F}(s)\overline{G}(s) &= \frac{1}{s} + \frac{2\overline{F}(s)s}{s^2+1} \\
\Rightarrow & & \overline{F}(s) - 2\frac{\overline{F}(s)s}{s^2+1} &= \frac{1}{s} \\
\Rightarrow & & \overline{F}(s)\frac{(s^2-2s+1)}{s^2+1} &= \frac{1}{s} \\
\Rightarrow & & \overline{F}(s) &= \frac{1}{s}\frac{s^2+1}{(s-1)^2} = \frac{1}{s}\frac{(s^2-1)+2}{(s-1)^2} \\
\Rightarrow & & \overline{F}(s) &= \frac{1}{s}\left[\frac{s+1}{s-1} + \frac{2}{(s-1)^2}\right] \\
\Rightarrow & & \overline{F}(s) &= \frac{1}{s}\left[\frac{(s-1)+2}{s-1} + \frac{2}{(s-1)^2}\right] \\
\Rightarrow & & \overline{F}(s) &= \frac{1}{s}\left[1 + \frac{2}{s-1} + \frac{2}{(s-1)^2}\right] \\
\Rightarrow & & \overline{F}(s) &= \frac{1}{s} + \frac{2}{s(s-1)} + \frac{2}{s(s-1)^2}
\end{aligned}$$

Applying inverse Laplace transform, we get

$$\begin{aligned}
F(t) &= 1 + 2\int_0^t e^u\,du + 2\int_0^t ue^u\,du \\
&= 1 + 2[e^u]_0^t + 2[ue^u - e^u]_0^t \\
&= 1 + 2e^t - 2 + 2te^t - 2e^t + 2 \\
&= 1 + 2te^t.
\end{aligned}$$

□

Example 3.5.10. Solve $F(t) = 1 + 2\int_0^t F(t-u)\sin u\,du$.

Solution. Let $G(t) = \sin t$. Then $\overline{G}(s) = \dfrac{1}{s^2+1}$. Then the given equation may be written as

$$F(t) = 1 + 2(F * G)(t).$$

Applying Laplace transform on the above equation, we get

$$\begin{aligned}
& & \overline{F}(s) = &\ \frac{1}{s} + 2\overline{F}(s)\overline{G}(s) = \frac{1}{s} + \frac{2\overline{F}(s)}{s^2+1} \\
\Rightarrow & & \overline{F}(s) - \frac{2\overline{F}(s)}{s^2+1} = &\ \frac{1}{s} \\
\Rightarrow & & \overline{F}(s)\frac{s^2-1}{s^2+1} = &\ \frac{1}{s}
\end{aligned}$$

$$\Rightarrow \qquad \overline{F}(s) = \frac{s^2+1}{s(s^2-1)} = \frac{1}{s}\left[\frac{s^2}{s^2-1} + \frac{1}{s^2-1}\right]$$

$$\Rightarrow \qquad \overline{F}(s) = \frac{s}{s^2-1} + \frac{1}{s(s^2-1)}.$$

Now, applying inverse Laplace transform we get

$$\begin{aligned} F(t) &= \cosh t + \int_0^t \sinh u \, du \\ &= \cosh t + [\cosh u]_0^t \\ &= \cosh t + \cosh t - 1 = 2\cosh t - 1. \end{aligned}$$

□

3.5.3 Applications to Partial Differential Equations

Definition 3.5.11. Let $u(x,t)$ be a function of two variables. The Laplace transform $L[u]$ of u in the variable t is defined as

$$L[u](s) = \int_0^\infty u(x,t)e^{-st}\,dt.$$

The above is defined for all s for which the above integral exists.

Lemma 3.5.12. *Let $u(x,t)$ be a function of two variables, and let $U(x,s)$ be the Laplace transform of $u(x,t)$ in the variable t.*

1. *Suppose that $\frac{\partial u^i}{\partial t^i}(x,t)$ is bounded at infinity for $0 \le i \le n-1$ and $\frac{\partial u^i}{\partial t^i}(x,t)$ are continuous at $(x,0)$ for all x and for all $0 \le i \le n-1$. Then $L\left[\frac{\partial^n u}{\partial t^n}(x,t)\right] = s^n U(x,s) - s^{n-1}u(x,0) - s^{n-2}u_t(x,0) - \ldots - \frac{\partial^{n-1}u}{\partial t^{n-1}}(x,0)$.*
2. *$L\left[\frac{\partial^n u}{\partial x^n}(x,t)\right](s) = \frac{\partial^n}{\partial x^n}U(x,s)$.*
3. *$L\left[\frac{\partial^2 u}{\partial x \partial t}(x,t)\right](s) = s\frac{\partial}{\partial x}U(x,s) - \frac{d}{dx}u(x,0)$.*

Proof. 1. We shall prove it by using Principle of Mathematical Induction. Let $n = 1$. Then

$$\begin{aligned} L\left[\frac{\partial u}{\partial t}(x,t)\right](s) &= \int_0^\infty \frac{\partial u}{\partial t}(x,t)e^{-st}dt \\ &= \left[e^{-st}u(x,t) + s\int u(x,t)e^{-st}dt\right]_0^\infty \\ &= sU(x,s) - u(x,0). \end{aligned}$$

Assume that the above is true for $n-1$. Now

$$L\left[\frac{\partial^n u}{\partial t^n}(x,t)\right](s) = \int_0^\infty \frac{\partial^n u}{\partial t^n}(x,t)e^{-st}dt$$

$$
\begin{aligned}
&= \left[e^{-st}\frac{\partial^{n-1}u}{\partial t^{n-1}}(x,t)+s\int \frac{\partial^{n-1}u}{\partial t^{n-1}}(x,t)e^{-st}dt\right]\\
&= -\frac{\partial^{n-1}u}{\partial t^{n-1}}(x,0)+s\int_0^\infty \frac{\partial^{n-1}u}{\partial t^{n-1}}(x,t)e^{-st}dt\\
&= -\frac{\partial^{n-1}u}{\partial t^{n-1}}(x,0)+s[s^{n-1}U(x,s)-s^{n-2}u(x,t)-\cdots-\frac{\partial^{n-2}u}{\partial t^{n-2}}(x,0)]\\
&= s^nU(x,s)-s^{n-1}u(x,0)-s^{n-2}u_t(x,0)-\ldots-\frac{\partial^{n-1}u}{\partial t^{n-1}}(x,0).
\end{aligned}
$$

Hence the proof.

2. We see that

$$L\left[\frac{\partial^n u}{\partial x^n}(x,t)\right](s)=\int_0^\infty \frac{\partial^n u}{\partial x^n}(x,t)e^{-st}dt=\frac{\partial^n}{\partial x^n}\left[\int_0^\infty u(x,t)e^{-st}dt\right]=\frac{\partial^n}{\partial x^n}U(x,s).$$

3.

$$
\begin{aligned}
L\left[\frac{\partial^2 u}{\partial x\partial t}(x,t)\right](s) &= \int_0^\infty \frac{\partial^2 u}{\partial x\partial t}(x,t)e^{-st}dt=\frac{\partial}{\partial x}\left[\int_0^\infty \frac{\partial u}{\partial t}(x,t)e^{-st}dt\right]\\
&= \frac{\partial}{\partial x}[sU(x,s)-u(x,0)]=s\frac{\partial}{\partial x}U(x,s)-\frac{d}{dx}u(x,0).
\end{aligned}
$$

□

Example 3.5.13. Solve $u_t=u_{xx}$, $0<x<1$, $t>0$ subject to the conditions $u(0,t)=1=u(1,t)$ for all $t>0$ and $u(x,0)=1+\sin\pi x$, $0<x<1$.

Solution. Let $U(x,s)$ be the Laplace transform of $u(x,t)$ in the variable t. Applying Laplace transform to $u_t=u_{xx}$, we have

$$sU(x,s)-u(x,0)=\frac{\partial^2}{\partial x^2}u(x,s)$$

i.e.,

$$sU(x,s)-(1+\sin\pi x)=\frac{\partial^2}{\partial x^2}U(x,s).$$

Fixing s, the above equation reduces to $\frac{d^2}{dx^2}U-sU=-(1+\sin\pi x)$. Solving the above differential equation, we have

$$\text{C.F. }=c_1(s)e^{\sqrt{s}x}+c_2e^{-\sqrt{s}x}.$$

$$
\begin{aligned}
\text{P.I. } &= \frac{1}{D^2-s}(-1-\sin\pi x)\\
&= -\frac{1}{D^2-s}1-\frac{1}{D^2-s}\sin\pi x\\
&= \frac{1}{s}\frac{1}{1-\frac{D^2}{s}}1-\frac{D^2-s}{\sin\pi x}\\
&= \frac{1}{s}-\frac{1}{-\pi^2-s}\sin\pi x \qquad \left(\because \frac{1}{f(D^2)}\sin ax=\frac{1}{f(-a^2)}\sin ax\right).
\end{aligned}
$$

lightatinfinite@gmail.com
jay_mehta@spuvvn.edu

Therefore

$$U(x,s) = \text{ C.F. } + \text{ P.I. } = c_1(s)e^{\sqrt{s}x} + c_2(s)e^{-\sqrt{s}x} + \frac{1}{s} + \frac{\sin \pi x}{\pi^2 + s},$$

where $c_1(s)$ and $c_2(s)$ are arbitrary constants. Since s is arbitrary, the above solution holds for all x and for all s. Since $u(0,t) = 1$, applying Laplace transform yields $U(0,s) = \frac{1}{s}$. Putting $U(0,s) = \frac{1}{s}$ in above equation, we get $c_1(s) = -c_2(s)$. Therefore,

$$U(x,s) = c_1(s)(e^{\sqrt{s}x} - e^{-\sqrt{s}x}) + \frac{1}{s} + \frac{\sin \pi x}{\pi^2 + s}.$$

Now, $u(1,t) = 1 \Rightarrow U(1,s) = \frac{1}{s}$. Then $c_1(s)(e^{\sqrt{s}} - e^{-\sqrt{s}}) = 0$. Since $s > 0$, we have $c_1(s) = 0$. Hence

$$U(x,s) = \frac{1}{s} + \frac{\sin \pi x}{\pi^2 + s}.$$

Applying inverse Laplace transform, one obtains

$$u(x,t) = 1 + e^{-\pi^2 t} \sin \pi x.$$

□

Example 3.5.14. Solve $u_{xx} = \frac{1}{c^2}u_{tt} - \cos \omega t$, $0 < x < \infty$, $0 < t < \infty$ subject to $u(0,t) = 0$ for all t, u is bounded in the variable x, and $u_t(x,0) = u(x,0) = 0$ for all x.

Solution. Let U be the Laplace transform of u in the second variable t. Applying Laplace transform on $u_{xx} = \frac{1}{c^2}u_{tt} - \cos \omega t$, we get

$$\frac{\partial^2}{\partial x^2}U(x,s) = \frac{1}{c^2}\left[s^2 U(x,s) - su(x,0) - u_t(x,0)\right] - \frac{s}{s^2 + \omega^2}.$$

Since $u(x,0) = u_t(x,0) = 0$, we have

$$\frac{\partial^2}{\partial x^2}U(x,s) = \frac{s^2}{c^2}U(x,s) - \frac{s}{s^2 + \omega^2}.$$

Fix s. Then the above equation will become

$$\frac{d^2U}{dx^2} - \frac{s^2}{c^2}U = -\frac{s}{s^2 + \omega^2}.$$

Therefore the solution is

$$\text{C.F. } = c_1(s)e^{\frac{sx}{c}} + c_2(s)e^{-\frac{sx}{x}}.$$

$$\begin{aligned}\text{P.I. } &= \frac{1}{D^2 - \frac{s^2}{c^2}}\left(-\frac{s}{s^2 + \omega^2}\right) \\ &= -\frac{s}{s^2 + \omega^2}\frac{1}{\frac{s^2}{c^2}}\left(1 - \frac{c^2 D^2}{s^2}\right)1 = \frac{c^2}{s(s^2 + \omega^2)}.\end{aligned}$$

Thus,

$$U(x,s) = c_1(s)e^{\frac{sx}{c}} + c_2(s)e^{-\frac{sx}{x}} + \frac{c^2}{s(s^2 + \omega^2)}.$$

Since u is bounded in the variable x, its Laplace transform U is bounded in the variable s. Since $s > 0$, it follows that $c_1(s) = 0$. Therefore,

$$U(x,s) = c_2(s)e^{-\frac{sx}{x}} + \frac{c^2}{s(s^2+\omega^2)}.$$

Since $u(0,t) = 0$, $U(0,s) = 0$. Therefore

$$0 = U(0,s) = c_2(s)e^{-\frac{sx}{x}} + \frac{c^2}{s(s^2+\omega^2)}.$$

Therefore $c_2(s) = \frac{-c^2}{s(s^2+\omega^2)}$. Then

$$U(x,s) = -\frac{c^2}{s(s^2+\omega^2)}e^{-\frac{sx}{x}} + \frac{c^2}{s(s^2+\omega^2)}. \tag{3.1}$$

Now, we have

$$\frac{1}{\omega}L^{-1}\left[\frac{\omega}{s(s^2+\omega^2)}\right](t) = \frac{1}{\omega}\int_0^t \sin\omega u\,du = -\frac{1}{\omega^2}\left[\cos\omega u\right]_0^t = -\frac{\cos\omega t}{\omega^2} + \frac{1}{\omega^2}$$

and

$$L^{-1}[e^{-as}\overline{f}(s)](t) = f(t-a)H(t-a).$$

Therefore applying inverse Laplace transform to equation (3.1), we get

$$u(x,t) = -\frac{c^2}{\omega^2}\left(1-\cos\omega\left(t-\frac{x}{x}\right)\right)H\left(t-\frac{x}{c}\right) + \frac{c^2}{\omega^2}(1-\cos\omega t).$$

□

Example 3.5.15. Solve $u_{tt} = u_{xx}$, $0 < x < 1$, $t > 0$ subject to $u(0,t) = 0 = u(1,t)$ for all t, $u(x,0) = \sin\pi x$ and $u_t(x,0) = -\sin\pi x$, for all x.

