Lecture notes on

Complex Analysis II

PS04CMTH51

$V(\gamma)$

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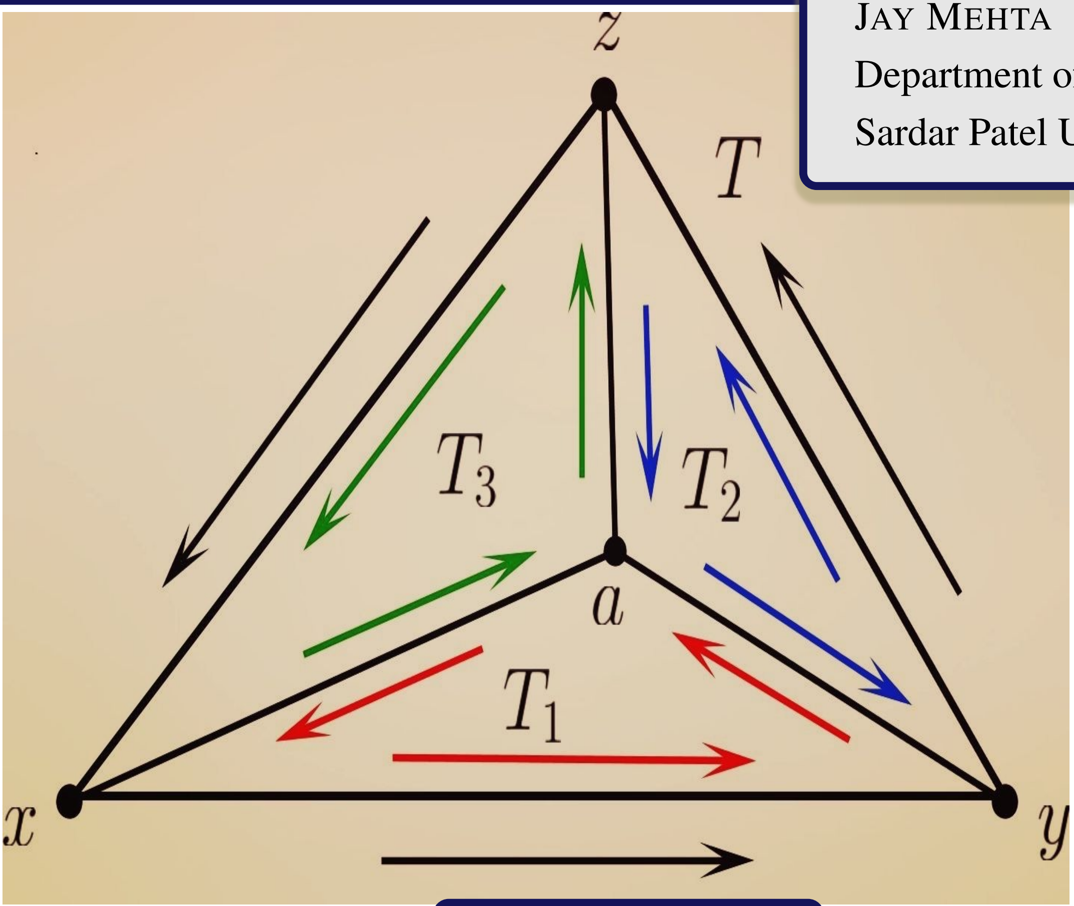

$F' = f$

$C(G,\Omega)$

$H(G)$

Semester - IV
2022-23

Contents

Preface and Acknowledgments

These lecture note is the outcome of the course “Complex Analysis II” (PS04CMTH51) new course offered to the M.Sc. (Semester - IV) students at Department of Mathematics, Sardar Patel University, 2022-23. The lecture notes on “Complex Analysis - II” by Dr. P. A. Dabhi’s are excellent and rich in solved exercises and were of great help while relearning and teaching this course. My notes, on the other hand, are strictly cohered to the syllabus and just consists of a more simplified version of the proofs of the results by Conway explained with more details. I have kept the numbering of theorems, definitions, etc. same as in the book by Conway so as to easily relate and refer. For ease of understanding the book, these notes includes all the detailed arguments and reasoning of the proofs of Conway along with some pictorial demonstrations wherever applicable.

These are aimed to provide a reading material to the students for their easy understanding, in addition to the references mentioned in the university syllabus, so as to save time of the teacher and the students in writing on the board and copying in the notebooks, respectively. The exercises and proofs left as exercise for the readers in the textbook of Conway are left as exercises in this notes as well for the students. I strongly encourage the student readers to solve these exercise. I also strongly recommend Dabhi sir’s lecture notes for better depth in the subject, specially to the problem solvers.

There may be a few errors/typos in this reading material. The students and interested readers are welcomed to give their valuable suggestions, comments or point out errors, if and whenever they find any. Hopefully there will be improvement and enhancement of the notes in the subsequent repetitions of the course in future.

Acknowledgment

I am indebted to Dr. P. A. Dabhi for providing his old course lecture notes, for all the helpful discussions, and his constant encouragement that created an enthusiasm in teaching this course. Thanks are due to Prof. Robert “Bob” Gardner and many other professors for selflessly putting out their lecture notes online that helped me a lot in better understanding of the proofs. I also thank Prof. D. J. Karia who always promptly responded and helped me solve my doubts whenever I had difficulties before giving the lecture.

Finally, I would like to thank my students for their implicit and explicit encouragement, their kind feedback, their interest and discussions, etc. that motivated me to write these notes. I would like to give a mention to my students Ms. Shweta Gohil, Ms. Shreya Chauhan, and Mr. Praharsh Patel (2018-20 batch) for volunteering to present a few results in the class. I would also like to thank my students Mr. Ameerraja Ansari, Ms. Shreya (2018-20 batch), and many other students for their interest in the subject and discussions during the course. I thank Harshit Ratandhara and his batchmates (2019-21 batch) for pointing out typos and errors in the lecture notes, most of which are corrected for the revised version of old course (PS04CMTH21) for 2021-22 batch.

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Syllabus

PS04CMTH51: Complex Analysis - II

Unit I: Riemann Stieltjes integral: a function of bounded variation on $[a,b]$, its total variation, rectifiable curve, smooth curve, piecewise smooth curve, polygonal path, integral of a continuous function on $[a,b]$ with respect to a function of bounded variation and its properties, integral of continuous function defined on $\{\gamma\}$ with respect to γ and its properties, zeros of an analytic function, multiplicity of zero of an analytic function, the index of a closed curve and its properties.

Unit II: Cauchy's Theorem (First version), Cauchy's Integral Formula (First and Second Version), Cauchy's Integral formula for derivatives, Morera's Theorem, existence of a primitive on simply connected region, characterization of non-vanishing analytic function on simply connected region, Counting zero principle and open mapping theorem, Classification of singularities namely removable singularity, pole and essential singularity, order of a pole, Casorati-Weierstrass theorem.

Unit III: Argument Principle, its generalization and examples, Rouche's theorem and deduction of Fundamental Theorem of Algebra, Maximum Modulus principle (statements only), Schwarz's lemma, its applications and consequences, Mobius transformation φ_a and its properties, the space of continuous functions $C(G,\Omega)$, the topology on $C(G,\Omega)$, normal family in $C(G,\Omega)$, equicontinuity of a family in $C(G,\Omega)$, Arzela Ascoli theorem in $C(G,\Omega)$.

Unit IV: The space $H(G)$ of analytic functions, locally bounded family in $H(G)$, Hurwitz's therorem, Montel's theorem, infinite product, convergence and absolute convergence of infinite product, convergence of infinite product of elements in $H(G)$, elementary factors and its properties, The Weierstrass Factorization Theorem, factorization of sin, cos, sinh and cosh, Walli's formula.

Reference Book

1. J. B. Conway, Functions of One Complex Variable, 2nd Edition, Narosa, New Delhi, 1978.

2. W. Rudin, Real and Complex Analysis, McGraw Hill, 1967.

3. R. Narasimhan and Y. Nievergelt, Complex Analysis in One Variable, Birkhauser, 2001.

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Unit 1

Chapter IV

Complex Integration

§1. Riemann-Stieltjes integrals

1.1 Definition. A function $\gamma : [a,b] \to \mathbb{C}$ is said to be of *bounded variation* if there is a constant $M > 0$ such that for any partition $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$ of $[a,b]$

$$v(\gamma;P) = \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \leq M.$$

If γ is of bounded variation, then the *total variation* of γ is defined by

$$V(\gamma) = \sup\{v(\gamma;P) : P \text{ a partition of } [a,b]\}.$$

Clearly, $V(\gamma) \leq M < \infty$.

Exercise. Show that γ is of bounded variation if and only if $\mathrm{Re}\gamma$ and $\mathrm{Im}\gamma$ are of bounded variation.

Exercise. If γ is real valued and is non-decreasing then γ is of bounded variation and $V(\gamma) = \gamma(b) - \gamma(a)$ (see Exercise 1 in book).

Let us see some properties of the functions of bounded variation that can be deduced easily.