Solution. Let U be the Laplace transform of u in the second variable t. Applying Laplace transform on $u_{tt} = u_{xx}$, we get

$$s^2U(x,s) - su(x,0) - u_t(x,0) = \frac{\partial^2}{\partial x^2}U(x,s),$$

$$\text{i.e. } s^2U(x,s) - s\sin\pi x + \sin\pi x = \frac{\partial^2}{\partial x^2}U(x,s).$$

Fix s. Then the above equation becomes

$$\frac{d^2U}{dx^2} - s^2U = \sin\pi x - s\sin\pi x.$$

Then

$$\text{C.F. } = c_1(s)e^{sx} + c_2(s)e^{-sx}.$$

$$\text{P.I. } = \frac{1}{D^2-s^2}\left[\sin\pi x - s\sin\pi x\right]$$

$$= \frac{1}{-\pi^2 - s^2}\sin\pi x - \frac{s}{-\pi^2 - s^2}\sin\pi x.$$

Therefore

$$U(x,s) = c_1(s)e^{sx} + c_2(s)e^{-sx} + \frac{\sin\pi x}{\pi^2 + s^2}(s-1).$$

Since $u(0,t) = 0 = u(1,t)$, we get $U(0,s) = 0 = U(1,s)$. Then

$$0 = U(0,s) = c_1(s) + c_2(s) \Rightarrow c_1(s) = -c_2(s).$$

Therefore $U(x,s) = c_1(s)(e^{sx} - e^{-sx}) + \frac{\sin\pi x}{\pi^2 + s^2}(s-1)$. Also $U(1,s) = 0$ implies

$$0 = U(1,s) = c_1(s)(e^{s} - e^{-s}) + \frac{\sin\pi}{\pi^2 + s^2}(s-1),$$

i.e. $c_1(s) = 0$ (since $s > 0$). Therefore

$$U(x,s) = \frac{\sin\pi x}{\pi^2 + s^2}(s-1).$$

Applying inverse Laplace transform, we get

$$u(x,t) = \sin\pi x\left[\cos\pi t - \frac{1}{\pi}\sin\pi t\right].$$

□

Example 3.5.16. Solve $u_{xx} = \frac{1}{c^2}u_{tt} + k$, $0 < x < l$, $t > 0$ subject to $u(0,t) = u_x(0,t) = 0$ for all $t > 0$ and $u(x,0) = u_t(x,0) = 0$ for all x.

Solution. Let U be the Laplace transform of u in the second variable t. Applying Laplace transform on the given equation, we get

$$\frac{\partial^2}{\partial x^2}U(x,s) = \frac{1}{c^2}\left[s^2U(x,s) - su(x,0) - u_t(x,0)\right] + k\frac{1}{s}.$$

Fix s. Then the above becomes

$$\frac{d^2U}{dx^2} - \frac{s^2}{c^2}U = \frac{k}{s}.$$

Therefore C.F. $= c_1(s)e^{\frac{sx}{c}} + c_2(s)e^{-\frac{sx}{c}}$.

$$\begin{aligned} \text{P.I. } &= \frac{1}{D^2 - \frac{s^2}{c^2}}\frac{k}{s} \\ &= -\frac{k}{s}\frac{1}{\frac{s^2}{c^2}\left(1 - \frac{c^2}{s^2}D^2\right)}1 = -\frac{kc^2}{s^3}. \end{aligned}$$

Therefore

$$U(x,s) = c_1(s)e^{\frac{sx}{c}} + c_2(s)e^{-\frac{sx}{c}} - \frac{kc^2}{s^3}.$$

Then

$$\frac{\partial}{\partial x}U(x,s) = \frac{s}{c}\left(c_1(s) - c_2(s)\right).$$

Since $u_x(0,t) = 0$, $\frac{\partial}{\partial x}U(0,s) = 0$. Therefore

$$0 = \frac{\partial U}{\partial x} = \frac{s}{c}(c_1(s) - c_2(s)) \Rightarrow c_1(s) = c_2(s).$$

Therefore,

$$U(x,s) = c_1(s)(e^{\frac{sx}{c}} + e^{-\frac{sx}{c}}) - \frac{kc^2}{s^3}.$$

Since $u(0,t) = 0$, $U(0,s) = 0$. Then $0 = U(0,s) = 2c_1(s) - \frac{kc^2}{s^3} \Rightarrow c_1(s) = \frac{kc^2}{2s^3}$. Therefore

$$U(x,s) = \frac{kc^2}{2s^3}\left(e^{\frac{sx}{c}} + e^{-\frac{sx}{c}}\right) - \frac{kc^2}{s^3} = \frac{kc^2}{2}\left[\frac{e^{\frac{sx}{c}}}{s^3} + \frac{e^{-\frac{sx}{c}}}{s^3}\right] - \frac{kc^2}{s^3}.$$

Now, applying the inverse Laplace transform, we get

$$u(x,t) = \frac{kc^2}{2}\frac{\left(t+\frac{x}{c}\right)^2}{2}H\left(t+\frac{x}{c}\right) + \frac{kc^2}{2}\frac{\left(t-\frac{x}{c}\right)^2}{2}H\left(t-\frac{x}{c}\right) - \frac{kc^2t^2}{2}.$$

Therefore

$$u(x,t) = \frac{kc^2}{4}\left(t+\frac{x}{c}\right)^2 H\left(t+\frac{x}{c}\right) + \frac{kc^2}{4}\left(t-\frac{x}{c}\right)^2 H\left(t-\frac{x}{c}\right) - \frac{kc^2t^2}{2}.$$

□

Example 3.5.17. Solve $u_{tt} = c^2 u_{xx}$, $x > 0$, $t > 0$ subject to $u(0,t) = A\sin\omega t$, $u_x(0,t) = 0$ for all t, $u(x,0) = u_t(x,0) = 0$ for all x.

Solution. Let U be the Laplace transform of u in the second variable t. Applying Laplace transform to $u_{tt} = c^2 u_{xx}$, we get

$$s^2 U(x,s) - su(x,0) - u_t(x,0) = c^2 \frac{\partial^2}{\partial x^2}U(x,s).$$

Fix s. Then the above equation becomes

$$s^2 U = c^2 \frac{d^2U}{dx^2} \Rightarrow \frac{d^2U}{dx^2} - \frac{s^2}{c^2}U = 0.$$

C.F. $= c_1(s)e^{\frac{sx}{c}} + c_2(s)e^{-\frac{sx}{c}}$. Therefore

$$U(x,s) = c_1(s)e^{\frac{sx}{c}} + c_2(s)e^{-\frac{sx}{c}}.$$

Since $u(0,t) = A\sin\omega t$, $U(0,s) = \frac{A\omega}{s^2+\omega^2}$. Also,

$$\frac{\partial}{\partial x}U(0,s) = c_1(s) - c_2(s) = 0 \Rightarrow c_1(s) = c_2(s).$$

Therefore $U(x,s) = c_1(s)\left(e^{\frac{sx}{c}} + e^{-\frac{sx}{c}}\right)$. Thus, $U(0,s) = \frac{A\omega}{s^2+\omega^2} = 2c_1(s)$. Therefore

$$U(x,s) = \frac{A}{2}\left[\frac{\omega e^{\frac{sx}{c}}}{s^2+\omega^2} + \frac{\omega e^{-\frac{sx}{c}}}{s^2+\omega^2}\right].$$

Applying inverse Laplace transform, we get

$$u(x,t) = \frac{A}{2}\left[\sin\left(\omega\left(t+\frac{x}{c}\right)\right)H\left(t+\frac{x}{c}\right) + \sin\left(\omega\left(t-\frac{x}{c}\right)\right)H\left(t-\frac{x}{c}\right)\right].$$

□

Example 3.5.18. Solve $u_{tt} = c^2 u_{xx}$, $0 < x < l$, $t > 0$ subject to $u(0,t) = u(l,t) = 0$, $t > 0$ and $u(x,0) = \lambda \sin\left(\frac{\pi x}{l}\right)$, $u_t(x,0) = 0$.

Solution. Let U be the Laplace transform of u in the second variable t. Applying Laplace transform to $u_{tt} = c^2 u_{xx}$, we get

$$s^2 U(x,s) - su(x,0) - u_t(x,0) = c^2 \frac{\partial^2}{\partial x^2} U(x,s).$$

Fix s, then the above equation becomes

$$\frac{d^2U}{dx^2} - \frac{s^2}{c^2} U = -s\lambda \sin\left(\frac{\pi x}{l}\right).$$

Therefore C.F. $= c_1(s) e^{\frac{sx}{c}} + c_2(s) e^{-\frac{sx}{c}}$. Also,

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - \frac{s^2}{c^2}} \left(-s\lambda \sin\left(\frac{\pi x}{l}\right)\right) \\ &= -s\lambda \frac{1}{\left(-\frac{\pi^2}{l^2} - \frac{s^2}{c^2}\right)} \sin\left(\frac{\pi x}{l}\right). \end{aligned}$$

Therefore

$$U(x,s) = c_1(s) e^{\frac{sx}{c}} + c_2(s) e^{-\frac{sx}{c}} + s\lambda \frac{1}{\left(\frac{\pi^2}{l^2} + \frac{s^2}{c^2}\right)} \sin\left(\frac{\pi x}{l}\right).$$

Since s was arbitrary, the above solution holds for all s and all x. Since $u(0,t) = 0$, $U(0,s) = 0$. Therefore $c_1(s) = -c_2(s)$. Then

$$U(x,s) = c_1(s)\left(e^{\frac{sx}{c}} - e^{-\frac{sx}{c}}\right) + s\lambda \frac{1}{\left(\frac{\pi^2}{l^2} + \frac{s^2}{c^2}\right)} \sin\left(\frac{\pi x}{l}\right).$$

Since $u(l,t) = 0$, $U(l,s) = 0$. Therefore

$$0 = U(l,s) = c_1(s)\left[e^{\frac{s}{c}l} - e^{-\frac{s}{c}l}\right], \text{ i.e. } c_1(s) \sinh\left(\frac{sl}{c}\right) = 0.$$

Since $s > 0$ and $c > 0$, $c_1(s) = 0$. Hence

$$U(x,s) = -\frac{\lambda s c^2}{s^2 - \frac{c^2 \pi^2}{l^2}} \sin\left(\frac{\pi x}{l}\right).$$

Applying inverse Laplace transform, we get

$$u(x,t) = -\sin\left(\frac{\pi x}{l}\right) \lambda c^2 \cosh\left(\frac{c\pi t}{l}\right).$$

□

3.5.4 Applications to Simultaneous Differential Equations

We consider some examples on solving simultaneous differential equations by an application of Laplace transform. The final answers are not given and the students are encourage to do the computations.

Example 3.5.19. Solve $(D^2-2)x-3y=e^{2t}$, $(D^2+2)y+x=0$ subject to $x(0)=y(0)=1$ and $x'(0)=y'(0)=0$.

Solution. Here $x''-2x-3y=e^{2t}$ and $y''+2y+x=0$. Let X and Y be the Laplace transform of x and y respectively. Applying Laplace transform to both the above equations, we get

$$\begin{aligned} s^2X-sx(0)-x'(0)-2X-3Y &= \frac{1}{s-2} \\ s^2Y-sy(0)-y'(0)+2Y+X &= 0. \end{aligned}$$

Therefore

$$\begin{aligned} s^2X-s-2X-3Y &= \frac{1}{s-2} \\ s^2Y-s+2Y+X &= 0. \end{aligned}$$

Then

$$(s^2-2)X-3Y = \frac{1}{s-2}+s \tag{3.2}$$

$$X+(s^2+2)Y = s. \tag{3.3}$$

Multiplying equation (3.3) by (s^2-2) and subtracting it from equation (3.2), we get

$$\begin{aligned} -3Y-(s^2-2)(s^2+2)Y &= \frac{1}{s-2}+s-s(s^2-2), \\ \Rightarrow (-s^4+1)Y &= \frac{1}{s-2}+2-s(s^2-2) \\ \Rightarrow Y &= -\frac{1}{(s^4-1)(s-2)}-\frac{s}{s^4-1}+\frac{s(s^2-2)}{s^4-1}. \end{aligned}$$

Now, $L^{-1}\left[\frac{1}{(s^2-1)(s^2+1)}\right](t)=\frac{1}{2}L^{-1}\left[\frac{-1}{s^2+1}+\frac{1}{s^2-1}\right](t)=-\frac{1}{2}\sin t+\frac{1}{2}\sinh t.$

Also, $L^{-1}\left[\frac{1}{s-2}\right](t)=e^{2t}$. Therefore

$$L^{-1}\left[\frac{1}{(s^4-1)(s-2)}\right](t)=\int_0^t e^{2(t-u)}\left(\frac{1}{2}\sin u+\frac{1}{2}\sinh u\right)\,du.$$

Also, $L^{-1}\left[\frac{s}{s^4-1}\right]=\frac{1}{2}\left[\frac{s}{s^2-1}-\frac{s}{s^2+1}\right]=\frac{1}{2}(\cosh t-\cos t)$.

Now, $\frac{s(s^2-1-1)}{(s^2-1)(s^2+1)}=\frac{s}{(s^2+1)}-\frac{s}{(s^2-1)(s^2+1)}$. Then

$$L^{-1}\left[\frac{s(s^2-2)}{s^4-1}\right]=L^{-1}\left[\frac{s}{s^2+1}\right](t)-L^{-1}\left[\frac{s}{(s^2-1)(s^2+1)}\right](t)$$

$$= \cos t - \frac{1}{2}(\cosh t - \cos t) = \frac{1}{2}(\cos t - \cosh t).$$

Therefore,

Now multiplying equation (3.2) by s^2+2 and equation (3.3) by -3, we get

$$(s^2-1)(s^2+2)X - 3(s^2+2)Y = (s^2+2)\frac{1}{s-2} + s(s^2+2)$$
$$\text{and } 3X - 3(s^2+2)Y = 3s.$$

Therefore

$$((s^4-4)+3)X = \frac{s^2+2}{s-2} + 3s + s(s^2+2)$$
$$\Rightarrow X = \frac{s^2+2}{(s-2)(s^4-1)} + \frac{s(s^2+2)}{s^4-1} + \frac{3s}{s^4-1}.$$

Applying inverse Laplace transform, we get

$$\frac{s(s^2+2)}{s^4-1} = \frac{s(s^2+1+1)}{(s^2-1)(s^2+1)} = \frac{s}{s^2-1} + \frac{s}{(s^2-1)(s^2+1)}$$
$$L^{-1}\left[\frac{s(s^2+2)}{s^4-1}\right](t) = \cosh t + \frac{1}{2}(\cosh t - \cos t) = \frac{3}{2}\cosh t - \frac{1}{2}\cos t.$$
$$\frac{3s}{s^4-1} = \frac{3}{2}\left[\frac{s}{s^2-1} - \frac{s}{s^2+1}\right].$$

Then $L^{-1}\left[\frac{3s}{s^4-1}\right] = \frac{3}{2}(\cosh t - \cos t)$.

Also for $\frac{s^2+2}{(s-2)(s^4-1)} = \frac{s^2-1+3}{(s-2)(s^4-1)}$.

`Complete it.` □

Example 3.5.20. Solve $(D-2)x-(D+1)y = 6e^{3t}$, $(2D-3)x+(D-3)y = 6e^{3t}$ subject to $x(0) = y(0) = 3$.

Solution. Here $x'-2x-y'-y = 6e^{3t}$ and $2x'-3x+y'-3y = 6e^{3t}$. Let X and Y be the Laplace transform of x and y respectively. Applying Laplace transform to both the above equations, we get

$$sX - x(0) - 2X - sY + y(0) - Y = L\left[6e^{3t}\right]$$
$$2sX - 2x(0) - 3x + sY - y(0) - 3Y = L\left[6e^{3t}\right].$$

Therefore

$$sX - 3 - 2x - sY + 3 - Y = \frac{6}{s-3}$$
$$2sX - 6 - 3X + sY - 3 - 3Y = \frac{6}{s-3}.$$

Then

$$(s-2)X - (s+1)Y = \frac{6}{s-3} \tag{3.4}$$

$$(2s-3)X+(s-3)Y=\frac{6}{s-3}+9. \tag{3.5}$$

From equations (3.4) and (3.5), we have

$$\begin{aligned}
[-(2s-3)(s+1)-(s-2)(s-3)]Y &= \frac{6(2s-3)}{s-3}-\frac{6(s-2)}{s-3}-9(s-2)\\
\Rightarrow [-2s^2-2s+3s+3-s^2+5s-6]Y &= \frac{6(2s-3)}{s-3}-\frac{6(s-2)}{s-3}-9(s-2)\\
\Rightarrow [-3s^2+6s-3]Y &= \frac{6(2s-3)}{s-3}-\frac{6(s-2)}{s-3}-9(s-2).
\end{aligned}$$

Therefore

$$\begin{aligned}
Y &= -\frac{6(2s-3)}{3(s-1)^2(s-3)}+\frac{6(s-2)}{3(s-3)(s-1)^2}+\frac{9(s-2)}{3(s-1)^2}\\
&= \frac{-4s+6+2s-4}{(s-3)(s-1)^2}+\frac{9(s-1-1)}{3(s-1)^2}\\
&= \frac{-2(s-1)}{(s-3)(s-1)^2}+\frac{3}{s-1}-\frac{3}{(s-1)^2}\\
&= \frac{1}{s-1}-\frac{1}{s-3}+\frac{3}{s-1}-\frac{3}{(s-1)^2}.
\end{aligned}$$

Applying inverse Laplace transform, we get

$$\begin{aligned}
y(t) &= L^{-1}\left[\frac{1}{s-1}-\frac{1}{s-3}+\frac{3}{s-1}-\frac{3}{(s-1)^2}\right](t)\\
&= e^t-e^{3t}+3e^t-3te^t\\
&= 4e^t-3e^{3t}-3te^t.
\end{aligned}$$

Now, multiplying equation (3.4) by -2 and adding it to equation (3.5) gives

$$x=y-3y'-6e^{3t}.$$

Using the solution $y(t)$ obtained above, we get

$$x(t)=e^t+6te^t+2e^{3t}.$$

□

Example 3.5.21. Solve $D^2x+Dy+3x=15e^{-t}$ and $D^2y-4Dx+3y=15\sin 2t$ subject to $x(0)=35$, $x'(0)=-48$, $y(0)=27$, $y'(0)=-55$.

Solution. Here $x''+y'+3x=15e^{-t}$ and $y''-4x'+3y=15\sin 2t$. Let X and Y be the Laplace transform of x and y respectively. Applying Laplace transform to both the above equations, we get

$$\begin{aligned}
s^2X-sx(0)-x'(0)+sY-y(0)+3X &= \frac{15}{s+1}\\
s^2Y-sy(0)-y'(0)+4sX-4x(0)+3Y &= \frac{30}{s^2+4}.
\end{aligned}$$

Then

$$\begin{aligned}(s^2+3)X-35s+48+sY-27 &= \frac{15}{s+1}\\ (s^2+3)Y-27s+55+4sX-140 &= \frac{30}{s^2+4}.\end{aligned}$$

`Complete it as an exercise.` □

Examples 3.5.22. 1. Solve $\frac{dx}{dt}-2x-\frac{dy}{dt}-y=6e^{3t}$, $2\frac{dx}{dt}-3x+\frac{dy}{dt}-3y=6e^{3t}$ subject to $x(0)=3$, $y(0)=0$ using Laplace transform methods.
Let X and Y be the Laplace transforms of x and y respectively. Applying Laplace transform on the equations $\frac{dx}{dt}-2x-\frac{dy}{dt}-y=6e^{3t}$ and $2\frac{dx}{dt}-3x+\frac{dy}{dt}-3y=6e^{3t}$, we have $sX-x(0)-2X-sY+y(0)-Y=\frac{6}{s-3}$ and $2sX-2x(0)-3X+sY-y(0)-3Y=\frac{6}{s-3}$. Substituting $x(0)=3$, $y(0)=0$ and simplifying we obtain $(s-2)X-(s+1)Y=\frac{6}{s-3}+3$ and $(2s-3)X+(s-3)Y=\frac{6}{s-3}+6$. Solving the last two equation for X and Y, we get $X=\frac{3s-1}{(s-1)^2}+\frac{4}{(s-1)(s-3)}$ and $Y=-\frac{1}{(s-1)^2}+\frac{1}{s-1}-\frac{1}{s-3}$. Applying inverse Laplace transform gives $x(t)=(1+2t)e^{2t}+2e^{3t}$ and $y(t)=(1-t)e^{t}-e^{3t}$.

2. Solve $\frac{dx}{dt}+\frac{dy}{dt}=t$, $\frac{d^2x}{dt^2}-y=e^{-t}$ subject to $x(0)=3$, $x'(0)=-2$, $y(0)=0$ using Laplace transform methods.
Let X and Y be the Laplace transforms of x and y respectively. Applying Laplace transform to both the differential equations and using the conditions given, we obtain $X+Y=\frac{3}{s}+\frac{1}{s^3}$ and $s^2X-Y=3s-2+\frac{1}{s+1}$. Using the last two equations, we have $(1+s^2)Y=\frac{1}{s}+2-\frac{1}{s+1}$, i.e., $Y=\frac{1}{s(1+s^2)}+\frac{2}{1+s^2}-\frac{1}{(1+s)(1+s^2)}$. Applying inverse Laplace transform, we get

$$\begin{aligned} y(t) &= \int_0^t \sin u\,du+2\sin t-\int_0^t e^{-(t-u)}\sin u\,du\\ &= 1-\cos t+2\sin t-e^{-t}\left[\frac{e^{-u}}{2}(\sin u-\cos u)\right]_0^t\\ &= 1-\frac{1}{2}\cos t+\frac{3}{2}\sin t-\frac{1}{2}e^{-t}.\end{aligned}$$

Since $X+Y=\frac{3}{s}+\frac{1}{s^3}$, $x(t)+y(t)=3+\frac{t^2}{2}$. Using $y(t)$ obtained above, $x(t)=2+\frac{1}{2}[t^2+\cos t-3\sin t+e^{t}]$.

3. Solve the simultaneous differential equations $(D-2)x+3y=0$ and $2x+(D-1)y=0$ subject to $x(0)=8$ and $y(0)=3$.
Applying Laplace transform to above differential equations and using the conditions gives, $(s-2)X+3Y=8$ and $2X+(s-1)Y=3$. Solving above equations for Y, we get $Y=\frac{3(s-2)}{(s-4)(s+1)}-\frac{16}{(s-4)(s+1)}$. Applying inverse Laplace transform gives $y(t)=-\frac{1}{5}e^{4t}+\frac{77}{10}e^{-t}$.

CHAPTER 4

Z- TRANSFORM

4.1 Green's Function and its Application

In what follows, we describe the method to solve the second order differential equation using Green's function.

Consider the differential equation

$$y'' + P(x)y' + Q(x)y = f(x)$$

on $[a,b]$ subject to conditions $B_1(y) = 0$ and $B_2(y) = 0$.

Steps:

1. Find the fundamental solutions u_1 and u_2 of the equation $y'' + P(x)y' + Q(x)y = f(x)$, i.e., u_1 and u_2 are solutions of given equations and they are linearly independent.
2. By taking appropriate linear combinations of u_1 and u_2, find y_1 and y_2 so that $B_1(y_1) = 0$ and $B_2(y_2) = 0$.
3. Define *Green's function* $G(x,s)$ as follows:

$$G(x,s) = \begin{cases} \frac{y_1(s)y_2(x)}{W(y_1,y_2)(s)}, & s \leq x; \\ \frac{y_1(x)y_2(s)}{W(y_1,y_2)(s)}, & s \geq x, \end{cases}$$

where $W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is the Wronskian of y_1 and y_2.

4. Evaluate $y(s) = \int_a^b G(x,s)f(s)\,ds$.

Example 4.1.1. Find the Green's function for $y''(x) = f(x)$, $y(0) = 0 = y(1)$ and hence find the solution of the above equation when $f(x) = x^2$.

Solution. The fundamental solution of $y''=0$ are $u_1(x)=1$ and $u_2(x)=x$ $(\because y(x)=c_1+c_2x)$.

Consider $y_1(x)=x$ and $y_2(x)=x-1$. Then $y_1(0)=0$ and $y_2(1)=0$.

Now the Wronskian of y_1 and y_2 is

$$W(y_1,y_2)=\begin{vmatrix} x & x-1 \\ 1 & 1 \end{vmatrix}=x-(x-1)=1.$$

Therefore the Green's function for the above problem is

$$G(x,s)=\begin{cases} \frac{s(x-1)}{1}, & s\le x \\ \frac{x(s-1)}{1}, & s\ge x \end{cases}.$$

When $f(x)=x^2$, the solution of the above differential equation is

$$\begin{aligned} y(x) &= \int_0^1 G(x,s)s^2\,ds \\ &= \int_0^x G(x,s)s^2\,ds+\int_x^1 G(x,s)s^2\,ds \\ &= \int_0^x s(x-1)s^2\,ds+\int_x^1 x(s-1)s^2\,ds \\ &= (x-1)\left[\frac{s^4}{4}\right]_0^x+x\left[\frac{s^4}{4}-\frac{s^3}{3}\right]_x^1 \\ &= \frac{x^5}{4}-\frac{x^4}{4}+\frac{x}{4}-\frac{x}{3}-\frac{x^5}{4}+\frac{x^4}{3} \\ &= \frac{x^4}{12}-\frac{x}{12}=\frac{x^4-x}{12}. \end{aligned}$$

□

Example 4.1.2. Find the Green's function for $y''(x)+y(x)=f(x)$, $y(0)=0$, $y'(1)=0$. Hence find the solution of the above equation when $f(x)=x$.

Solution. The fundamental solution of $y''(x)+y(x)=0$ are

$$u_1(x)=\cos x,\quad u_2(x)=\sin x \qquad (\text{as } y(x)=c_1\cos x+c_2\sin x).$$

Consider $y_1(x)=\sin x$. Then $y_1(0)=0$. For y_2, consider $y_2(x)=c_1\cos x+c_2\sin x$. Then $y_2'(x)=-c_1\sin x+c_2\cos x$. Thus, $y_2'(1)=0$ implies $-c_1\sin 1+c_2\cos 1=0$. Take $c_1=\cos 1$ and $c_2=\sin 1$. Then we have

$$y_2(x)=\cos 1\cos x+\sin 1\sin x=\cos(x-1).$$

Then $y_1(0)=0$ and $y_2'(1)=0$. Now, the Wronskian of y_1 and y_2 is

$$\begin{aligned} W(y_1,y_2) &= \begin{vmatrix} \sin x & \cos(x-1) \\ \cos x & -\sin(x-1) \end{vmatrix} \\ &= -\sin x\sin(x-1)-\cos x\cos(x-1) \\ &= -\cos 1. \end{aligned}$$

Therefore Green's function for the above problem is

$$G(x,s) = \begin{cases} \frac{\sin s \cos(x-1)}{-\cos 1}, & s \le x \\ \frac{\sin x \cos(s-1)}{-\cos 1}, & s \ge x. \end{cases}$$

When $f(x) = x$ the solution of the above differential equation is

$$\begin{aligned}
y(x) &= \int_0^1 G(x,s) s\, ds \\
&= \int_0^x \frac{\sin s \cos(x-1)}{-\cos 1} s\, ds + \int_x^1 \frac{\sin x \cos(s-1)}{-\cos 1} s\, ds \\
&= \frac{\cos(x-1)}{-\cos 1}\left[s[-\cos s] - 1[-\sin s]\right]_0^x + \frac{\sin x}{-\cos 1}\left[s[\sin(s-1)] - [-\cos(s-1)]\right]_x^1 \\
&= \frac{\cos(x-1)}{-\cos 1}\left[-x\cos x + \sin xr\right] - \frac{\sin x}{\cos 1}\left[1 - x\sin(x-1) - \cos(x-1)\right] \\
&= x\frac{\cos 1}{\cos 1} - \frac{\sin x}{\cos 1} = x - \frac{\sin x}{\cos 1}.
\end{aligned}$$

□

Example 4.1.3. Solve $y''(x) + 9y(x) = x\cos x$ subject to $y(0) = 0$, $y'(\pi) = 0$.