1.2 Proposition. *Let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation. Then:*

(a) If P and Q are partitions of $[a,b]$ and $P \subset Q$ then $v(\gamma;P) \leq v(\gamma;Q)$;

(b) If $\sigma : [a,b] \to \mathbb{C}$ is also of bounded variation and $\alpha, \beta \in \mathbb{C}$ then $\alpha\gamma + \beta\sigma$ is of bounded variation and $V(\alpha\gamma + \beta\sigma) \leq |\alpha| V(\gamma) + |\beta| V(\sigma)$.

Proof. Exercise □

Definition. A function $\gamma : [a,b] \to \mathbb{C}$ is called *smooth* if γ' is continuous on $[a.b]$.

γ is called *piecewise smooth* if there are points $a = t_0 < t_1 < \cdots < t_m = b$ such that γ is smooth on each interval (t_k, t_{k+1}).

Exercise. If $\gamma : [a,b] \to \mathbb{C}$ is of bounded variation and if $a \leq s_1 < s_2 < s_3 \leq b$, then $V_{s_1}^{s_2}(\gamma) + V_{s_2}^{s_3}(\gamma) = V_{s_1}^{s_3}(\gamma)$.

The following proposition gives a wealthy subcollection of the functions of bounded variation.

1.3 Proposition. *If $\gamma : [a,b] \to \mathbb{C}$ is piecewise smooth then γ is of bounded variation and*

$$V(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

Proof. First assume that γ is smooth on $[a.b]$. Let $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$ be any partition of $[a,b]$. Then

$$\begin{aligned} v(\gamma;P) &= \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \\ &= \sum_{k=1}^{m} \left| \int_{t_{k-1}}^{t_k} \gamma'(t)\,dt \right| && (\because \gamma \text{ is smooth \& by FTC}) \\ &\leq \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} |\gamma'(t)|\,dt \\ &= \int_a^b |\gamma'(t)|\,dt. \end{aligned}$$

Recall FTC

Fundamental Theorem of Calculus. Let $g : [a,b] \to \mathbb{C}$ be differentiable, then

$$\int_a^b g'(t)\,dt = g(b) - g(a).$$

Hence, γ is of bounded variation. Taking supremum over all such partitions P of $[a,b]$, we get

$$V(\gamma) \leq \int_a^b |\gamma'(t)|dt.$$

Since γ' is continuous on $[a,b]$ and $[a,b]$ is compact, γ' is uniformly continuous on $[a,b]$. Then given $\varepsilon > 0$ there is $\delta_1 > 0$ such that $|s-t| < \delta_1$ implies $|\gamma'(s) - \gamma'(t)| < \varepsilon$ for all $s,t \in [a,b]$. Since γ' is integrable, by the definition of integral, there is $\delta_2 > 0$ such that if $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$ and $\|P\| = \max\{(t_k - t_{k-1}) : 1 \leq k \leq m\} < \delta_2$ then

$$\left| \int_a^b |\gamma'(t)|dt - \sum_{k=1}^m |\gamma'(\tau_k)|(t_k - t_{k-1}) \right| < \varepsilon,$$

where τ_k is any point in $[t_{k-1}, t_k]$. Hence

$$\begin{aligned}
\int_a^b |\gamma'(t)|\,dt &< \varepsilon + \sum_{k=1}^m |\gamma'(\tau_k)|(t_k - t_{k-1}) \\
&= \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(\tau_k)\,dt \right| && (\because \gamma'(\tau_k) \text{ is constant}) \\
&= \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} [\gamma'(\tau_k) - \gamma'(t) + \gamma'(t)]\,dt \right| && (t \in [t_{k-1}, t_k]) \\
&\leq \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} [\gamma'(\tau_k) - \gamma'(t)]\,dt \right| + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(t)\,dt \right|.
\end{aligned}$$

Take $\delta = \min\{\delta_1, \delta_2\}$. If $\|P\| < \delta$ then $|\gamma'(\tau_k) - \gamma'(t)| < \varepsilon$ for $t \in [t_{k-1}, r_k]$ and

$$\begin{aligned}
\int_a^b |\gamma'(t)|\,dt &\leq \varepsilon + \varepsilon(b-a) + \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})| \\
&= \varepsilon + \varepsilon(b-a) + v(\gamma; P) \\
&\leq \varepsilon[1 + (b-a)] + V(\gamma).
\end{aligned}$$

Let $\varepsilon \to 0+$ gives

$$\int_a^b |\gamma'(t)|\,dt \le V(\gamma)$$

which gives the equality.

Now, if γ is piecewise smooth then there are points $a = c_0 < c_1 < \cdots < c_n = b$ such that γ is smooth on each of (c_{k-1}, c_k). Then

$$\begin{aligned}\int_a^b |\gamma'(t)|\,dt &= \sum_{k=1}^{n} \int_{c_{k-1}}^{c_k} |\gamma'(t)|\,dt \\ &= \sum_{k=1}^{n} V_{c_{k-1}}^{c_k}(\gamma) \\ &= V_a^b(\gamma).\end{aligned}$$

□

1.4 Proposition. *Let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation and suppose that $f : [a,b] \to \mathbb{C}$ is continuous. Then there is a complex number I such that for every $\varepsilon > 0$ there is $\delta > 0$ such that when $P = \{t_0 < t_1 < \cdots < t_m\}$ is a partition of $[a,b]$ with $\|P\| = \max\{(t_k - t_{k-1}) : 1 \le k \le m\} < \delta$ then*

$$\left| I - \sum_{k=1}^{m} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \right| < \varepsilon$$

for whatever choice of points τ_k, $t_{k-1} \le \tau_k \le t_k$.

Definition. This number I is called the *integral of f* or the *Riemann-Stieltjes integral of f* with respect to γ over $[a,b]$ and is denoted by

$$I = \int_a^b f\,d\gamma = \int_a^b f(t)\,d\gamma(t).$$

Proof. Since f is continuous on compact set $[a,b]$, it is uniformly continuous. Then for every $\varepsilon = \frac{1}{m}$ ($m \in \mathbb{N}$), we can (inductively) find $\delta_1 > \delta_2 > \delta_3 > \cdots$ such that if $|s-t| < \delta_m$ then $|f(s) - f(t)| < \frac{1}{m}$. For each $m \ge 1$ let $\mathscr{P}_m$ be the set of all partitions P of $[a,b]$ with $\|P\| < \delta_m$. So $\mathscr{P}_1 \supset \mathscr{P}_2 \supset \mathscr{P}_3 \supset \cdots$. Finally define F_m to be the closure of the set

1.5 $$\left\{ \sum_{n=1}^{m} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] : P \in \mathscr{P}_m \text{ and } t_{k-1} \le \tau_k \le t_k \right\}.$$

The following are claimed to hold:

1.6 $$\begin{cases} F_1 \supset F_2 \supset \cdots \text{ and} \\ \operatorname{diam} F_m \le \frac{2}{m} V(\gamma). \end{cases}$$

The fact that $F_1 \supset F_2 \supset \cdots$ follows trivially from the fact that $\mathscr{P}_1 \supset \mathscr{P}_2 \supset \cdots$. Since the diameter of a set and its closure is same, for the second claim it suffices to show the diameter of the set in (1.5) is $\leq \frac{2}{m}V(\gamma)$.

For any partition $P = \{t_0 < t_1 < \cdots < t_m\}$ denote by $S(P)$, the sum of the form $\sum_{k=1}^{m} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})]$ where τ_k is any point with $t_{k-1} \leq \tau_k \leq t_k$. Fix $m \geq 1$ and let $P \in \mathscr{P}_m$. If $P \subset Q$, then $\|Q\| \leq \|P\|$. Then $Q \in \mathscr{P}_m$ ($\because \|Q\| \leq \|P\| < \delta_m$). First we prove that if $P \subset Q$ then $|S(P) - S(Q)| \leq \frac{1}{m}V(\gamma)$. It is sufficient to prove for partition Q obtained from P by adding one extra point to P.

For $1 \leq p \leq m$, let $t_{p-1} < t^* < t_p$ and $Q = P \cup \{t^*\}$. If $t_{p-1} \leq \sigma \leq t^*$ and $t^* \leq \sigma' \leq t_p$ and $t_{k-1} \leq \tau_k, \sigma_k \leq t_k$, $k \neq p$, then

$$\begin{aligned}
|S(P) - S(Q)| &= \Bigg|\sum f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] - \Bigg(\sum_{k\neq p} f(\sigma_k)[\gamma(t_k) - \gamma(t_{k-1})] \\
&\quad + f(\sigma)[\gamma(t^*) - \gamma(t_{p-1})] + f(\sigma')[\gamma(t_p) - \gamma(t^*)]\Bigg)\Bigg| \\
&= \Bigg|\sum_{k\neq p} [f(\tau_k) - f(\sigma_k)][\gamma(t_k) - \gamma(t_{k-1})] + f(\tau_p)[\gamma(t_p) - \gamma(t_{p-1})] \\
&\quad - f(\sigma)[\gamma(t^*) - \gamma(t_{p-1})] - f(\sigma')[\gamma(t_p) - \gamma(t^*)]\Bigg| \\
&\leq \sum_{k\neq p} |f(\tau_k) - f(\sigma_k)||\gamma(t_k) - \gamma(t_{k-1})| + |f(\tau_p) - f(\sigma)||\gamma(t^*) - \gamma(t_{p-1})| \\
&\quad + |f(\tau_p) - f(\sigma')||\gamma(t_p) - \gamma(t^*)| \\
&< \frac{1}{m}\sum_{k\neq p} |\gamma(t_k) - \gamma(t_{k-1})| + \frac{1}{m}|\gamma(t^*) - \gamma(t_{p-1})| + \frac{1}{m}|\gamma(t_p) - \gamma(t^*)| \\
&\qquad \left(\because |\tau_k - \sigma_k| < \delta_m \Rightarrow |f(\tau_k) - f(\sigma_k)| < \frac{1}{m}\right) \\
&\leq \frac{1}{m}V(\gamma).
\end{aligned}$$

Let P and R be any two partitions in $\mathscr{P}_m$. Then $Q = P \cup R$ is a partition that contains both P and R. Then from above argument,

$$\begin{aligned}
|S(P) - S(R)| &\leq |S(P) - S(Q)| + |S(Q) - S(R)| \\
&\leq \frac{2}{m}V(\gamma) \qquad\qquad (\because P \subset Q, R \subset Q).
\end{aligned}$$

Thus, $\{F_m\}$ is a decreasing sequence of closed sets and $\operatorname{diam} F_m \to 0$ as $m \to \infty$. Then by Cantor's Intersection Theorem (II.3.7), $\bigcap_{m=1}^{\infty} F_m = \{I\}$ for some $I \in \mathbb{C}$. That is, $I \in F_m, \forall\, m$.

Recall

3.7 Cantor's Theorem (Chapter II, page no. 19 in Conway). *A metric space (X,d) is complete if and only if for any sequence $\{F_n\}$ of non-empty closed sets with $F_1 \supset F_2 \supset \cdots$ and* $\operatorname{diam} F_n \to 0$, $\bigcap_{n=1}^{\infty} F_n$ *consists of a single point.*

If $\varepsilon > 0$ is given then take $M \in \mathbb{N}$ such that $M > \frac{2}{\varepsilon}V(\gamma)$. Then $\varepsilon > \frac{2}{M}V(\gamma) \geq \operatorname{diam} F_M$. So $F_M \subset B(I,\varepsilon)$. Choosing $\delta = \delta_M$, we get the result. □

The next result follows from routine $\varepsilon - \delta$ argument, the proof of which is left as an exercise.

1.7 Proposition. *Let f and g be continuous functions on $[a,b]$ and γ and σ be functions of bounded variation on $[a,b]$. Then for any scalars α and β:*

(a) $\int_a^b (\alpha f + \beta g)\, d\gamma = \alpha \int_a^b f\, d\gamma + \beta \int_a^b g\, d\gamma$

(b) $\int_a^b f\, d(\alpha\gamma + \beta\sigma) = \alpha \int_a^b f\, d\gamma + \beta \int_a^b f\, d\sigma$.

Proof. Exercise □

The following result is a useful tool in computing these integrals.

1.8 Proposition. *Let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation and let $f : [a,b] \to \mathbb{C}$ be continuous. If $a = t_0 < t_1 < \cdots < t_n = b$ then*

$$\int_a^b f\, d\gamma = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f\, d\gamma.$$

Proof. Exercise □

If γ is piecewise smooth and f is continuous then the following theorem says that $\int f\, d\gamma$ can be found by the methods of integration learned in calculus.

1.9 Theorem. *If γ is piecewise smooth and $f : [a,b] \to \mathbb{C}$ is continuous then*

$$\int_a^b f\, d\gamma = \int_a^b f(t)\gamma'(t)\, dt.$$

Proof. First we assume that γ is smooth. Since $\gamma = \gamma_1 + i\gamma_2$ and $\int_a^b f\, d\gamma = \int_a^b f\, d\gamma_1 + i \int_a^b f\, d\gamma_2$, it is sufficient to prove the result for real valued γ, i.e., $\gamma([a,b]) \subset \mathbb{R}$.

Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that if $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$ is a partition of $[a,b]$ with $\|P\| < \delta$ then

1.10
$$\left| \int_a^b f\, d\gamma - \sum_{k=1}^{n} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \right| < \tfrac{1}{2}\varepsilon$$

and

1.11 $$\left|\int_a^b f(t)\gamma'(t)\,dt - \sum_{k=1}^{m} f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1})\right| < \tfrac{1}{2}\varepsilon$$

for any choice of τ_k in $[t_{k-1}, t_k]$. By applying Mean Value Theorem for derivatives, we get that there is $\tau_k \in [t_{k-1}, t_k]$ such that

(∗) $$\gamma'(\tau_k) = \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}}.$$

Now,

$$\begin{aligned}
&\left|\int_a^b f\,d\gamma - \int_a^b f(t)\gamma'(t)\,dt\right| \\
&= \left|\int_a^b f\,d\gamma - \sum_{k=1}^{m} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] + \sum_{k=1}^{m} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] - \int_a^b f(t)\gamma'(t)\,dt\right| \\
&\le \left|\int_a^b f\,d\gamma - \sum_{k=1}^{m} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})]\right| + \left|\int_a^b f(t)\gamma'(t)\,dt - \sum_{k=1}^{m} f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1})\right| \quad \text{(by (}*\text{))} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{(by (1.10) \& (1.11)).}
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we get

$$\int_a^b f\,d\gamma = \int_a^b f(t)\gamma'(t)\,dt.$$

□

Definition. A *path* is a continuous function $\gamma : [a,b] \to \mathbb{C}$.

If $\gamma : [a.b] \to \mathbb{C}$ is a path then the set $\{\gamma(t) : a \le t \le b\}$ is called the *trace of* γ and is denoted by $\{\gamma\}$. Note that since γ is continuous on compact set $[a,b]$, the trace of γ is a compact (and bounded) subset of $\mathbb{C}$.

γ is said to be *rectifiable path* if γ is a function of bounded variation. If P is a partition of $[a,b]$ then $v(\gamma;P)$ is exactly the sum of the length of the line-segment connecting points on the trace of γ. To say that γ is rectifiable is to say that γ has finite length and its length is $V(\gamma)$. In particular, if γ is piecewise smooth then γ is rectifiable and its length is $\int_a^b |\gamma'(t)|\,dt$.

If $\gamma : [a,b] \to \mathbb{C}$ is a rectifiable path with $\{\gamma\} \subset E \subset \mathbb{C}$ and $f : E \to \mathbb{C}$ is continuous then $f \circ \gamma : [a,b] \to \mathbb{C}$ is a continuous function on $[a,b]$. With this in mind, the following definition makes sense.

1.12 Definition. If $\gamma : [a,b] \to \mathbb{C}$ is a rectifiable path and f is a function defined and continuous on the trace of γ then the (line) integral of f along γ is

$$\int_a^b f(\gamma(t))\, d\gamma(t).$$

This line integral is also denoted by $\int_\gamma f = \int_\gamma f(z)\, dz$.

Examples.

1. Let $\gamma : [0,2\pi] \to \mathbb{C}$ to be $\gamma(t) = e^{it}$ and define $f(z) = \frac{1}{z}$, $z \neq 0$.
   Since γ is smooth,

$$\begin{aligned}
\int_\gamma f(z)\, dz &= \int_0^{2\pi} f(\gamma(t))\, d\gamma(t) \\
&= \int_0^{2\pi} f(\gamma(t))\gamma'(t)\, dt \\
&= \int_0^{2\pi} \frac{1}{e^{it}} ie^{it}\, dt \\
&= \int_0^{2\pi} i\, dt = 2\pi i.
\end{aligned}$$

2. Let γ be same as above and $f(z) = z^m$ for $m \in \mathbb{N} \cup \{0\}$.

$$\begin{aligned}
\int_\gamma f(z)\, dz &= \int_0^{2\pi} f(\gamma(t))\, d\gamma(t) \\
&= \int_0^{2\pi} f(\gamma(t))\gamma'(t)\, dt \\
&= \int_0^{2\pi} e^{imt}(ie^{it})\, dt \\
&= i\int_0^{2\pi} e^{i(m+1)t}\, dt \\
&= i\int_0^{2\pi} \cos(m+1)t\, dt - \int_0^{2\pi} \sin(m+1)t\, dt \\
&= 0.
\end{aligned}$$

3. Let $a, b \in \mathbb{C}$, $\gamma : [0,1] \to \mathbb{C}$ be defined by $\gamma(t) = tb + (1-t)a$ and $f(z) = z^n$, $n \geq 0$.

$$\begin{aligned}
\int_\gamma f(z)\, dz &= \int_0^1 f(\gamma(t))\, d\gamma(t) \\
&= \int_0^1 f(\gamma(t))\gamma'(t)\, dt \\
&= \int_0^1 [tb + (1-t)a]^n (b-a)\, dt
\end{aligned}$$

$$= (b-a)\left[\frac{[tb+(1-t)a]^{n+1}}{(n+1)}\cdot\frac{1}{(b-a)}\right]_0^1$$
$$= \frac{b^{n+1}-a^{n+1}}{n+1}.$$

Remark. Let $\gamma : [a,b] \to \mathbb{C}$ be a rectifiable path and $\varphi : [c,d] \to \mathbb{C}$ be a continuous non-decreasing function whose image is all of $[a,b]$, $\varphi(c) = a$, and $\varphi(d) = b$. Then $\gamma \circ \varphi : [c,d] \to \mathbb{C}$ is a path with the same trace as that of $\{\gamma\}$. Also, $\gamma \circ \varphi$ is rectifiable because if $\{c = s_0 < s_1 < \cdots < s_n = d\}$ is a partition of $[c,d]$, then since φ is non-decreasing (and onto), $\{a = \varphi(c) = \varphi(s_0) \le \varphi(s_1) \le \cdots \le \varphi(s_n) = \varphi(d) = b\}$ is a partition of $[a,b]$. Hence

$$\sum_{k=1}^{n} |\gamma(\varphi(s_k)) - \gamma(\varphi(s_{k-1}))| \le V(\gamma)$$
$$\Rightarrow \sum_{k=1}^{n} |(\gamma\circ\varphi)(s_k) - (\gamma\circ\varphi)(s_{k-1})| \le V(\gamma)$$
$$\Rightarrow V(\gamma\circ\varphi) \le V(\gamma).$$

So if f is continuous on $\{\gamma\} = \{\gamma\circ\varphi\}$, then $\int_{\gamma\circ\varphi} f$ is well defined.