Solution. The fundamental solutions of $y''(x) + 9y(x) = x\cos x$ are

$$u_1(x) = \cos 3x \text{ and } u_2(x) = \sin 3x.$$

Consider $y_1(x) = \sin 3x$ and $y_2(x) = \cos 3x$. Then $y_1(0) = 0$ and $y_2'(\pi) = 0$. Now,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin 3x & \cos 3x \\ 3\cos 3x & -3\sin 3x \end{vmatrix} = -3.$$

Therefore Green's function for the problem is

$$G(x,s) = \begin{cases} \frac{\sin 3s \cos 3x}{-3}, & s \le x \\ \frac{\sin 3x \cos 3s}{-3}, & s \ge x. \end{cases}$$

When $f(x) = x\cos x$, the solution of the above differential equation is

$$\begin{aligned}
y(x) &= \int_0^\pi G(x,s) s\cos s\, ds \\
&= \int_0^x \frac{\sin 3s \cos 3x}{-3} s\cos s\, ds + \int_x^\pi \frac{\sin 3x \cos 3s}{-3} s\cos s\, ds \\
&= \frac{\cos 3x}{-6}\int_0^x [\sin 4s + \sin 2s] s\, ds + \frac{\sin 3x}{-6}\int_x^\pi [\cos 4s + \cos 2s] s\, ds \\
&= \frac{\cos 3x}{-6}\left[s\frac{-\cos 4s}{4} + 1\frac{\sin 4s}{16} + s\frac{-\cos 2s}{2} + \frac{\sin 2s}{4}\right]_0^x \\
&\qquad - \frac{\sin 3x}{-6}\left[s\frac{\sin 4s}{4} + 1\frac{\cos 4s}{16} + s\frac{\sin 2s}{2} + \frac{\cos 2s}{4}\right]_x^\pi \\
&= -\frac{\cos 3x}{6}\left[-x\left(\frac{\cos 4x}{4} + \frac{\cos 2x}{2}\right) + \left(\frac{\sin 4x}{16} + \frac{\sin 2x}{4}\right)\right] \\
&\qquad - \frac{\sin 3x}{6}\left[\frac{1}{16} + \frac{1}{4} - x\left(\frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right) - \left(\frac{\cos 4x}{16} + \frac{\cos 2x}{4}\right)\right].
\end{aligned}$$

□

4.2 Gram Schmidt Orthonormalization

Definition 4.2.1. Let V be a vector space over $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$). A map $\langle\cdot\,,\cdot\rangle : V\times V\to\mathbb{K}$ is called an *inner product* if for all $x,y,z\in V$ and $\alpha\in\mathbb{K}$,

1. $\langle x,x\rangle\geq 0$ and if $\langle x,x\rangle=0$ then $x=0$
2. $\langle x,y\rangle=\overline{\langle y,x\rangle}$
3. $\langle\alpha x,y\rangle=\alpha\langle x,y\rangle$
4. $\langle x+y,z\rangle=\langle x,z\rangle+\langle y,z\rangle$

A vector space V with an inner product is called an *inner product space*.

Examples 4.2.2. 1. $\mathbb{C}^n$ is an inner product space with the inner product

$$\langle(x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots y_n)\rangle:=\sum_{k=1}^{n}x_k\bar{y}_k.$$

2. Let $\ell^2=\{(x_n)_{n\in\mathbb{N}}:\sum_n|x_n|^2<\infty\}$ Then ℓ^2 is an inner product space with an inner product

$$\langle(x_n),(y_n)\rangle=\sum_n x_n\bar{y}_n.$$

3. Let $C[a,b]=\{f:[a,b]\to\mathbb{C}: f \text{ is continuous}\}$. Then $C[a,b]$ is an inner product space with an inner product

$$\langle f,g\rangle:=\int_a^b f(x)\overline{g(x)}\,dx.$$

Definition 4.2.3. Let V be an inner product space and let $x,y\in V$. Then elements x and y are called *orthogonal* if $\langle x,y\rangle=0$.

Definition 4.2.4. A non-empty subset A of an inner product space V is called orthogonal if $\langle x,y\rangle=0$ for every $x,y\in A$ with $x\neq y$.

Example 4.2.5. Show that any orthogonal subset of an inner product space, which does not contain zero, is linearly independent.

Solution. Let $A=\{x_1,x_2,\ldots,x_n\}$ be an orthogonal set such that $x_i\neq 0$ for all i. Let $\alpha_1,\alpha_2,\ldots,\alpha_n\in\mathbb{K}$ such that

$$\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n=0.$$

Then

$$\begin{aligned}0&=\langle 0,x_i\rangle\quad\forall\, i=1,2,\ldots,n\\&=\langle\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n,x_i\rangle\\&=\alpha_1\langle x_1,x_i\rangle+\alpha_2\langle x_2,x_i\rangle+\cdots+\alpha_n\langle x_n,x_i\rangle\\&=\alpha_i\langle x_i,x_i\rangle.\end{aligned}$$

Since $\langle x_i,x_i\rangle\neq 0$, $\alpha_i=0$ for all i and hence A is linearly independent. □

Definition 4.2.6. Let V be an inner product space. Then the map $\|\cdot\| : V \to \mathbb{K}$ defined by $\|x\| = \sqrt{\langle x,x\rangle}$, $(x \in V)$ is called a *norm* on V.

Definition 4.2.7. A non-empty subset of an inner product space V is called *orthonormal* if the set is orthogonal and the norm of each element of the set is 1.

Ex Show that every orthonormal set is linearly independent.

Theorem 4.2.8 (Gram Schmidt orthonormalization theorem)**.** *Let V be an inner product space and let $\{x_1, x_2, \dots\}$ be a linearly independent subset of V. Then there exists an orthonormal subset $\{e_1, e_2, \dots\}$ of V such that*

$$\mathrm{sp}\{e_1, e_2, \dots\} = \mathrm{sp}\{x_1, x_2, \dots\}.$$

[Gram Schmidt orthonormalization process]
Let $\{x_1, x_2, \dots\}$ be a linearly independent subset of an inner product space V. Let $e_1 = \dfrac{x_1}{\|x_1\|}$ (and $y_1 = x_1$). Let

$$y_2 = x_2 - \langle x_2, e_1\rangle e_1.$$

Take $e_2 = \dfrac{y_2}{\|y_2\|}$. Let

$$y_3 = x_3 - \langle x_3, e_1\rangle e_1 - \langle x_3, e_2\rangle e_2.$$

Take $e_3 = \dfrac{y_3}{\|y_3\|}$. In general, we get

$$\boxed{y_n = x_n - \sum_{j=1}^{n-1} \langle x_n, e_j\rangle e_j \quad \text{and} \quad e_n = \frac{y_n}{\|y_n\|}}.$$

Example 4.2.9. Orthonormalize the set $\{1, x, x^2, x^3\}$ over $[-1, 1]$.

Solution. Let $f_0(x) = 1$, $f_1(x) = x$, $f_2(x) = x^2$ and $f_3(x) = x^3$. We have to orthonormalize the set $\{f_0, f_1, f_2, f_3\}$ of $C[-1,1]$. Let $h_0(x) = f_0(x)$. Then

$$\|h_0\| = \|f_0\| = \left(\int_{-1}^{1} 1^2\, dx\right)^{\frac{1}{2}} = \sqrt{2}.$$

Take $g_0(x) = \dfrac{f_0(x)}{\|f_0(x)\|} = \dfrac{1}{\sqrt{2}}$. Let

$$h_1(x) = f_1(x) - \langle f_1, g_0\rangle g_0(x) = x - \left(\int_{-1}^{1} x\frac{1}{\sqrt{2}}\, dx\right)\frac{1}{\sqrt{2}} = x.$$

Then $\|h_1\| = \left(\int_{-1}^{1} x^2\, dx\right)^{\frac{1}{2}} = \sqrt{\dfrac{2}{3}}$. Let $g_1(x) = \dfrac{h_1(x)}{\|h_1\|} = \sqrt{\dfrac{3}{2}}x$. Let

$$h_2(x) = f_2(x) - \langle f_2, g_0\rangle g_0(x) - \langle f_2, g_1\rangle g_1(x)$$

$$= x^2 - \int_{-1}^{1} \left(x^2 \frac{1}{\sqrt{2}}\, dx \right) \frac{1}{\sqrt{2}} - \int_{-1}^{1} \left(x \sqrt{\frac{3}{2}} x\, dx \right) \sqrt{\frac{3}{2}} x = x^2 - \frac{1}{3}.$$

Then

$$\|h_2\| = \left(\int_{-1}^{1} \left(x^2 - \frac{1}{3} \right)^2 dx \right)^{\frac{1}{2}} = \left(2 \int_{0}^{1} \left(x^2 - \frac{1}{3} \right)^2 dx \right)^{\frac{1}{2}} = \left(\frac{8}{45} \right)^{\frac{1}{2}} = \frac{2}{3} \sqrt{\frac{2}{5}}.$$

Let $g_2(x) = \frac{h_2(x)}{\|h_2\|} = \frac{3}{2}\sqrt{\frac{5}{2}} \left(x^2 - \frac{1}{3} \right) = \sqrt{\frac{5}{2}} \left(\frac{3}{2} x^2 - \frac{1}{2} \right)$. Let

$$\begin{aligned} h_3(x) &= f_3(x) - \langle f_3, g_0 \rangle g_0(x) - \langle f_3, g_1 \rangle g_1(x) - \langle f_3, g_2 \rangle g_2(x) \\ &= x^3 - \left(\int_{-1}^{1} x^3 \frac{1}{\sqrt{2}}\, dx \right) \frac{1}{\sqrt{2}} - \left(\int_{-1}^{1} x^3 \sqrt{\frac{3}{2}} x\, dx \right) \sqrt{\frac{3}{2}} x \\ &\quad - \left(\int_{-1}^{1} x^3 \frac{3}{2} \sqrt{\frac{5}{2}} \left(x^2 - \frac{1}{3} \right) \right) \frac{3}{2} \sqrt{\frac{5}{2}} \left(x^2 - \frac{1}{3} \right) \\ &= x^3 - \frac{3}{5} x. \end{aligned}$$

Therefore

$$\begin{aligned} \|h_3\|^2 &= \left(\int_{-1}^{1} \left(x^3 - \frac{3}{5} x \right)^2 dx \right) \\ &= 2 \int_{0}^{1} \left(x^3 - \frac{3}{5} x \right)^2 dx \\ &= 2 \left[\frac{1}{7} - \frac{6}{5}\frac{1}{5} + \frac{9}{25}\frac{1}{3} \right] \\ &= 2 \left[\frac{1}{7} - \frac{3}{25} \right] = \frac{8}{175}. \end{aligned}$$

Therefore $\|h_3\| = \frac{2}{5}\sqrt{\frac{2}{7}}$. Let $g_3(x) = \frac{h_3(x)}{\|h_3\|}$. Thus, $\{g_0, g_1, g_2, g_3\}$ is the required orthonormal set.

Hence, by Gram-Schmidt orthonormalization method, the orthonormal set obtained by orthonormalizing the set $\{1, x, x^2, x^3\}$ over $[-1, 1]$ is

$$\left\{ \sqrt{\frac{1}{2}} P_0(x), \sqrt{\frac{3}{2}} P_1(x), \sqrt{\frac{5}{2}} P_2(x), \sqrt{\frac{7}{2}} P_3(x) \right\}.$$

□

Example 4.2.10. Show that $\mathrm{sp}\{1, x, x^2, x^3\} = \mathrm{sp}\left\{ \sqrt{\frac{2n+1}{2}} P_n(x) : n \in \mathbb{N} \cup \{0\} \right\}$ in $C[-1, 1]$.

Solution. Let $n \in \mathbb{N} \cup \{0\}$. Since $P_n(x)$ is the polynomial, we have

$$\sqrt{\frac{2n+1}{2}} P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in L(\{x^n : n \in \mathbb{N} \cup \{0\}\}),$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$. Hence, it follows that

$$\operatorname{sp}\left\{\sqrt{\frac{2n+1}{2}}P_n(x) : n \in \mathbb{N}\cup\{0\}\right\} \subset \operatorname{sp}\{1, x, x^2, x^3\}.$$

Since any polynomial can be expressed as a linear combination of Legendre polynomials, it follows that

$$\operatorname{sp}\{1, x, x^2, x^3\} \subset \operatorname{sp}\left\{\sqrt{\frac{2n+1}{2}}P_n(x) : n \in \mathbb{N}\cup\{0\}\right\}.$$

Hence the result. □

Example 4.2.11. The set $\left\{\sqrt{\frac{2n+1}{2}}P_n(x) : n \in \mathbb{N}\cup\{0\}\right\}$ is orthonormal set in $C[-1,1]$ (or $L^2[-1,1]$).