1.13 Proposition. *If $\gamma : [a,b] \to \mathbb{C}$ is a rectifiable path and $\varphi : [c,d] \to [a,b]$ is a continuous non-decreasing function with $\varphi(c) = a$, $\varphi(d) = b$; then for any function f continuous on $\{\gamma\}$*

$$\int_\gamma f = \int_{\gamma\circ\varphi} f.$$

Proof. Let $\varepsilon > 0$ be given. Then there is $\delta_1 > 0$ such that if $\{c = s_0 < s_1 < \cdots < s_n = d\}$ is a partition of $[c,d]$ with $|s_k - s_{k-1}| < \delta_1$ for $1 \le k \le n$, and $s_{k-1} \le \sigma_k \le s_k$ we have

1.14 $$\left|\int_{\gamma\circ\varphi} f - \sum_{k=1}^{n} f(\gamma\circ\varphi(\sigma_k))[\gamma\circ\varphi(s_k) - \gamma\circ\varphi(s_{k-1})]\right| < \tfrac{1}{2}\varepsilon.$$

Similarly there is $\delta_2 > 0$ such that if $\{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a,b]$ with $(t_k - t_{k-1}) < \delta_2$ and $t_{k-1} \le \tau_k \le t_k$ then

1.15 $$\left|\int_\gamma f - \sum_{k=1}^{n} f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})]\right| < \tfrac{1}{2}\varepsilon.$$

Since φ is continuous on compact set $[c,d]$, it is uniformly continuous. So given $\delta_2 > 0$ there is $\delta > 0$ which we can choose to be $< \delta_1$ such that $|\varphi(s) - \varphi(s')| < \delta_2$ whenever $|s - s'| < \delta$.

If $\{s_0 < s_1 < \cdots < s_n\}$ is a partition of $[c,d]$ with $s_k - s_{k-1} < \delta < \delta_1$ and $t_k = \varphi(s_k)$, then $\{t_0 \le t_1 \le \cdots \le t_n\}$ is a partition of $[a,b]$ with $t_k - t_{k-1} = \varphi(s_k) - \varphi(s_{k-1}) < \delta_2$. If

$s_{k-1} \leq \sigma_k \leq s_k$ and $\tau_k = \varphi(\sigma_k)$ then both the above inequalities hold and we have

$$\begin{aligned}\left|\int_\gamma f - \int_{\gamma\circ\varphi} f\right| &= \left|\int_\gamma f - \sum_{k=1}^n f(\gamma(\tau_k))[\gamma(t_k)-\gamma(t_{k-1})] + \sum_{k=1}^n f(\gamma(\tau_k))[\gamma(t_k)-\gamma(t_{k-1})] - \int_{\gamma\circ\varphi} f\right| \\ &\leq \left|\int_\gamma f - \sum_{k=1}^n f(\gamma(\tau_k))[\gamma(t_k)-\gamma(t_{k-1})]\right| \\ &\quad + \left|\sum_{k=1}^n f(\gamma\circ\varphi(\sigma_k))[\gamma\circ\varphi(s_k)-\gamma\circ\varphi(s_{k-1})] - \int_{\gamma\circ\varphi} f\right| \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Since ε is arbitrary, the equality holds and the result follows. □

We aim to give an equivalence relation on the collection of rectifiable paths so that each member of an equivalence class has the same trace and hence the line integral of any continuous function on this trace is same for each path in the class, i.e. each member of the class.

We might be tempted to define σ and γ to be equivalent if $\sigma = \gamma\circ\varphi$ for some function φ as above. However, this fails to be an equivalence relation (**Verify**)! The actual equivalence relation is defined in the following way.

1.16 Definition. Let $\sigma : [c,d] \to \mathbb{C}$ and $\gamma : [a,b] \to \mathbb{C}$ be rectifiable paths. The path σ is *equivalent* to γ if there is a function $\varphi : [c,d] \to [a,b]$ which is continuous, strictly increasing, and with $\varphi(c) = a$, $\varphi(d) = b$; such that $\sigma = \gamma\circ\varphi$. We call the function φ, a *change of parameter*.

A *curve* is an equivalence class of paths. The trace of a curve is the trace of one of its members in the class. If f is continuous on the trace of the curve, then f is integrable and the integral of f over the curve is the integral of f over any member of the curve (i.e. equivalence class).

A curve is *smooth* (piecewise smooth) if and only if one of its representatives is smooth (piecewise smooth).

From now on we will not make a distinction between a path and a curve.

Let $\gamma : [a,b] \to \mathbb{C}$ be a rectifiable path and for $a \leq t \leq b$ let $|\gamma|(t) = V(\gamma;[a,t])$, i.e.,

$$|\gamma|(t) = \sup\left\{\sum_{k=1}^n |\gamma(t_k)-\gamma(t_{k-1})| : \{t_0 < t_1 < \cdots < t_n\} \text{ is a partition of } [a,t]\right\}.$$

Clearly, $|\gamma|(t)$ is increasing (**Verify!**) and so $|\gamma| : [a,b] \to \mathbb{C}$ is a function of bounded variation (by Exercise 1 stated before). If f is continuous on $\{\gamma\}$ define

$$\int_\gamma f\,|dz| = \int_a^b f(\gamma(t))\,d|\gamma|(t).$$

If γ is rectifiable curve then denote by $-\gamma$ the curve defined by $-\gamma(t) = \gamma(-t)$ for $-b \le t \le -a$. Another notation for this is γ^{-1}. Also if $c \in \mathbb{C}$ then $\gamma + c$ denotes the curve defined by $(\gamma+c)(t) = \gamma(t) + c$. The following proposition gives many basic properties of the line integral.

1.17 Proposition. *Let γ be a rectifiable curve and suppose that f is a function continuous on γ. Then:*

(a) $\int_\gamma f = -\int_{-\gamma} f$;
(b) $\left|\int_\gamma f\right| \le \int_\gamma |f|\,|dz| \le V(\gamma)\sup\left[|f(z)| : z \in \{\gamma\}\right]$.
(c) If $c \in \mathbb{C}$ then $\int_\gamma f(z)\,dz = \int_{\gamma+c} f(z-c)\,dz$.

Proof. Exercise. □

Recall that a function F is a *primitive* of f if $F' = f$. Our aim is to prove the next result which is the analogue of the Fundamental Theorem of Calculus for the line integrals.

1.18 Theorem. *Let G be open in $\mathbb{C}$ and γ be a rectifiable path in G with initial and end points α and β respectively. If $f : G \to \mathbb{C}$ is a continuous function with a primitive $F : G \to \mathbb{C}$, then*

$$\int_\gamma f = F(\beta) - F(\alpha).$$

Before proving the theorem, we prove the following lemma which is needed in the proof of the theorem. Recall that a *polygonal path* is a curve made up of finitely many line-segments joined end to end.

1.19 Lemma. *If G is an open set in $\mathbb{C}$, $\gamma : [a,b] \to G$ is a rectifiable path, and $f : G \to \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path Γ in G such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and $\left|\int_\gamma f - \int_\Gamma f\right| < \varepsilon$.*

Proof. *Case I.* Suppose G is an open disk.

Since $\{\gamma\}$ is compact, $d = \mathrm{dist}(\{\gamma\}, \partial G) > 0$.

Why $d = \mathrm{dist}(\{\gamma\}, \partial G) > 0$?

This follows from the following result.

5.17 Theorem (Chapter II, page no. 28 in Conway). *If A and B are disjoint sets in X with B closed and A compact then $d(A,B) > 0$.*

Note that here $A = \{\gamma\}$ is a compact subset of G and $B = \partial G = \overline{G} \cap \overline{\mathbb{C} \setminus G}$ is closed.

Now, we show that $\{\gamma\}$ and ∂G are disjoint. Let $z \in \{\gamma\} \cap \partial G$. Since $z \in \{\gamma\} \subset G$ (i.e. $z = \gamma(t) \in G$) and G is open, there is $r > 0$ such that $B(z,r) \subset G$, i.e. $B(z,r) \cap (\mathbb{C} \setminus G) = \emptyset$. Therefore, $z \notin \overline{\mathbb{C} \setminus G}$ and hence $z \notin \partial G$ which is a contradiction. Hence $\{\gamma\} \cap \partial G = \emptyset$.

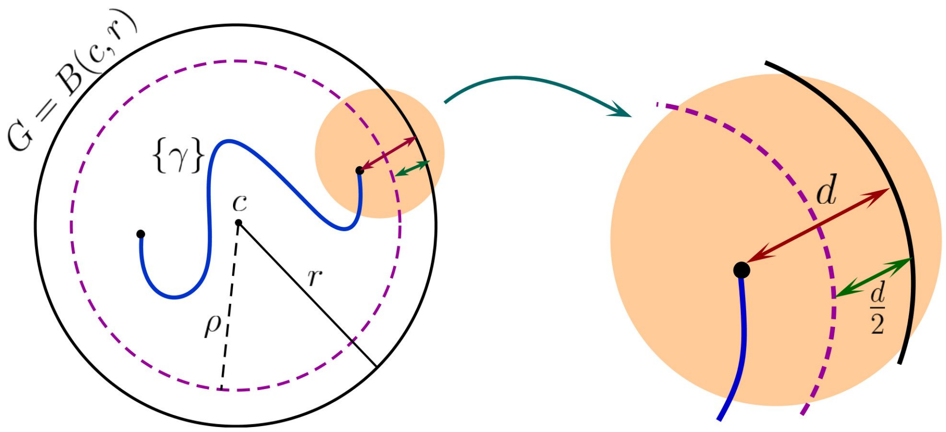


Figure IV.1: Here $d = \text{dist}(\{\gamma\}, \partial G)$ is attained at an endpoint of γ which may not be the case. The figure is for demonstration purpose only.

It follows that if $G = B(c,r)$ for some $c \in \mathbb{C}$, then $\{\gamma\} \subset B(c,\rho)$, where $\rho = r - \frac{1}{2}d$. The reason for considering this smaller disk is that f is uniformly continuous on $\overline{B}(c,\rho) \subset G$ (since continuous functions are uniformly continuous on compact sets). So (if necessary, taking G to be this appropriately smaller disk) without the loss of generality, we may assume that f is uniformly continuous on G. Then given $\varepsilon > 0$ there is $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $|z - w| < \delta$ (for all $z, w \in G$).

Since γ is continuous on(compact set) $[a,b]$, it is uniformly continuous. Therefore there is a partition $\{t_0 < t_1 < \cdots < t_n\}$ of $[a,b]$ such that

1.19a $$|\gamma(s) - \gamma(t)| < \delta$$

if $t_{k-1} \leq s,t \leq t_k$; and such that for $t_{k-1} \leq \tau_k \leq t_k$ we have

1.20 $$\left| \int_\gamma f - \sum_{k=1}^{n} f(\gamma(\tau_k)) \big[\gamma(t_k) - \gamma(t_{k-1})\big] \right| < \varepsilon.$$

Define $\Gamma : [a,b] \to \mathbb{C}$ by

(∗) $$\Gamma(t) = \frac{1}{t_k - t_{k-1}} [(t_k - t)\gamma(t_{k-1}) + (t - t_{k-1})\gamma(t_k)]$$

if $t_{k-1} \leq t \leq t_k$. That is, $\Gamma(t)$ is the line-segment joining $\gamma(t_{k-1})$ and $\gamma(t_k)$. Since all the endpoint $\gamma(t_k)$ $(1 \leq k \leq n)$ are in G and G is convex (being open disk), the polygonal path Γ is in G, and $\Gamma(a) = \Gamma(t_0) = \gamma(t_0) = \gamma(a)$ and similarly $\Gamma(b) = \gamma(b)$. Now for $t_{k-1} \leq \tau_k \leq t_k$,

$$|\Gamma(t) - \gamma(\tau_k)|$$

$$= \left| \frac{1}{t_k - t_{k-1}} \left[(t_k - t)\gamma(t_{k-1}) + (t - t_{k-1})\gamma(t_k) \right] - \gamma(\tau_k) \right|$$
$$= \frac{1}{t_k - t_{k-1}} \left| \left[(t_k - t)\gamma(t_{k-1}) + (t - t_{k-1})\gamma(t_k) \right] - \left[(t_k - t)\gamma(\tau_k) + (t - t_{k-1})\gamma(\tau_k) \right] \right|$$
$$= \frac{1}{t_k - t_{k-1}} \left| (t_k - t) \left[\gamma(t_{k-1}) - \gamma(\tau_k) \right] + (t - t_{k-1}) \left[\gamma(\tau_k) - \gamma(t_k) \right] \right|$$
$$< \frac{1}{t_k - t_{k-1}} \left[(t_k - t)\delta + (t - t_{k-1})\delta \right] = \delta \qquad \text{(by (1.19a))}.$$

Thus,

1.21 $$|\Gamma(t) - \gamma(\tau_k)| < \delta \text{ for } t_{k-1} \le t \le t_k.$$

Since Γ is polygonal, it is piecewise smooth and so $\int_\gamma f = \int_a^b f(\Gamma(t))\Gamma'(t)\,dt$. By $(*)$, putting the value of $\Gamma'(t)$, it follows that

$(**)$ $$\int_\Gamma f = \sum_{k=1}^n \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\Gamma(t))\,dt.$$

Now,

$$\left| \int_\gamma f - \int_\Gamma f \right|$$
$$= \left| \int_\gamma f - \sum_{k=1}^n f(\gamma(\tau_k)) \left[\gamma(t_k) - \gamma(t_{k-1}) \right] + \sum_{k=1}^n f(\gamma(\tau_k)) \left[\gamma(t_k) - \gamma(t_{k-1}) \right] - \int_\Gamma f \right|$$
$$< \varepsilon + \left| \sum_{k=1}^n f(\gamma(\tau_k)) \left[\gamma(t_k) - \gamma(t_{k-1}) \right] - \int_\Gamma f \right| \qquad \text{(by (1.20))}$$
$$= \varepsilon + \left| \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \underbrace{f(\gamma(\tau_k))}_{\text{const. wrt } t}\,dt \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} - \sum_{k=1}^n \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\Gamma(t))\,dt \right| \qquad \text{(by } (**))$$
$$\le \varepsilon + \sum_{k=1}^n \frac{|\gamma(t_k) - \gamma(t_{k-1})|}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} |f(\gamma(\tau_k)) - f(\Gamma(t))|\,dt$$
$$\le \varepsilon + \varepsilon \sum_{k=1}^n |\gamma(t_k) - \gamma(t_{k-1})| \qquad \text{(by (1.21) and uniform continuity of } f)$$
$$\le \varepsilon(1 + V(\gamma)).$$

Case II. G is arbitrary subset of $\mathbb{C}$.

Since $\{\gamma\}$ and ∂G are disjoint, $\{\gamma\}$ is compact and ∂G is closed, by Theorem II.5.17, there is a number r with $0 < r < d(\{\gamma\}, \partial G)$. Since γ is uniformly continuous, (given $r > 0$) choose $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < r$ whenever $|s - t| < \delta$.

If $P = \{t_0 < t_1 < \cdots < t_n\}$ is a partition of $[a,b]$ with $\|P\| < \delta$, then $|\gamma(t) - \gamma(t_{k-1})|$ for $t_{k-1} \leq t \leq t_k$. That is, if $\gamma_k : [t_{k-1}, t_k] \to G$ is defined by $\gamma_k(t) = \gamma(t)$ then $\{\gamma_k\} \subset B(\gamma(t_{k-1}), r)$ for $1 \leq k \leq n$. Then by Case I, there is a polygonal path $\Gamma_k : [t_{k-1}, t_k] \to B(\gamma(t_{k-1}), r)$ such that $\Gamma_k(t_{k-1}) = \gamma_k(t_{k-1}) = \gamma(t_{k-1})$, $\Gamma_k(t_k) = \gamma_k(t_k) = \gamma(t_k)$, and $\left|\int_{\gamma_k} f - \int_{\Gamma_k} f\right| < \frac{\varepsilon}{n}$.

Define $\Gamma : [a,b] \to G$ by $\Gamma(t) = \Gamma_k(t)$ on $[t_{k-1}, t_k]$. Then Γ has the desired properties. □

Proof of Theorem 1.18. Case I. $\gamma : [a,b] \to \mathbb{C}$ is piecewise smooth with initial enpoints α and β respectively (i.e. $\gamma(a) = \alpha$ and $\gamma(b) = \beta$). Then

$$\begin{aligned}
\int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\
&= \int_a^b F'(\gamma(t))\gamma'(t)\,dt && (\because F' = f) \\
&= \int_a^b (F\circ\gamma)'(t)\,dt && \left(\substack{\text{Chain rule} \\ (g\circ f)'(z) = g'(f(z))f'(z)}\right) \\
&= (F\circ\gamma)(b) - (F\circ\gamma)(a) && \text{(by FTC)} \\
&= F(\gamma(b)) - F(\gamma(a)) \\
&= F(\beta) - F(\alpha).
\end{aligned}$$

Case II. The General Case.