Solution. We know that

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0, & n \neq m \\ \frac{2}{2n+1}, & n = m \end{cases}.$$

Let $n \in \mathbb{N}\cup\{0\}$. Then

$$\begin{aligned} &\int_{-1}^{1} \sqrt{\frac{2n+1}{2}}P_n(x)\sqrt{\frac{2m+1}{2}}P_m(x)\,dx \\ &= \frac{\sqrt{(2n+1)(2m+1)}}{2}\int_{-1}^{1} P_n(x)P_m(x)\,dx \\ &= \begin{cases} 0, & m \neq n \\ \frac{2n+1}{2}\frac{2}{2n+1} = 1, & m = n. \end{cases} \end{aligned}$$

Hence, the given set is orthonormal. □

Remark 4.2.12. Note that the above orthonormal set is an orthonormal basis for the Hilbert space $L^2[-1,1]$ and

$$f(x) = \sum_{n=0}^{\infty}\left\langle f, \sqrt{\frac{2n+1}{2}}P_n\right\rangle\sqrt{\frac{2n+1}{2}}P_n(x).$$

4.3 Least Square Approximation

Theorem 4.3.1 (Least square approximation)**.** *Let $f \in L^2[-1,1]$ and let $p(x) = b_0P_0(x) + b_1P_1(x) + \cdots + b_nP_n(x)$ be a polynomial of degree n. Then the integral $\int_{-1}^{1} |f(x)-p(x)|^2\,dx$ is minimum if and only if*

$$b_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)P_k(x)\,dx, \qquad k = 0, 1, 2, \ldots, n.$$

Proof. For $k = 0,1,\dots,n$, let $a_k = \dfrac{2k+1}{2}\displaystyle\int_{-1}^{1} f(x)P_k(x)\,dx$. Now,

$$\begin{aligned}
\int_{-1}^{1} |f(x)-p(x)|^2\,dx &= \int_{-1}^{1}\left(f(x)-\sum_{j=0}^{n} b_jP_j(x)\right)^2 dx \\
&= \int_{-1}^{1} f(x)^2\,dx - 2\sum_{j=0}^{n} b_j\int_{-1}^{1} f(x)P_j(x)\,dx + \int_{-1}^{1}\left(\sum_{j=0}^{n} b_jP_j(x)\right)^2 dx \\
&= \int_{-1}^{1} f(x)^2\,dx - 2\sum_{j=0}^{n} b_j\frac{2a_j}{2j+1} + \sum_{j=0}^{n} b_j^2\frac{2}{2j+1} + \sum_{j=0}^{n}\frac{2a_j^2}{2j+1} - \sum_{j=0}^{n}\frac{2a_j^2}{2j+1} \\
&= \int_{-1}^{1} f(x)^2dx + 2\sum_{j=0}^{n}\frac{(a_j-b_j)^2}{2j+1} - \sum_{j=0}^{n}\frac{2a_j^2}{2j+1}.
\end{aligned}$$

Since the first and the last term on the right hand side of the above equation are fixed, the above integral will be minimum if and only if $\displaystyle\sum_{j=0}^{n}\frac{(a_j-b_j)^2}{2j+1} = 0$, i.e. if and only if $b_j = a_j$ for all j. □

Example 4.3.2. Find a polynomial of degree two so that $\displaystyle\int_{-1}^{1}|e^x - p(x)|^2\,dx$ is minimum.

Solution. Let $p(x) = b_0P_0(x) + b_1P_1(x) + b_2P_2(x)$, where

$$b_k = \frac{2k+1}{2}\int_{-1}^{1} e^xP_k(x)\,dx, \qquad (k = 0,1,2).$$

Then $p(x) = b_0 + b_1x + b_2\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$. Here

$$b_0 = \frac{1}{2}\int_{-1}^{1} e^x\cdot 1\,dx = \frac{1}{2}[e^1 - e^{-1}] = \sinh 1.$$

$$b_1 = \frac{3}{2}\int_{-1}^{1} e^x\,x\,dx = \frac{3}{2}[xe^x - e^x]_{-1}^{1} = 3e^{-1}.$$

$$\begin{aligned}
b_2 &= \frac{5}{2}\int_{-1}^{1} e^x\left(\frac{3}{2}x^2 - \frac{1}{2}\right) dx \\
&= \frac{15}{4}\int_{-1}^{1} e^xx^2\,dx - \frac{5}{4}\int_{-1}^{1} e^x\,dx \\
&= \frac{15}{4}\left[x^2e^x - 2xe^x + 2e^x\right]_{-1}^{1} - \frac{5}{2}\sinh 1 \\
&= \frac{15}{4}\left[e^1 - 2e^1 + 2e^1 - (e^{-1} + 2e^{-1} + 2e^{-1})\right] - \frac{5}{2}\sinh 1 \\
&= \frac{15}{4}[e - e^{-1} - 4e^{-1}] - \frac{5}{2}\sinh 1 \\
&= \frac{15}{2}\sinh 1 - 15e^{-1} - \frac{5}{2}\sinh 1 = 5(\sinh 1 - 3e^{-1}).
\end{aligned}$$

□

Example 4.3.3. Show that

$$\min\left\{\int_{-1}^{1} p(x)^2\,dx : p(x) \text{ is a monic polynomial of degree } n\right\} = \int_{-1}^{1} (cP_n(x))^2\,dx,$$

where $c \in \mathbb{R}$, so that $cP_n(x)$ is a monic polynomial.

Solution. Let $p(x)$ be a monic polynomial of degree n. Then there exists $c_0, c_1, \ldots, c_{n-1}$ such that

$$p(x) = c_0P_0(x) + c_1P_1(x) + \cdots + c_{n-1}P_{n-1}(x) + cP_n(x).$$

Therefore

$$\begin{aligned}\int_{-1}^{1} p(x)^2\,dx &= \int_{-1}^{1} (c_0P_0(x) + c_1P_1(x) + \cdots + c_{n-1}P_{n-1}(x) + c_nP_n(x))^2\,dx \\ &= c_0^2\int_{-1}^{1} P_0(x)^2dx + \cdots + c_{n-1}^2\int_{-1}^{1} P_{n-1}(x)^2dx + c^2\int_{-1}^{1} P_n(x)^2dx.\end{aligned}$$

Since c is fixed, above integral will be minimum if and only if $c_0 = c_1 = \cdots = c_{n-1} = 0$. Hence the result. □

Example 4.3.4. Find $\min\left\{\int_{-1}^{1} p(x)^2\,dx : p(x) \text{ is a monic polynomial of degree } 2\right\}$

Solution. By above example, we have to find $\int_{-1}^{1}(cP_2(x))^2dx$, where $c \in \mathbb{R}$ so that $cP_2(x)$ is a monic polynomial. Thus, $c = \frac{2}{3}$. Therefore

$$\begin{aligned}\int_{-1}^{1}\left(\frac{2}{3}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\right)^2 dx &= \left[\frac{1}{5} - \frac{2}{3}\frac{1}{3} + \frac{1}{9}]\right] \\ &= 2\left[\frac{1}{5} - \frac{1}{9}\right] = \frac{8}{45}.\end{aligned}$$

□

Example 4.3.5. Find a polynomial of degree 2 so that $\int_{-1}^{1}|\sin x - p(x)|^2dx$ is minimum.

Solution. For this, let $b_k = \frac{2k+1}{2}\int_{-1}^{1}\sin xP_k(x)\,dx$, $k = 0,1,2$. Then $p(x) = b_0 + b_1x + b_2\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$. Now,

$$b_0 = \frac{1}{2}\int_{-1}^{1}\sin x\,dx = 0.$$

$$b_1 = \frac{3}{2}\int_{-1}^{1}\sin x\,x\,dx = 3\int_{0}^{1} x\sin x\,dx = 3\,[x(-\cos x) + \sin x]_0^1 = 3(\sin 1 - \cos 1).$$

$$b_2 = \frac{5}{2}\int_{-1}^{1}\sin x\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\,dx = \frac{15}{4}\int_{-1}^{1} x^2\sin x\,dx - \frac{5}{4}\int_{-1}^{1}\sin x\,dx = 0.$$

Hence $p(x) = 3(\sin 1 - \cos 1)x$. □

Example 4.3.6. Find a polynomial of degree 2 so that $\int_{-1}^{1}|\cos x - p(x)|^2dx$ is minimum.

lightatinfinite@gmail.com
jay_mehta@spuvvn.edu

Solution. For this, let $b_k = \frac{2k+1}{2}\int_{-1}^{1}\cos x P_k(x)\,dx$, $k=0,1,2$. Then $p(x) = b_0 + b_1 x + b_2\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$. Now,

$$b_0 = \frac{1}{2}\int_{-1}^{1}\cos x\,dx = \sin 1.$$

$$b_1 = \frac{3}{2}\int_{-1}^{1}\cos x\,x\,dx = 0.$$

$$\begin{aligned}
b_2 &= \frac{5}{2}\int_{-1}^{1}\cos x\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\,dx\\
&= \frac{15}{4}\int_{-1}^{1}x^2\cos x\,dx - \frac{5}{4}\int_{-1}^{1}\cos x\,dx\\
&= \frac{15}{2}\int_{0}^{1}x^2\cos x\,dx - \frac{5}{2}\int_{0}^{1}\cos x\,dx\\
&= \frac{15}{2}\left[x^2\sin x + 2x\cos x - 2\sin x\right]_0^1 - \frac{5}{2}\sin 1\\
&= \frac{15}{2}[\sin 1 + 2\cos 1 - 2\sin 1] - \frac{5}{2}\sin 1\\
&= 15\cos 1 - 10\sin 1.
\end{aligned}$$

Hence $p(x) = \sin 1 + 5(3\cos 1 - 2\sin 1)x^2$. □

4.4 Orthogonality with respect to weight functions

Let ω be a non-negative "weight function" on $[a,b]$. Two functions f and g are called orthogonal with respect to the weight function ω if

$$\int_a^b f(x)g(x)\omega(x)\,dx = 0 \quad \text{i.e.,} \quad \int_a^b f(x)\sqrt{\omega(x)}g(x)\sqrt{\omega(x)}\,dx = 0.$$

We know that the set $\{1, x, x^2, \ldots\}$ is linearly independent over $[a,b]$. Let ω be a weight function on $[a,b]$. Then $\{\sqrt{\omega(x)}, x\sqrt{\omega(x)}, \ldots\}$ is linearly independent over $[a,b]$. When we orthonormalize the above set on $[a,b]$, we get an orthonormal set

$$\{Q_0(x)\sqrt{\omega(x)}, Q_1(x)\sqrt{\omega(x)}, Q_2(x)\sqrt{\omega(x)}, \ldots\},$$

where $Q_n(x)$ is a polynomial of degree n. That is the set $\{Q_0(x), Q_1(x), \ldots\}$ is an orthonormal set on $[a,b]$ with respect to the weight function ω. For example,

1. If $a=-1$, $b=1$ and $\omega(x) = \frac{1}{\sqrt{1-x^2}}$, then we obtain Tchebycheff polynomials $T_n(x)$,

$$T_n(x) = \frac{1}{2^n - 1}\cos(n\cos^{-1}(x)).$$

2. If $a=-1$, $b=1$ and $\omega(x)=1$, then we get Legendre polynomials.
3. If $a=-1$, $b=1$ and $\omega(x)=\sqrt{1-x^2}$, then we get the polynomials

$$Q_n(x) = \frac{\sin((n+1)\cos^{-1}x)}{\sqrt{1-x^2}}.$$

4. If $a=0$, $b=1$ and $\omega(x)=x^{q-1}(1-x)^{p-q}$, $q>0$ and $p-q>1$, then we get Jacobi polynomials.
5. If $a=0$, $b=\infty$ and $\omega(x)=e^{-x^2}$, then we get Hermite polynomials.
6. If $a=0$, $b=\infty$ and $\omega(x)=e^{x}$, then we get Laguerre polynomials.

Example 4.4.1. Orthonormalize the set $\{1,x,x^2,x^3\}$ on $(0,\infty)$ with respect to the weight function e^{-x^2}.

Solution. We need to orthonormalize the set

$$\{\sqrt{e^{-x^2}},x\sqrt{e^{-x^2}},x^2\sqrt{e^{-x^2}},x^3\sqrt{e^{-x^2}}\}$$

on $(0,\infty)$. Let $f_i(x)=x^i\sqrt{e^{-x^2}}$, $i=0,1,2,3$. Now,

$$\|f_0\|^2=\int_0^\infty \sqrt{e^{-x^2}}\sqrt{e^{-x^2}}\,dx=\int_0^\infty e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}.$$

Let $g(x)=\dfrac{f_0(x)}{\|f_0\|}=\dfrac{\sqrt{2}}{\pi^{\frac{1}{4}}}\sqrt{e^{-x^2}}$. Let

$$\begin{aligned} h_1(x)&=f_1(x)-\langle f_1,g_0\rangle g_0(x)\\ &=x\sqrt{e^{-x^2}}-\left(\int_0^\infty xe^{-x^2}\,dx\right)\frac{2}{\sqrt{\pi}} \end{aligned}$$

`Complete it as exercise.` □

4.5 Hermite Polynomials

Definition 4.5.1. Hermite polynomials $H_n(x)$ of degree n are orthogonal polynomials over $(0,\infty)$ with respect to the weight function e^{-x^2}.

They are most conveniently defined by means of a generating function $\psi(x,t)$ given by

$$\psi(x,t)=e^{-t^2+2xt}:=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n.$$

Theorem 4.5.2. *Let $n\in\mathbb{N}\cup\{0\}$. Then $H_n(x)=(-1)^n e^{x^2}\dfrac{d^n}{dx^n}e^{-x^2}$.*

Proof. We have $\psi(x,t)=e^{-t^2+2xt}=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n$. We note that $H_n(x)=\frac{\partial^n}{\partial t^n}\psi(x,t)\Big|_{t=0}$.

We have

$$\psi(x,t)=e^{-t^2+2xt-x^2+x^2}=e^{x^2}e^{-(t-x)^2}=e^{x^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(t-x)^{2k}.$$

Therefore

$$\frac{\partial^n}{\partial t^n}\psi(x,t)=e^{x^2}\sum_{2k\geq n}\frac{(-1)^k 2k(2k-1)\cdots(2k-n+1)}{k!}(t-x)^{2k-n}.$$

Therefore

$$\begin{aligned}
\frac{\partial^n}{\partial t^n}\psi\Big|_{t=0} &= e^{x^2}\sum_{2k\geq n}\frac{(-1)^k 2k(2k-1)\cdots(2k-n+1)}{k!}x^{2k-n}\\
&= (-1)^n e^{x^2}\sum_{2k\geq n}\frac{(-1)^k}{k!}\frac{d^n}{dx^n}x^{2k}\\
&= (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(\sum_{2k\geq n}\frac{(-1)^k}{k!}x^{2k}\right)\\
&= (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(\sum_{2k<n}\frac{(-1)^k}{k!}x^{2k}+\sum_{2k\geq n}\frac{(-1)^k}{k!}x^{2k}\right)\\
&= (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(\sum_{k=0}^{\infty}\frac{(-x^2)^k}{k!}\right).
\end{aligned}$$

Hence,

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right).$$

□

Example 4.5.3. Find $H_2(x)$ and hence compute $H_2(0)$.