Let $\varepsilon > 0$ be given. Then by Lemma 1.19, there is a polygonal path $\Gamma : [a,b] \to \mathbb{C}$ such that $\Gamma(a) = \gamma(a) = \alpha$, $\Gamma(b) = \gamma(b) = \beta$, and

$$\left|\int_\gamma f - \int_\Gamma f\right| < \varepsilon.$$

Since Γ is a polygonal path, it is piecewise smooth. So by Case I,

$$\int_\Gamma f = F(\beta) - F(\alpha).$$

Therefore

$$\left|\int_\gamma f - [F(\beta) - F(\alpha)]\right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\int_\gamma f = F(\beta) - F(\alpha)$. □

Definition. A curve $\gamma : [a.b] \to \mathbb{C}$ is said to be *closed* if $\gamma(a) = \gamma(b)$.

1.22 Corollary. *Let G, γ, and f satisfy the same hypothesis as in Theorem 1.18. If γ is a closed curve then*

$$\int_\gamma f = 0.$$

Proof. Since γ is closed, $\alpha = \gamma(a) = \gamma(b) = \beta$. Then by above theorem,

$$\int_\gamma f = F(\beta) - F(\alpha) = 0.$$

□

Remark. The Fundamental Theorem of Calculus says that each continuous function has a primitive. This is not true in the case of function of complex variable. For example, $f(z) = |z|^2 = x^2 + y^2$ has no primitive.

Suppose, if possible, F is a primitive of f, then F is analytic (by definition of primitive) and $F' = f$. If $F = U + iV$, then $F'(x+iy) = F'(z) = f(z) = |z|^2 = x^2 + y^2$. Now, using the Cauchy-Riemann equations,

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} = x^2 + y^2, \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} = 0.$$

Since $\frac{\partial U}{\partial y} = 0$, it follows that $U(x,y) = u(x)$ for some differentiable function u of x. But then $u'(x) = \frac{\partial U}{\partial x} = x^2 + y^2$ which is a contradiction as $u(x) = U(x,y)$ is a function of x only and independent of y whereas $u'(x) = \frac{\partial U}{\partial x}$ is a function of x and y.

An alternative way to see that $|z|^2$ does not have a primitive is to apply Theorem 1.18 (see Exercise 8).

§3. Zeros of an analytic function

3.1 Definition. If $f : G \to \mathbb{C}$ is analytic and $a \in G$ satisfies $f(a) = 0$, then a is called a *zero of f of multiplicity $m \geq 1$* if there is an analytic function $g : G \to \mathbb{C}$ such that $f(z) = (z-a)^m g(z)$ where $g(a) \neq 0$.

3.2 Definition. An *entire function* is a function which is defined and analytic in the whole complex plane $\mathbb{C}$.

The following is a corollary of the Fundamental Theorem of Algebra. Since it is listed in the section numbering of the syllabus, we include the statement.

3.6 Corollary. *If $p(z)$ is a polynomial and $a_1, \ldots, a_m$ are its zeros with a_j having multiplicity k_j then $p(z) = c(z-a_1)^{k_1} \cdots (z-a_m)^{k_m}$ for some constant c and $k_1 + \cdots + k_m$ is the degree of p.*

The following is an important result in Complex Analysis which is also known as the *Identity Theorem.*

3.7 Theorem. *Let G be a connected open set and let $f : G \to \mathbb{C}$ be an analytic function. Then the following are equivalent statements:*

(a) $f \equiv 0$;
(b) there is a point a in G such that $f^{(n)}(a) = 0$ for each $n \geq 0$;
(c) $\{z \in G : f(z) = 0\}$ *has a limit point in G.*

Proof. Clearly (a) implies (b) and (a) implies (c).

Proof for (c) ⇒ (b). Let $a \in G$ be a limit point of the set $Z = \{z \in G : f(z) = 0\}$. Since a is a limit point of Z and f is continuous it follows that $f(a) = 0$.

Why $f(a) = 0$?

Since a is a limit point of Z, there is a sequence $\{z_n\}$ in Z such that $a = \lim_{n\to\infty} z_n$. Then $f(z_n) = 0$ for all n. Since f is continuous on G (being analytic), $f(a) = \lim_{n\to\infty} f(z_n) = 0$.

Suppose there is an integer $n \geq 1$ such that $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$ but $f^{(n)}(a) \neq 0$. Since G is open and $a \in G$, there is $R > 0$ such that $B(a,R) \subset G$. Since f is analytic, expanding f in the power series about a gives

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k,$$

where $a_k = \frac{f^{(k)}(a)}{k!}$ for $|z-a| < R$. Since $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$ (i.e., $a_0 = a_1 = \cdots = a_{n-1} = 0$), and the power series of f is of the form

$$f(z) = \sum_{k=n}^{\infty} a_k (z-a)^k.$$

If

$$g(z) = \sum_{k=n}^{\infty} a_k (z-a)^{k-n},$$

then

- g is analytic in $B(a,R)$ (since it is defined as a convergent power series),
- $f(z) = (z-a)^n g(z)$ (clear from the definition of f and g), and
- $g(a) = a_n = \frac{f^{(n)}(a)}{n!} \neq 0$ (since n is assumed such that $f^{(n)}(a) \neq 0$).

Since g is continuous and $g(a) \neq 0$, we can find r with $0 < r < R$ such that $g(z) \neq 0$ for $|z-a| < r$. But since a is a limit point of Z, there is a point b with $0 < |b-a| < r$ such that $f(b) = 0$. This gives $g(b) = (b-a)^n f(b) = 0$ which is a contradiction. Hence no such integer n can be found, i.e., $f^{(n)}(a) = 0$ for all $n \geq 1$, which proves (b).

Proof for (b) $\Rightarrow$ (a). Let

$$A = \{z \in G : f^{(n)}(z) = 0 \text{ for all } n \geq 0\}.$$

By our assumption (b), it follows that $A \neq \emptyset$.

We show that $f \equiv 0$ by showing that $A = G$. We know that a connected set does not have a non-empty proper subset which is both open and closed. Since $A \subset G$ is non-empty, to show that $A = G$ it suffices to show that A is both open and closed.

Claim. A is closed.

Let $z \in A^-$ (here A^- denotes $\overline{A}$, the closure of A, a notation used by Conway). Then there is a sequence $\{z_k\}$ in A such that $z = \lim_{k\to\infty} z_k$. Since each $f^{(n)}$ is continuous (f being analytic), by the definition of A it follows that

$$f(z) = \lim_{k\to\infty} f^{(n)}(z_k) = 0.$$

So $z \in A$ and A is closed.

Claim. A is open.

Let $a \in A$. Since G is open, there is $R > 0$ such that $B(a,R) \subset G$. Since f is analytic, expanding f in power series about the point a in $B(a,R)$, we get

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

for $|z-a| < R$, where $a_n = \frac{f^{(n)}(a)}{n!} = 0$ for all n (by our assumption that $a \in A$). Hence, $f(z) = 0$ for all z in $B(a,R)$ and consequently $B(a,R) \subset A$. Thus, A is open. □

3.8 Corollary. *If f and g are analytic on a region G then $f \equiv g$ if and only if $\{z \in G : f(z) = g(z)\}$ has a limit point in G.*

Proof. Applying above theorem for analytic function $f-g$, we get the result. □

3.9 Corollary. *If f is analytic on an open connected set G and f is not identically zero then for each a in G with $f(a) = 0$ there is an integer $n \geq 1$ and an analytic function $g : G \to \mathbb{C}$ such that $g(a) \neq 0$ and $f(z) = (z-a)^n g(z)$ for all z in G.*

That is, each zero of f has finite multiplicity.

Proof. Since $f \not\equiv 0$, by Theorem 3.7 there is a largest integer $n \geq 1$ such that $f^{(n-1)}(a) = 0$, i.e., $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$. Define $g : G \to \mathbb{C}$ by

$$g(z) = \begin{cases} \frac{f(z)}{(z-a)^n} & z \neq a \\ \frac{f^{(n)}(a)}{n!} & z = a. \end{cases}$$

Then clearly g is analytic in $G \setminus \{a\}$. To show that g is analytic at a we show that g is differentiable at a.