Solution. We have

$$\begin{aligned}
H_2(x) &= e^{x^2}\frac{d^2}{dx^2}(e^{x^2})\\
&= e^{x^2}\frac{d}{dx}\left(e^{-x^2}(-2x)\right)\\
&= -e^{x^2}\left[e^{-x^2}(2)+2xe^{-x^2}(-2x)\right]\\
&= -e^{x^2}[2e^{-x^2}-4x^2e^{-x^2}]\\
&= -(-4x^2+2) = 4x^2-2.
\end{aligned}$$

Hence $H_2(0) = -2$. □

Example 4.5.4. Show that $H_n'(x) = 2nH_{n-1}(x)$, $n \geq 1$.

Solution. We have $e^{-t^2+2xt} = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n$. Then differentiating both sides with respect to x, we get

$$\begin{aligned}
e^{-t^2+2xt}(2t) &= \sum_{n=1}^{\infty}\frac{H_n'(x)}{n!}t^n \qquad (\because \text{ first term is constant})\\
\Rightarrow 2t\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n &= \sum_{n=1}^{\infty}\frac{H_n'(x)}{n!}t^n\\
\Rightarrow \sum_{n=0}^{\infty}2\frac{H_n(x)}{n!}t^{n+1} &= \sum_{n=1}^{\infty}\frac{H_n'(x)}{n!}t^n
\end{aligned}$$

$$\Rightarrow \sum_{n=1}^{\infty} 2\frac{H_{n-1}(x)}{(n-1)!}t^n = \sum_{n=1}^{\infty} \frac{H_n'(x)}{n!}t^n.$$

Comparing the coefficients of t^n on both the sides, we get

$$2\frac{H_{n-1}(x)}{(n-1)!} = \frac{H_n'(x)}{n!}$$
$$\Rightarrow 2nH_{n-1}(x) = H_n'(x), \quad n \geq 1.$$

□

Example 4.5.5. Show that $H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$, $\forall\, n \geq 1$.

Solution. We know that $e^{-t^2+2xt} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!}t^n$. Then differentiating both sides with respect to t, we get

$$(-2t+2x)e^{-t^2+2xt} = \sum_{n=1}^{\infty} \frac{H_n(x)}{n!}nt^{n-1} \qquad \text{(differentiating wrt } t\text{)}$$
$$\Rightarrow -2\sum_{n=0}^{\infty} \frac{H_n(x)}{n!}t^{n+1} + 2x\sum_{n=0}^{\infty} \frac{H_n(x)}{n!}t^n = \sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!}t^{n-1}$$
$$\Rightarrow -2\sum_{n=1}^{\infty} \frac{H_{n-1}(x)}{(n-1)!}t^n + 2x\sum n = 0^{\infty}\frac{H_n(x)}{n!}t^n = \sum_{n=0}^{\infty} \frac{H_{n+1}(x)}{n!}t^n.$$

Comparing the coefficients of t^n on both the sides, we get

$$-2\frac{H_{n-1}(x)}{(n-1)!} + 2x\frac{H_n(x)}{n!} = \frac{H_{n+1}(x)}{n!}, \quad n \geq 1.$$

Hence multiplying above equation by $n!$, we get

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0, \quad \forall\, n \geq 1.$$

□

Example 4.5.6. Show that $H_n(x)$ satisfies $y'' - 2xy' + 2ny = 0$.

Solution. We know that

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$$
$$\Rightarrow H_{n+1}(x) - 2xH_n(x) + H_n'(x) = 0$$
$$\Rightarrow H_{n+1}'(x) - 2H_n(x) - 2xH_n'(x) + H_n''(x) = 0 \qquad \text{(differentiating wrt } x\text{)}$$
$$\Rightarrow 2(n+1)H_n(x) - 2H_n(x) - 2xH_n'(x) + H_n''(x) = 0 \qquad \text{(by previous example)}$$
$$\Rightarrow H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0.$$

Hence $H_n(x)$ satisfies $y'' - 2xy' + 2ny = 0$ which is known as *Hermite equation*. □

4.5.1 Orthogonality of Hermite polynomials

Theorem 4.5.7. *Hermite polynomials are orthogonal. More precisely,*

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}\,dx = \begin{cases} 0, & m \neq n \\ 2^n n!\sqrt{\pi}, & m = n. \end{cases}$$

Proof. Let $m,n \in \mathbb{N}\cup\{0\}$ and let $m \neq n$. We may assume that $m < n$. Now,

$$\begin{aligned}
&\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}\,dx \\
&= \int_{-\infty}^{\infty} H_m(x)e^{-x^2}\left[(-1)^n\left(\frac{d^n}{dx^n}\left(e^{-x^2}\right)\right)e^{x^2}\right]dx \\
&= (-1)^n\int_{-\infty}^{\infty} H_m(x)\frac{d^n}{dx^n}\left(e^{-x^2}\right)dx \\
&= (-1)^n\left[H_m(x)\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right) - \int H_m'(x)\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right)dx\right]_{-\infty}^{\infty} \\
&= 0 - (-1)^n\int_{-\infty}^{\infty} 2mH_{m-1}(x)\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right)dx \\
&= (-1)^n 2^2 m(m-1)\int_{-\infty}^{\infty} H_{m-2}(x)\frac{d^{n-2}}{dx^{n-2}}\left(e^{-x^2}\right)dx \qquad \text{(repeating once more).}
\end{aligned}$$

Repeating this process m-times, we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}\,dx \\
&= (-1)^n(-1)^m 2^m m(m-1)\cdots 2\cdot 1\int_{-\infty}^{\infty} H_0(x)\frac{d^{n-m}}{dx^{n-m}}\left(e^{-x^2}\right)dx \qquad (4.1)\\
&= (-1)^n(-1)^m 2^m m! H_0(x)\left[\frac{d^{n-m-1}}{dx^{n-m-1}}\left(e^{-x^2}\right)\right]_{-\infty}^{\infty} \\
&= 0.
\end{aligned}$$

This is because, $n > m \Rightarrow n-m-1 \geq 0$ and hence $\frac{d^m}{dx^m}\left(e^{-x^2}\right) = p(x)e^{-x^2} \to 0$ as $|x| \to \infty$. Also for $n = 0$, $H_0(x) = e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right) = e^{x^2}e^{-x^2} = 1$.

Now, if $m = n$ then by equation (4.1), we have

$$\int_{-\infty}^{\infty} H_n(x)H_n(x)e^{-x^2}\,dx = 2^n n!\int_{-\infty}^{\infty} dx = 2^n n!\sqrt{\pi}.$$

□

4.6 *Z*-transform

Definition 4.6.1. Let $(a_n)_{n\geq 0} = (a_0, a_1, a_2, \ldots)$ be a sequence of complex numbers. The *Z*-

transform, $Z[(a_n)]$, of (a_n) is defined by

$$Z[(a_n)](z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}.$$

Remark 4.6.2. Note the the domain of $Z[(a_n)]$ is the set of all complex numbers where the series $\sum_{n=0}^{\infty} \frac{a_n}{z^n}$ converges.

Now onwards whenever we write (a_n) means $(a_n) = (a_0, a_1, a_2, \ldots)$. We see below some examples in which we compute the Z-transform of some standard series.

Compute the Z-transform of the following:

Example 4.6.3. $(a^n)_{n\geq 0}$ or simply written as (a^n).

Solution.
$$\begin{aligned} Z[(a^n)](z) &= \sum_{n=0}^{\infty} \frac{a^n}{z^n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \\ &= \frac{1}{1-\frac{a}{z}} \qquad (|z| > |a|) \\ &= \frac{z}{z-a}, \qquad (|z| > |a|). \end{aligned}$$
□

Example 4.6.4. $((-1)^n)_{n\geq 0}$.

Solution.
$$\begin{aligned} Z[(-1)^n](z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} = \sum_{n=0}^{\infty} \left(\frac{-1}{z}\right)^n \\ &= \frac{1}{1+\frac{1}{z}} \qquad (|z| > 1) \\ &= \frac{z}{z+1}, \qquad (|z| > 1). \end{aligned}$$
□

Example 4.6.5. $(e^{-\alpha n})$.

Solution. $$Z[(e^{-\alpha n})](z) = \sum_{n=0}^{\infty} \frac{e^{-\alpha n}}{z^n} = \sum_{n=0}^{\infty} \left(\frac{e^{-\alpha}}{z}\right)^n = \frac{1}{1-\frac{e^{-\alpha}}{z}} = \frac{z}{z-e^{-\alpha}}.$$ □

Example 4.6.6. $(\cos(n\alpha))$, where $\alpha \in \mathbb{R}$.

lightatinfinite@gmail.com
jay_mehta@spuvvn.edu

Solution.
$$
\begin{aligned}
Z[(\cos(n\alpha))](z) &= \sum_{n=0}^{\infty}\frac{\cos\alpha n}{z^n}\\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{e^{i\alpha n}+e^{-i\alpha n}}{z^n}\\
&= \frac{1}{2}\left[\sum_{n=0}^{\infty}\left(\frac{e^{i\alpha}}{z}\right)^n+\sum_{n=0}^{\infty}\left(\frac{e^{-i\alpha}}{z}\right)^n\right]\\
&= \frac{1}{2}\left(\frac{1}{1-\frac{e^{i\alpha}}{z}}+\frac{1}{1-\frac{e^{-i\alpha}}{z}}\right)\\
&= \frac{1}{2}\left[\frac{z}{z-e^{i\alpha}}+\frac{z}{z-e^{-i\alpha}}\right]\\
&= \frac{1}{2}\frac{z(z-e^{-i\alpha}+z-e^{-i\alpha})}{z^2+ze^{-i\alpha}-ze^{i\alpha}+1}\\
&= \frac{1}{2}\frac{z(2z-2\cos\alpha)}{z^2-2z\cos\alpha+1}\\
&= \frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}.
\end{aligned}
$$
□

Corollary 4.6.7. $Z[\cos n\pi](z)=\dfrac{z}{z+1}$ *and* $Z\left[\cos\frac{n\pi}{2}\right](z)=\dfrac{z^2}{z^2+1}$.

Example 4.6.8. $(\sin(n\alpha))$, where $\alpha\in\mathbb{R}$.

Solution.
$$
\begin{aligned}
Z[(\sin(n\alpha))](z) &= \sum_{n=1}^{\infty}\frac{\sin\alpha n}{z^n}\\
&= \frac{1}{2i}\sum_{n=1}^{\infty}\frac{e^{i\alpha n}-e^{-i\alpha n}}{z^n}\\
&= \frac{1}{2i}\left[\sum_{n=1}^{\infty}\left(\frac{e^{i\alpha}}{z}\right)^n-\sum_{n=1}^{\infty}\left(\frac{e^{-i\alpha}}{z}\right)^n\right]\\
&= \frac{1}{2i}\left[\frac{1}{1-\frac{e^{i\alpha}}{z}}-\frac{1}{1-\frac{e^{-i\alpha}}{z}}\right]\\
&= \frac{1}{2i}\left[\frac{z}{z-e^{i\alpha}}-\frac{z}{z-e^{-i\alpha}}\right]\\
&= \frac{1}{2i}\frac{z(z-e^{-i\alpha}-z+e^{-i\alpha})}{z^2-ze^{-i\alpha}-ze^{i\alpha}+1}\\
&= \frac{z(\sin\alpha)}{z^2-2z\cos\alpha+1}.
\end{aligned}
$$
□

Corollary 4.6.9. $Z[\sin n\pi](z)=0$ *and* $Z\left[\sin\frac{n\pi}{2}\right](z)=\dfrac{z}{z^2+1}$.

Ex Compute the Z-transform of the following:

1. $(\cosh\alpha n)$
   Answer: $\dfrac{z(z-\cosh\alpha)}{z^2-2z\cosh\alpha+1}$.
2. $(\sinh\alpha n)$
   Answer: $\dfrac{z(\sinh\alpha)}{z^2-2z\cosh\alpha+1}$.

4.6.1 Properties of Z-transform

We note some of the properties of the Z- transform.

Proposition 4.6.10.

1. ***Z-transform is linear.*** *Let (a_n) and (b_n) be complex sequences, and $\alpha,\beta\in\mathbb{C}$. Then*
$$Z[\alpha(a_n)+\beta(b_n)] = \alpha Z[(a_n)]+\beta Z[(b_n)].$$
2. *Let (a_n) be a complex sequence, and let $0\neq\alpha\in\mathbb{C}$. Then $Z[(\alpha^n a_n)](z)=Z[(a_n)][(\frac{z}{\alpha})$. Similarly $Z[(\alpha^{-n}a_n)](z)=Z[(a_n)](\alpha z)$.*
3. *Let $p\in\mathbb{N}$. Then $Z[(n^p)](z)=-z\frac{d}{dz}Z[(n^{p-1})](z)$.*
4. *Let (a_n) be a complex sequence, and let $k\in\mathbb{N}\cup\{0\}$. Then*
$$Z[(a_{n+k})](z)=z^kZ[(a_n)](z)-a_0z^k-a_1z^{k-1}-\cdots-a_{k-1}z.$$

Proof.

1. Let z be in the intersection of Z-transforms of both (a_n) and (b_n). Then
$$\begin{aligned}
Z[\alpha(a_n)+\beta(b_n)](z) &= \sum_{n=0}^{\infty}\frac{\alpha a_n+\beta b_n}{z^n}\\
&= \sum_{n=0}^{\infty}\frac{\alpha a_n}{z^n}+\sum_{n=0}^{\infty}\frac{\beta b_n}{z^n}\\
&= \alpha\sum_{n=0}^{\infty}\frac{a_n}{z^n}+\beta\sum_{n=0}^{\infty}\frac{b_n}{z^n}\\
&= \alpha Z[(a_n)](z)+\beta Z[(b_n)](z).
\end{aligned}$$
2. We have
$$Z[(\alpha^n a_n)](z)=\sum_{n=0}^{\infty}a_n\left(\frac{\alpha}{z}\right)^n=\sum_{n=0}^{\infty}\frac{a_n}{\left(\frac{z}{\alpha}\right)^n}=Z[(a_n)]\left(\frac{z}{\alpha}\right).$$
   Similarly,
$$Z[(\alpha^{-n}a_n)](z)=\sum_{n=0}^{\infty}\frac{\alpha^{-n}a_n}{z^n}=\sum_{n=0}^{\infty}\frac{a_n}{(\alpha z)^n}=Z[(a_n)](\alpha z).$$

3. $$\begin{aligned} z\frac{d}{dz}Z[(n^{p-1})](z) &= z\frac{d}{dz}\left(\sum_{n=0}^{\infty} n^{p-1}z^{-n}\right) \\ &= z\sum_{n=1}^{\infty}(-n)n^{p-1}z^{-n-1} \\ &= -\sum_{n=1}^{\infty} n^p z^{-n} = -\sum_{n=0}^{\infty} n^p z^{-n} \\ &= Z[(n^p)](z). \end{aligned}$$

4. By definition

$$\begin{aligned} Z[(a_{n+k})](z) &= \sum_{n=0}^{\infty} a_{n+k}z^{-n} \\ &= z^k\sum_{n=0}^{\infty} a_{n+k}z^{-n-k} \\ &= z^k\sum_{n=k}^{\infty} a_n z^{-n} \\ &= z^k\left(\sum_{n=0}^{\infty} a_n z^{-n} - a_0 - \frac{a_1}{z} - \cdots - \frac{a_{k-1}}{z^{k-1}}\right) \\ &= z^k Z[(a_n)](z) - a_0 z^k - a_1 z^{k-1} - \cdots - a_{k-1}z. \end{aligned}$$

□

We shall compute the Z-transform of following sequences.

Example 4.6.11. $Z[(1)](z) = \dfrac{z}{z-1}$.

Example 4.6.12. $Z[(n)](z) = \dfrac{z}{(z-1)^2}$.

Solution. $$\begin{aligned} Z[(n)](z) = Z[(n^1)](z) &= -z\frac{d}{dz}Z[(1)](z) \\ &= -z\frac{d}{dz}\left(\frac{z}{z-1}\right) \\ &= -z\left[\frac{(z-1)-z(1)}{(z-1)^2}\right] = \frac{z}{(z-1)^2}. \end{aligned}$$ □

Example 4.6.13. $Z[(n^2)](z) = \dfrac{z(z+1)}{(z-1)^3}$.

Solution. $$\begin{aligned} Z[(n^2)](z) &= -z\frac{d}{dz}Z[(n)](z) \\ &= -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) \\ &= -z\left[\frac{(z-1)^2 - z\cdot 2(z-1)}{(z-1)^4}\right] \\ &= z\frac{z^2-1}{(z-1)^4} = \frac{z(z+1)}{(z-1)^3}. \end{aligned}$$ □

Example 4.6.14. $Z[(n+1)^2](z)$

Solution.
$$\begin{aligned} Z[(n+1)^2](z) &= Z[(n^2)](z) + 2Z[(n)](z) + Z[(1)](z) \\ &= \frac{z(z+1)}{(z-1)^3} + \frac{2z}{(z-1)^2} + \frac{z}{z-1} \\ &= \frac{(z^2+z) + 2z(z-1) + z(z-1)^2}{(z-1)^3} \\ &= \frac{z^2+z+2z^2-2z+z^3-2z^2+z}{(z-1)^3} \\ &= \frac{z^3+z^2}{(z-1)^3} = \frac{z^2(z+1)}{(z-1)^3}. \end{aligned}$$
□

Example 4.6.15. $Z[(2^n \cdot n)](z)$

Solution. $Z[(2^n \cdot n)](z) = Z[(n)]\left(\frac{z}{2}\right) = \frac{\frac{z}{2}}{\left(\frac{z}{2}-1\right)^2} = \frac{2z}{(z-2)^2}.$ □

Example 4.6.16. $Z[\left(\frac{a^n}{n!}\right)](z)$

Solution. Here $a_n = \frac{1}{n!}$. Now

$$Z[(a_n)](z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n} = \sum_{n=0}^{\infty} \frac{\frac{1}{n!}}{z^n} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = e^{\frac{1}{z}}.$$

Therefore

$$Z[(a^n a_n)](z) = Z[(a_n)]\left(\frac{z}{a}\right) = e^{\frac{1}{z/a}} = e^{\frac{a}{z}}.$$

□

Example 4.6.17.
$$\begin{aligned} Z[(a_{n+1})](z) &= \sum_{n=0}^{\infty} \frac{a_{n+1}}{z^n} \\ &= z\sum_{n=0}^{\infty} \frac{a_{n+1}}{z^{n+1}} \\ &= z\left(\sum_{n=1}^{\infty} \frac{a_{n+1}}{z^{n+1}} - a_0\right) \\ &= zZ[(a_n)](z) - a_0 z. \end{aligned}$$

Example 4.6.18.
$$\begin{aligned} Z[(a_{n+2})](z) &= \sum_{n=0}^{\infty} \frac{a_{n+2}}{z^n} \\ &= z^2\sum_{n=0}^{\infty} \frac{a_{n+2}}{z^{n+2}} \\ &= z^2\left(\sum_{n=2}^{\infty} \frac{a_{n+2}}{z^{n+2}} - a_0 - \frac{a_1}{z}\right) \\ &= z^2 Z[(a_n)](z) - a_0 z^2 - a_1 z. \end{aligned}$$

In general
$$Z[(a_{n+p})](z) = z^p Z[(a_n)](z) - a_0 z^p - a_1 z^{p-1} - \cdots - a_{p-1} z.$$

Proof by induction.

Examples 4.6.19. 1. $a_n = \frac{1}{3^n}$ if $0 \le n \le 5$ and $a_n = 0$ if $n > 5$.
By definition $Z[(a_n)](z) = \sum_{n=0}^{5} (3z)^{-n}$. Thus the domain of the Z- transform of this sequence is $\mathbb{C} \setminus \{0\}$.

4.7 Inverse Z-transform

Definition 4.7.1. Let f be a complex function. If there is a sequence (a_n) such that $Z[(a_n)] = f$, i.e. if f is the Z-transform of the sequence (a_n), then (a_n) is called the inverse Z-transform of f. In this case, we write $Z^{-1}[f] = (a_n)$.

Since Z-transform is linear, it follows that inverse Z-transform is linear, i.e.

$$Z^{-1}[\alpha f + \beta g] = \alpha Z^{-1} f + \beta Z^{-1} g.$$

Proof. Let $Z^{-1}[f] = (a_n)$ and $Z^{-1}[g] = (b_n)$. Then

$$Z[(\alpha a_n + \beta b_n)] = \alpha Z[(a_n)] + \beta Z[(b_n)] = \alpha f + \beta g.$$

Therefore

$$Z^{-1}[\alpha f + \beta g] = (\alpha a_n + \beta b_n) = \alpha (a_n) + \beta (b_n) = \alpha Z^{-1}[f] + \beta Z^{-1}[(b_n)].$$

□

4.7.1 Convolution product of sequences

Definition 4.7.2. Let (a_n) and (b_n) be sequences of complex numbers. Then the *convolution*, $(a_n) * (b_n)$, of (a_n) and (b_n) is the sequence (c_n), where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Theorem 4.7.3 (Convolution theorem)**.** *Let (a_n) and (b_n) be complex sequences. Then*

$$Z[(a_n) * (b_n)] = Z[(a_n)] Z[(b_n)].$$

Proof. Let $z \in \mathbb{C}$ be in the intersection of domains of Z- transforms of (a_n) and (b_n). Then

$$(Z[(a_n)] Z[(b_n)])(z) = Z[(a_n)](z) Z[(b_n)](z)$$

$$= \left(\sum_{n=0}^{\infty} \frac{a_n}{z^n}\right)\left(\sum_{n=0}^{\infty} \frac{b_n}{z^n}\right)$$
$$= \sum_{n=0}^{\infty} \frac{c_n}{z^n}$$
$$= Z[(c_n)](z) = Z[(a_n)*(b_n)](z),$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ for all $n \in \mathbb{N} \cup \{0\}$. □

Corollary 4.7.4. $Z^{-1}[Z(a_n)z(b_n)] = (a_n)*(b_n)$.

Thus, it follows from Convolution Theorem that if f and g are Z- transforms of (a_n) and (b_n), then $Z^{-1}[fg] = (a_n)*(b_n)$.

Remark 4.7.5. We know that, if $|r| < 1$ then $\sum_{k=1}^{n} a_k = a_1 \left(\frac{1-r^n}{1-r}\right)$. Therefore

$$\begin{aligned}
\sum_{k=0}^{n} a^k b^{n-k} &= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k \\
&= b^n \left(1 + \sum_{k=1}^{n} \left(\frac{a}{b}\right)^k\right) \\
&= b^n \left(1 + \frac{a}{b}\left(\frac{1-\left(\frac{a}{b}\right)^n}{1-\frac{a}{b}}\right)\right) \\
&= b^n \left(1 + \frac{a}{b^n}\frac{(b^n-a^n)}{b-a}\right) \\
&= b^n \left(\frac{b-a+\frac{a}{b^n}(b^n-a^n)}{b-a}\right) \\
&= \frac{b^{n+1}-ab^n+ab^n-a^{n+1}}{b-a} = \frac{b^{n+1}-a^{n+1}}{b-a}.
\end{aligned}$$

Thus,

$$\sum_{k=0}^{n} a^k b^{n-k} = \begin{cases} \left(\frac{b^{n+1}-a^{n+1}}{b-a}\right) & \text{if } a \neq b \\ ((n+1)a^n) & \text{if } a = b. \end{cases}$$

Example 4.7.6. Compute the inverse Z-transform of $\frac{z^2}{(z-a)(z-b)}$.

Solution. Let $(x_n) = (a^n)$ and $(y^n) = (b_n)$. Then $Z[(x_n)](z) = \frac{z}{z-a}$ and $Z[(y_n)](z) = \frac{z}{z-b}$. Now

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-b}\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-b}\right]$$

$$= (a^n) * (b^n) \qquad \left(= \sum_{k=0}^{n} a^k b^{n-k}\right)$$

$$= \begin{cases} \frac{b^{n+1}-a^{n+1}}{b-a} & \text{if } a \neq b \\ (n+1)a^n & \text{if } a = b. \end{cases}$$

□

Example 4.7.7. Compute the inverse Z-transform of $\frac{z^2}{(z-2)(z-3)}$.

Solution. $Z^{-1}\left[\frac{z^2}{(z-2)(z-3)}\right] = (2^n) * (3^n) = (c_n)$. Then

$$c_n = \sum_{k=0}^{\infty} 2^k 3^{n-k}\left(\frac{3-2}{3-2}\right) = 3^{n+1} - 2^{n+1}.$$

□

Examples 4.7.8. 1. Since the Z-transform of (a^n), where $a \in \mathbb{C}$, is $\frac{z}{z-a}$, $Z^{-1}\left[\frac{z}{z-a}\right] = (a^n)$.

2. Now we shall show that $Z^{-1}\left[\frac{z}{(z+5)(z+6)}\right] = ((-1)^n(5^n - 6^n))$.
   We note that $\frac{z}{(z+5)(z+6)} = \frac{z}{z+5} - \frac{z}{z+6}$. Hence

$$\begin{aligned} Z^{-1}\left[\frac{z}{(z+5)(z+6)}\right] &= Z^{-1}\left[\frac{z}{z+5}\right] - Z^{-1}\left[\frac{z}{z+6}\right] \\ &= ((-5)^n) - ((-6)^n) = ((-1)^n(5^n - 6^n)). \end{aligned}$$

3. We shall compute the inverse Z-transform of $\frac{z(z+1)}{(z-1)^2}$.
   We first note that $Z^{-1}\left[\frac{z}{(z-1)^2}\right] = (n)$. Now

$$\begin{aligned} Z^{-1}\left[\frac{z(z+1)}{(z-1)^2}\right] &= Z^{-1}\left[\frac{z}{z-1} + \frac{2z}{(z-1)^2}\right] \\ &= Z^{-1}\left[\frac{z}{z-1}\right] + Z^{-1}\left[\frac{2z}{(z-1)^2}\right] \\ &= (1) + (2n) = (2n+1). \end{aligned}$$

4. Next we find the inverse Z-transform of $\frac{z^2-3z+5}{(z-1)(z+2)}$.
   We notice that $\frac{z^2-3z+5}{(z-1)(z+2)} = \frac{z}{z-1} - \frac{5}{z+2}$. Now

$$\frac{1}{z+2} = \frac{1}{z}\frac{1}{1+\frac{2}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n(2z^{-1})^n = \sum_{n=1}^{\infty}(-1)^{n+1}2^{n-1}z^{-n}.$$

   Therefore $Z^{-1}\left[\frac{1}{z+2}\right] = (a_n)$, where $a_0 = 0$ and $a_n = (-1)^{n+1}2^{n-1}$ for all $n \in \mathbb{N}$. Now

$$Z^{-1}\left[\frac{z^2-3z+5}{(z-1)(z+2)}\right] = Z^{-1}\left[\frac{z}{z-1} - \frac{5}{z+2}\right]$$

$$= Z^{-1}\left[\frac{z}{z-1}\right] - 5Z^{-1}\left[\frac{1}{z+2}\right]$$
$$= (1) - 5(a_n) = (1 - 5a_n).$$

5. We shall find the inverse Z-transform of $\frac{8z-19}{z(z-2)(z-3)}$.
   It can be seen easily that $\frac{8z-19}{z(z-2)(z-3)} = -\frac{19}{6} + \frac{3}{2}\frac{z}{z-2} + \frac{5}{3}\frac{z}{z-3}$. Therefore $Z^{-1}\left[\frac{8z-19}{z(z-2)(z-3)}\right] = -\frac{19}{6}e_0 + \left(\frac{3}{2}2^n + \frac{5}{3}3^n\right)$, where $e_0 = (1,0,0,\ldots)$.
6. $\frac{1}{(z-a)^2}$.
   If $a = 0$, then $Z^{-1}\left[\frac{1}{(z-a)^2}\right] = Z^{-1}\left[\frac{1}{z^2}\right] = e_2$.
   Suppose that $a \neq 0$. If $|z| > |a|$, then
$$\begin{aligned}\frac{1}{(z-a)^2} &= \frac{1}{z^2}\frac{1}{(1-\frac{a}{z})^2} = \frac{1}{z^2}\left(1 + 2\frac{a}{z} + 3\frac{a^2}{z^2} + 4\frac{a^3}{z^3} + \cdots\right) \\ &= \frac{1}{a^2}\left(\frac{a^2}{z^2} + 2\frac{a^3}{z^3} + 3\frac{a^4}{z^4} + 4\frac{a^5}{z^5} + \cdots\right) \\ &= \sum_{n=2}^{\infty}(n-1)\frac{a^{n-2}}{z^n}.\end{aligned}$$
   Thus $Z^{-1}\left[\frac{1}{(z-a)^2}\right] = (y_n)$, where $y_0 = y_1 = 0$ and $y_n = (n-1)a^{n-2}$ for all $n \geq 2$.