Since f is analytic (at a), it has power series expansion about a in some neighborhood of a, i.e. there is some $R > 0$ such that $B(a,R) \subset G$ and

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$$

for $|z-a| < R$, where $a_k = \frac{f^{(k)}(a)}{k!}$. Since $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$,

$$f(z) = \sum_{k=n}^{\infty} a_k (z-a)^k.$$

Then

$$g(z) = \sum_{k=0}^{\infty} a_k (z-a)^{k-n} \qquad (z \neq a).$$

Now,

$$\begin{aligned}
\frac{g(z)-g(a)}{z-a} &= \frac{\sum\limits_{k=n}^{\infty} a_k(z-a)^{k-n} - \frac{f^{(n)}(a)}{n!}}{z-a} \\
&= \frac{1}{z-a}\left[a_n - \frac{f^{(n)}(a)}{n!} + \sum_{k=n+1}^{\infty} a_k(z-a)^{k-n}\right] \\
&= \sum_{k=n+1} a_k (z-a)^{k-n-1} \\
&\longrightarrow a_{n+1} \text{ as } z \to a.
\end{aligned}$$

Also, $g(a) = \frac{f^{(n)}(a)}{n!} \neq 0$ (since n is largest integer such that $f^{(n-1)}(a) = 0$) and (by definition of g and power series expansion f and g)

$$(z-a)^n g(z) = \begin{cases} f(z) & z \neq a \\ 0 \; (= f(a)) & z = a \end{cases},$$

i.e., $f(z) = (z-a)^n g(z)$ for all $z \in G$. □

3.10 Corollary. *If $f : G \to \mathbb{C}$ is analytic and not constant, $a \in G$, and $f(a) = 0$ then there is an $R > 0$ such that $B(a,R) \subset G$ and $f(z) \neq 0$ for $0 < |z-a| < R$.*

Proof. Since f is non-constant, $\not\equiv 0$. By Theorem 3.7, the set $Z = \{z \in G : f(z) = 0\}$ has no limit point in G. In particular, a is not a limit point of Z. Then by the definition of limit point, there is $R > 0$ such that

$$(B(a,R) \setminus \{a\}) \cap Z = \emptyset.$$

That is, $f(z) \neq 0$ for all z with $0 < |z-a| < R$. □

Corollary 3.10 says that if f is analytic and non-constant, then the zeros of f are isolated.

§4. The index of a closed curve

4.1 Proposition. *If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$ then*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a}$$

is an integer.

Proof. By Lemma 1.19, we know that γ can be approximated by a polygonal path $\Gamma : [0,1] \to \mathbb{C}$ such that given $\varepsilon > 0$

$$\left| \int_\gamma \frac{dz}{z-a} - \int_\Gamma \frac{dz}{z-a} \right| < \varepsilon.$$

Further since Γ is a polygonal path, it is piecewise smooth. By Proposition 1.8, $\int\limits_\Gamma \frac{dz}{z-a}$ can be written as a sum of integrals along the smooth segments of Γ. Therefore it suffices to prove the result for smooth curve γ.

Assume γ is smooth. Define $g : [0,1] \to \mathbb{C}$ by

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s) - a}\, ds.$$

Then $g(0) = 0$ and (since γ is assumed to be smooth)

$$(*) \qquad g(1) = \int_0^1 \frac{\gamma'(s)}{\gamma(s) - a}\, ds = \int_\gamma \frac{1}{z-a}\, dz.$$

Also by Fundamental Theorem of Calculus, we have

$$g'(t) = \frac{\gamma'(t)}{\gamma(t) - a} \text{ for } 0 \le t \le 1.$$

But then

$$\begin{aligned}
\frac{d}{dt}\left[e^{-g(t)}(\gamma(t) - a)\right] &= e^{-g(t)}\gamma'(t) - e^{-g(t)} g'(t)\,[\gamma(t) - a] \\
&= e^{-g(t)}\left[\gamma'(t) - g'(t)(\gamma(t) - a)\right] \\
&= e^{-g(t)}[\gamma'(t) - \gamma'(t)] && \left(\because g'(t) = \tfrac{\gamma'(t)}{\gamma(t)-a}\right) \\
&= 0.
\end{aligned}$$

Thus, $e^{-g(t)}(\gamma(t) - a)$ is a constant function and so

$$e^{-g(0)}(\gamma(0) - a) = e^{-g(1)}(\gamma(1) - a).$$

This implies

$$\begin{aligned}
& \gamma(0) - a = e^{-g(1)}(\gamma(1) - a) && (\because g(0) = 0) \\
\Rightarrow\ & \gamma(1) - a = e^{-g(1)}(\gamma(1) - a) && \left(\begin{smallmatrix}\because \gamma \text{ is closed} \\ \gamma(0) = \gamma(1)\end{smallmatrix}\right) \\
\Rightarrow\ & e^{-g(1)} = 1 && (\because a \ne \gamma(1) \text{ as } a \notin \{\gamma\}) \\
\Rightarrow\ & g(1) = 2\pi i k && (\text{ for some } k \in \mathbb{Z}) \\
\Rightarrow\ & \int_\gamma \frac{dz}{z-a} = 2\pi i k && (\text{by } (*)).
\end{aligned}$$

Hence, $\dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{dz}{z-a} = k$, which is an integer. □

4.2 Definition. If γ is a closed rectifiable curve in $\mathbb{C}$ then for $a \notin \{\gamma\}$

$$n(\gamma;a) = \frac{1}{2\pi i}\int_\gamma (z-a)^{-1}\,dz$$

is called the *index of* γ with respect to the point a. It is also sometimes called the *winding number* of γ around a.

If $\gamma : [0,1] \to \mathbb{C}$ is a curve then $-\gamma$ or γ^{-1} is a curve defined by $(-\gamma)(t) = \gamma(1-t)$, $0 \le t \le 1$. This is actually a reparametrization of the original definition. Also if γ and σ are curves defined on $[0,1]$ with $\gamma(1) = \sigma(0)$ then $\gamma+\sigma$ is the curve defined on $[0,1]$ by

$$(\gamma+\sigma)(t) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \sigma(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

Exercise. Let $\gamma, \sigma : [0,1] \to \mathbb{C}$ with $\gamma(1) = \sigma(0)$ then show that $(\gamma+\sigma)$ is rectifiable if and only if γ and σ are rectifiable.

4.3 Proposition. *If γ and σ are closed rectifiable curves having the same initial points then*

(a) $n(\gamma;a) = -n(-\gamma;a)$ for every $a \notin \{\gamma\}$;

(b) $n(\gamma+\sigma;a) = n(\gamma;a) + n(\sigma;a)$ for every $a \notin \{\gamma\} \cup \{\sigma\}$.

Proof. Exercise. □

Why is $n(\gamma;a)$ called the winding number of γ about a?

Note that if $\gamma(t) = a + e^{2\pi i n t}$ for $0 \le t \le 1$, then $n(\gamma;a) = n$. Let $b \in \mathbb{C}$. It can be checked (by direct computation) that if $|b-a| < 1$, then $n(\gamma;b) = n$ and if $|b-a| > 1$, then $n(\gamma;b) = 0$. This also follows from Theorem 4.4 given below. In this case $n(\gamma;b)$ measures the number of times γ wraps around $b-$ with the minus sign indicating that the curve goes in the clockwise direction.

Let γ be a closed rectifiable curve and consider the open set $G = \mathbb{C} \setminus \{\gamma\}$. Since $\{\gamma\}$ is compact, the set $\{z \in \mathbb{C} : |z| > R\} \subset G$ for some sufficiently large $R > 0$. This says that G has one and only one unbounded component.

4.4 Theorem. *Let γ be a closed rectifiable curve in $\mathbb{C}$. Then $n(\gamma;a)$ is constant for a belonging to a component of $G = \mathbb{C} \setminus \{\gamma\}$. Also, $n(\gamma;a) = 0$ for a belonging to the unbounded component of G.*

Proof. Define $f : G \to \mathbb{C}$ by $f(a) = n(\gamma;a)$. We will show that f is continuous. If this is done then $f(D)$ is connected for each (connected) component D of G. But since $f(G) \subset \mathbb{Z}$, it follows that $f(D)$ reduces to a singleton set.

Now we show that f is continuous. Note that the components of G are open (for a disconnected set, the components are both open and closed). Fix $a \in G$ and let $r = d(a, \{\gamma\})$. If $|a-b| < \delta < \frac{r}{2}$ then

$$|f(a) - f(b)| = |n(\gamma;a) - n(\gamma;b)|$$

$$= \frac{1}{2\pi}\left|\int_\gamma \frac{1}{z-a}\,dz - \int_\gamma \frac{1}{z-b}\,dz\right|$$
$$= \frac{1}{2\pi}\left|\int_\gamma \frac{a-b}{(z-a)(z-b)}\,dz\right|$$
$$(*) \qquad \leq \frac{|a-b|}{2\pi}\int_\gamma \frac{|dz|}{|z-a||z-b|}.$$

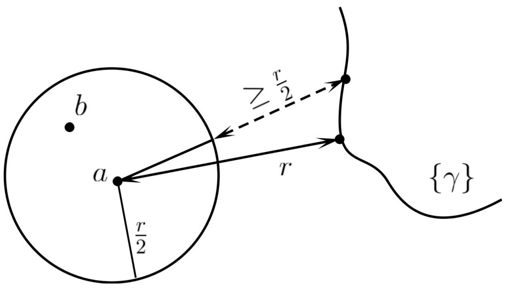
Since $r = d(a,\{\gamma\})$, for any $z \in \{\gamma\}$, $|z-a| \geq r > \frac{1}{2}r$. Thus, for $|a-b| < \frac{1}{2}r$ and z on $\{\gamma\}$ we have $|z-b| > \frac{1}{2}r$.

Why $|z-b| > \frac{1}{2}r$?