4.8 Applications of Z-transform

Example 4.8.1. Solve the difference equation $y_{n+2} - 7y_{n+1} + 12y_n = 0$ subject to $y_0 = 1$ and $y_1 = 2$.

Solution. Let $Y(z)$ be the Z-transform of (y_n). Applying the Z-transform on the given equation $y_{n+2} - 7y_{n+1} + 12y_n = 0$, we get
$$z^2Y(z) - y_0z^2 - y_1z - 7zY(z) + 7y_0z + 12Y(z) = 0.$$
Using given condition, it will become $(z^2 - 7z + 12)Y(z) - z^2 - 2z + 7z = 0$. Further simplification gives $(z^2 - 7z + 12)Y(z) = z^2 - 5z$. Therefore
$$\begin{aligned}Y(z) &= \frac{z^2}{(z-4)(z-3)} - \frac{5z}{(z-4)(z-3)} \\ &= \frac{z^2}{(z-4)(z-3)} - 5\left[\frac{z}{(z-4)} - \frac{z}{(z-3)}\right] \\ &= \frac{z^2}{(z-4)(z-3)} - \frac{5z}{(z-4)} + \frac{5z}{(z-3)}.\end{aligned}$$
Applying the inverse Z-transform, we get
$$\begin{aligned}(y_n) &= (4^{n+1} - 3^{n+1}) - 5\cdot(4^n) + 5\cdot(3^n) \\ &= (4^{n+1} - 3^{n+1} - 5\cdot 4^n + 5\cdot 3^n) \\ &= (-4^n + 2\cdot 3^n).\end{aligned}$$
□

Example 4.8.2. Solve $y_{n+2} - 3y_{n+1} + 2y_n = 0$ subject to $y_0 = -1$, $y_1 = 2$.

Solution. Let $Y(z)$ be the Z-transform of (y_n). Applying Z-transform to the given equation, we get

$$z^2Y(z) - y_0z^2 - y_1z - 3zY(z) + 3y_0z + 2Y(z) = 0,$$
$$\text{i.e. } (z^2 - 3z + 2)Y(z) + z^2 - 2z - 3z = 0.$$

Therefore

$$\begin{aligned} Y(z) &= -\frac{z^2 - 5z}{(z-2)(z-1)} \\ &= -\left(\frac{z^2}{(z-2)(z-1)} - 5\left[\frac{z}{z-2} - \frac{z}{z-1}\right]\right). \end{aligned}$$

Applying the inverse Z-transform, we get

$$\begin{aligned} (y_n) &= -(2^{n+1} - 1^{n+1}) + 5\cdot(2^n) - 5\cdot(1^n) \\ &= -2\cdot(2^n) + (1^n) + 5\cdot(2^n) - 5\cdot(1^n) \\ &= 3\cdot 2^n - 4 = 3\cdot(2^n) - 4\cdot(1^n). \end{aligned}$$

□

Example 4.8.3. Find the $(100)^{\text{th}}$ term of the Fibonacci sequence.

Solution. Here we have to solve the difference equation $y_{n+2} - y_{n+1} - y_n = 0$ subject to $y_0 = 0$ and $y_1 = 1$. Let $Y(z)$ be the Z-transform of (y_n). Applying the Z-transform on the given equation, we get

$$z^2Y(z) - y_0z^2 - y_1z - zY(z) + y_0z - Y(z) = 0,$$
$$\text{i.e. } z^2Y(z) - z - zY(z) - Y(z) = 0,$$
$$\text{i.e. } (z^2 - z - 1)Y(z) - z = 0.$$

Therefore

$$\begin{aligned} Y(z) &= \frac{z}{z^2 - z - 1} \\ &= \frac{z}{\left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right)} \\ &= \frac{z}{(z-w_1)(z-w_2)} \\ &= \frac{1}{w_1 - w_2}\left[\frac{z}{z-w_1} - \frac{z}{z-w_2}\right] \\ &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right). \end{aligned}$$

Take $n = 99$ (this gives the 100^{th} term of the Fibonacci sequence). Then we have

$$y_{99} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{99} - \left(\frac{1-\sqrt{5}}{2}\right)^{99}\right).$$

□

$$Z[(na_n)](z) = -z\frac{d}{dz}Z[(a_n)](z).$$

Proof. We have $Z[(a_n)](z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}$. Then

$$\begin{aligned}\frac{d}{dz}Z[(a_n)](z) &= \sum_{n=1}^{\infty}(-n)\frac{a_n}{z^{n+1}} \\ &= -\frac{1}{z}\sum_{n=1}^{\infty}\frac{na_n}{z^n} = -\frac{1}{z}Z[(na_n)](z).\end{aligned}$$

□

Example 4.8.4.

1. $$\begin{aligned}Z[(na^n)](z) &= -z\frac{d}{dz}Z[(a^n)](z) \\ &= -z\frac{d}{dz}\frac{z}{z-a} \\ &= -z\frac{z-a-z}{(z-a)^2} = \frac{az}{(z-a)^2}.\end{aligned}$$
2. $$\begin{aligned}Z[(n^2a^n)](z) &= -z\frac{d}{dz}Z[(a^n)](z) \\ &= -z\frac{d}{dz}\left[\frac{az}{(z-a)^2}\right](z) \\ &= -z\frac{(z-a)^2a-2az(z-a)}{(z-a)^4} \\ &= -\frac{az(z-a+2)}{(z-a)^3}.\end{aligned}$$

Example 4.8.5. Solve $y_{n+2} - 6y_{n+1} + 5y_n = 0$ subject to $y_0 = 1$ and $y_1 = -2$.

Solution. Let $Y(z)$ be the Z-transform of (y_n). Applying Z-transform to the given equation, we get

$$\begin{aligned}& z^2Y(z) - y_0z^2 - y_1z - 6zY(z) + 6zy_0 + 5Y(z) = 0 \\ \Rightarrow\ & (z^2-6z+5)Y(z) - z^2 + 2z + 6z = 0 \\ \Rightarrow\ & Y(z) = \frac{z^2-8z}{z^2-6z+5} \\ \Rightarrow\ & \frac{Y(z)}{z} = \frac{z-8}{z^2-6z+5} \\ \Rightarrow\ & \frac{Y(z)}{z} = \frac{A}{z-5} + \frac{B}{z-1}.\end{aligned}$$

Then $(z-8) = A(z-1) + B(z-5)$. Taking $z = 5$, we get $A = -\frac{3}{4}$ and taking $z = 1$, we get $B = \frac{7}{4}$. Hence,

$$\frac{Y(z)}{z} = -\frac{3}{4}\left(\frac{1}{z-5}\right) + \frac{7}{4}\left(\frac{1}{z-1}\right)$$

$$\Rightarrow Y(z) = -\frac{3}{4}\left(\frac{z}{z-5}\right) + \frac{7}{4}\left(\frac{z}{z-1}\right).$$

Applying inverse Z-transform, we get

$$(y_n) = -\frac{3}{4}(5^n) + \frac{7}{4}(1^n).$$

□

Example 4.8.6. Solve $y_{n+2} - y_n = 1$ subject to $y_0 = 1$ and $y_1 = 2$.

Solution. Let $Y(z)$ be the Z-transform of (y_n). Applying Z-transform to the given equation, we get

$$z^2 Y(z) - y_0 z^2 - y_1 z - Y(z) = \frac{z}{z-1}$$
$$\Rightarrow (z^2 - 1)Y(z) = z^2 + 2z + \frac{z}{z-1}$$
$$\Rightarrow Y(z) = \frac{z^2 + 2z}{z^2 - 1} + \frac{z}{(z-1)^2(z+1)}$$
$$\Rightarrow Y(z) = \frac{z^2}{(z-1)(z+1)} + \left[\frac{z}{z-1} - \frac{z}{z+1}\right] + z\left[\frac{-\frac{1}{4}z + \frac{3}{4}}{(z-1)^2} + \frac{\frac{1}{4}}{z+1}\right] \qquad \text{(by partial fraction)}$$
$$\Rightarrow Y(z) = \frac{z^2}{(z-1)(z+1)} + \frac{z}{z-1} - \frac{z}{z+1} - \frac{1}{4}\frac{z^2}{(z-1)^2} + \frac{3}{4}\frac{z}{(z-1)^2} + \frac{1}{4}\frac{z}{z+1}.$$

Applying the inverse Z-transform, we get

$$(y_n) = \left(\frac{1^{n+1} - (-1)^{n+1}}{2}\right) + (1) - ((-1)^n) + \frac{3}{4}(n) - \frac{1}{4}((-1)^n) - \frac{1}{4}(n+1).$$

□

Example 4.8.7. Solve $y_{n+2} - 4y_{n+1} + 3y_n = 5^n$ subject to $y_0 = 1$, $y_1 = 1$.

Solution. Let $Y(z)$ be the Z-transform of (y_n). Applying Z-transform to the given equation, we get

$$z^2 Y(z) - y_0 z^2 - y_1 z - 4zY(z) + 4y_0 z + 3Y(z) = \frac{z}{z-5}$$
$$\Rightarrow (z^2 - 4z + 3)Y(z) = z^2 - 3z + \frac{z}{z-5}.$$

Therefore, applying partial fractions, we get

$$Y(z) = \frac{z^2}{(z-3)(z-1)} - \frac{3}{2}\left[\frac{z}{z-3} - \frac{z}{z-1}\right] + z\left[\frac{1}{8}\frac{1}{z-5} - \frac{5}{16}\frac{1}{z-1} + \frac{1}{4}\frac{1}{z-1}\right].$$

Applying inverse Z-transform, we get

$$(y_n) = (3^{n+1} - 1^{n+1}) - \frac{3}{2}(3^n) + \frac{3}{2}(3^n) + \frac{3}{2}(1^n) + \frac{1}{8}(5^n) - \frac{5}{16}(1^n) + \frac{1}{4}(1^n).$$

□

Examples 4.8.8. We shall solve the following difference equations.

1. $y_{n+2}+5y_{n+1}+4y_n=2^n$ subject to $y_0=1$ and $y_1=-4$.
   Let Y be the Z- transform of (y_n). Applying Z- transform to above equation we get $z^2Y(z)-z^2y_0-zy_1+5zY(z)-5zy_0+4Y(z)=\frac{z}{z-2}$. Substituting $y_0=1$, $y_1=-4$ and simplifying, we get $Y(z)=\frac{19}{18}\frac{z}{z+4}+\frac{1}{18}\frac{z}{z-2}-\frac{1}{9}\frac{z}{z+1}$. Applying inverse Z- transform,
$$(y_n)=\left(\frac{19}{18}(-4)^n+\frac{1}{18}2^n-\frac{1}{9}(-1)^n\right).$$
2. $y_{n+2}-y_n=1$ subject to $y_0=1$ and $y_1=2$.
   Let Y be the Z- transform of (y_n). Applying Z- transform to $y_{n+2}-y_n=1$ and using given constraints, one obtains $(z^2-1)Y(z)-z^2-2z=\frac{z}{z-1}$. Further simplification gives $Y(z)=\frac{5}{4}\frac{z}{z-1}-\frac{1}{4}\frac{z}{z+1}+\frac{1}{2}\frac{z}{(z-1)^2}$. An application of inverse Z- transform give $(y_n)=(\frac{5}{4}-\frac{1}{4}(-1)^n+\frac{n}{2})$.
3. $y_{n+1}-ay_n=a^n$ subject to $y_0=x_0$.
   If $a=0$, then $y_{n+1}=0$ for all $n\in\mathbb{N}\cup\{0\}$. Thus $(y_n)=x_0e_0$.
   Let $a\neq 0$. Then applying Z- transform, one has $zY(z)-zx_0-aY(z)=\frac{z}{z-a}$, i.e., $Y(z)=x_0\frac{z}{z-a}+\frac{1}{z}\frac{z}{(z-a)^2}$. When we apply inverse transform, we get $(y_n)=(x_0a^n+na^{n-1})$.
4. $y_{n+1}-y_n=a(1-y_n)$ subject to $y_0=x_0$. If $a=0$. Then $y_{n+1}=y_n$ for all $n\in\mathbb{N}\cup\{0\}$ and $y_0=x_0$ implies that (y_n) is a constant sequence (x_0).
   Suppose that $a\neq 0$. Then applying Z- transform, we have $zY(z)-zx_0+(a-1)Y(z)=a$.
5. $y_{n+2}-y_{n+1}-6y_n=0$ subject to $y_0=0$ and $y_1=3$.
   An application of Z- transform on the above sequence gives $z^2Y(z)-3z-zY(z)-6Y(z)=0$. Therefore $Y(z)=\frac{3}{5}\frac{z}{z-3}-\frac{3}{5}\frac{z}{z+2}$. Applying inverse Z- transform, one obtains $(y_n)=\left(\frac{3}{5}(3^n-(-2)^n)\right)$.
6. $y_{n+2}+4y_{n+1}+3y_n=0$ subject to $y_0=y_1=1$.
   Applying Z- transform to the sequence and using given conditions, we get $z^2Y(z)-z^2-z+4zY(z)-4z+zY(z)=0$. Therefore $Y(z)=\frac{2z}{z+1}-\frac{z}{z+3}$. Hence $(y_n)=(2(-1)^n-(-3)^n)$.

INDEX

N

O

P

W