For if $|z-b| \leq \frac{1}{2}r$, then

$$|z-a| \leq |z-b| + |a-b| < \frac{r}{2} + \frac{r}{2} = r$$

which is a contradiction as $r = d(a,\{\gamma\}) \leq |z-a|$.



Then $\frac{1}{|z-a|} < \frac{r}{2}$ and $\frac{1}{|z-b|} < \frac{r}{2}$ and so by $(*)$

$$|f(a) - f(b)| < \frac{\delta}{2\pi}\frac{4}{r^2}\int_\gamma |dz|$$
$$\leq \frac{2\delta}{\pi r^2}V(\gamma) \qquad \text{(by Proposition 1.17 (b))}.$$

Let $\varepsilon > 0$ be given. Choose $\delta < \min\{\frac{r}{2}, \frac{\pi r^2 \varepsilon}{2V(\gamma)}\}$. Then

$$|a-b| < \delta \Rightarrow |f(a) - f(b)| < \varepsilon.$$

Therefore, f is continuous on the components of G and hence $f(a) = n(\gamma;a)$ is constant on the components of G.

Now let U be the unbounded component of G. Then there is $R > 0$ such that $\{z : |z| > R\} \subset U$. Let $\varepsilon > 0$ be given. Choose $a \in G$ with $|a| > R$ and $|z-a| > \frac{V(\gamma)}{2\pi\varepsilon}$ for all z on $\{\gamma\}$. Then

$$|n(\gamma;a)| = \frac{1}{2\pi}\left|\int_\gamma \frac{dz}{z-a}\right|$$

$$
\begin{aligned}
&\leq \frac{1}{2\pi}\int_{\gamma}\frac{|dz|}{|z-a|}\\
&\leq \frac{V(\gamma)}{2\pi}\cdot\frac{2\pi\varepsilon}{V(\gamma)}\\
&= \varepsilon.
\end{aligned}
$$

Thus, $|n(\gamma;a)| < \varepsilon$ whenever $|a| > R$ (with $|z-a| > \frac{V(\gamma)}{2\pi\varepsilon}$). That is, $n(\gamma;a) \to 0$ as $a \to \infty$. Since $n(\gamma;a)$ is constant on U (on any component of G), it must be zero on U. □

2

Unit 2

§5. Cauchy's Theorem and Integral Formula

5.1 Lemma. *Let γ be a rectifiable curve and suppose φ is a function defined and continuous on $\{\gamma\}$. For each $m \geq 1$ let $F_m(z) = \int_\gamma \varphi(w)(w-z)^{-m}\, dw$ for $z \notin \{\gamma\}$. Then each F_m is analytic on $\mathbb{C} \setminus \{\gamma\}$ and $F_m'(z) = mF_{m+1}(z)$.*

Proof. First we show that F_m is continuous for each $m \geq 1$. Since φ is continuous on the compact set $\{\gamma\}$, it is bounded. That is, there is $M > 0$ such that $|\varphi(w)| \leq M$ for all $w \in \{\gamma\}$.

Let $a \in \mathbb{C} \setminus \{\gamma\}$ and $r = d(a, \{\gamma\})$. Let $\varepsilon > 0$ be given. Choose $\delta = \min\left\{\frac{r}{2}, \frac{\varepsilon r^{m+1}}{MmV(\gamma)2^{m+1}}\right\}$. Then for any $z \in G = \mathbb{C} \setminus \{\gamma\}$ with $|z-a| < \delta \leq \frac{r}{2}$, we have $|w-a| \geq r > \frac{r}{2}$ and so $|w-z| \geq \frac{r}{2}$. The factorization

$$x^m - y^m = (x-y)\sum_{k=1}^{m} x^{m-k}y^{k-1}$$

gives

$$\frac{1}{(w-z)^m} - \frac{1}{(w-a)^m} = \left[\frac{1}{w-z} - \frac{1}{w-a}\right]\sum_{k=1}^{m}\frac{1}{(w-z)^{m-k}}\frac{1}{(w-a)^{k-1}}$$

5.2 $$= (z-a)\left[\frac{1}{(w-z)^m(w-a)} + \frac{1}{(w-z)^{m-1}(w-a)^2} + \cdots + \frac{1}{(w-z)(w-a)^m}\right].$$

Now,

$$|F_m(z) - F_m(a)|$$

$$= \left|\int_\gamma \frac{\varphi(w)}{(w-z)^m}\, dw - \int_\gamma \frac{\varphi(w)}{(w-a)^m}\, dw\right|$$

$$
\begin{aligned}
&= \left| \int_\gamma \varphi(w) \left[\frac{1}{(w-z)^m} - \frac{1}{(w-a)^m} \right] dw \right| \\
&\leq \int_\gamma |\varphi(w)||z-a| \sum_{k=1}^{m} \frac{1}{|w-z|^{m-k+1}|w-a|^k} |dw| \qquad \text{(by (5.2))} \\
&\leq M\delta V(\gamma) \frac{2^{m+1}}{r^{m+1}} \cdot m \\
&\leq \varepsilon.
\end{aligned}
$$

Hence, each F_m is continuous.

Fix $a \in G = \mathbb{C} \setminus \{\gamma\}$. Then for $z \in G$, $z \neq a$,

$$
\begin{aligned}
\frac{F_m(z) - F_m(a)}{z-a} &= \int_\gamma \varphi(w) \sum_{k=1}^{m} \frac{dw}{(w-z)^{m-k+1}(w-a)^k} \\
&= \sum_{k=1}^{m} \int_\gamma \frac{\varphi(w) dw}{(w-z)^{m-k+1}(w-a)^k} \\
&= \sum_{k=1}^{m} \int_\gamma \frac{\frac{\varphi(w)}{(w-a)^k} dw}{(w-z)^{m-k+1}}. \qquad \textbf{5.3}
\end{aligned}
$$

Since $a \notin \{\gamma\}$, $\frac{\varphi(w)}{(w-a)^k}$ is continuous for $1 \leq k \leq m$. Therefore the integrals on the right hand side of (5.3) are continuous functions of z for all $z \in G = \mathbb{C} \setminus \{\gamma\}$. Hence, letting $z \to a$, we get

$$
\begin{aligned}
F_m'(a) &= \lim_{z \to a} \frac{F_m(z) - F_m(a)}{z-a} \\
&= \sum_{k=1}^{m} \int_\gamma \frac{\varphi(w)}{(w-a)^{m+1}} dw \\
&= mF_{m+1}(a).
\end{aligned}
$$

Since $a \in G$ was arbitrary,

$$F_m'(z) = mF_{m+1}(z) \qquad \forall\, z \in \mathbb{C} \setminus \{\gamma\}.$$

□

Exercise (Exercise IV.5.1)**.** Let G be an open subset of $\mathbb{C}$ and $f : G \to \mathbb{C}$ be analytic. Define $\varphi : G \times G \to \mathbb{C}$ by $\varphi(z,w) = \frac{f(z)-f(w)}{z-w}$ if $z \neq w$ and $\varphi(z,z) = f'(z)$. Then show that φ is continuous and for each fixed $w \in G$, the map $z \mapsto \varphi(z,w)$ is analytic.

5.4 Cauchy's Integral Formula (First Version). *Let G be an open subset of the plane and $f : G \to \mathbb{C}$ an analytic function. If γ is a closed rectifiable curve in G such that $n(\gamma; w) = 0$ for all w in $\mathbb{C} \setminus G$, then for a in $G \setminus \{\gamma\}$*

$$n(\gamma; a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz.$$

Proof. Define $\varphi : G\times G \to \mathbb{C}$ by $\varphi(z,w) = \frac{f(w)-f(z)}{w-z}$ if $z \neq w$ and $\varphi(z,z) = f'(z)$. Then it follows from the above exercise that φ is continuous and for each fixed $w \in G$ the map $z \mapsto \varphi(z,w)$ is analytic.

Let $H = \{w \in \mathbb{C} : n(\gamma;w) = 0\}$. Then $\mathbb{C} \setminus G \subset H$. Also $H = n(\gamma;\cdot)^{-1}\big(\{0\}\big)$. Since $n(\gamma;\cdot)$ is continuous integer valued function on the components of $\mathbb{C}\setminus\{\gamma\}$, H is open in $\mathbb{C}\setminus\{\gamma\}$ an hence open in $\mathbb{C}$. Moreover, $H\cup G = \mathbb{C}$. Define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \int\limits_\gamma \varphi(z,w)\,dw & \text{if } z \in G \\ \int\limits_\gamma (w-z)^{-1} f(w)\,dw & \text{if } z \in H \end{cases}$$

We shall show that this piecewise definition of g is consistent on $H\cap G$. If $z \in H \cap G$, then

$$\begin{aligned} g(z) &= \int\limits_\gamma \varphi(z,w)\,dw \\ &= \int\limits_\gamma \frac{f(w)-f(z)}{w-z}\,dw \\ &= \int\limits_\gamma \frac{f(w)}{w-z}\,dw - f(z)\int\limits_\gamma \frac{1}{w-z}\,dw \\ &= \int\limits_\gamma \frac{f(w)}{w-z}\,dw - 2\pi i f(z)\frac{1}{2\pi i}\int\limits_\gamma \frac{1}{w-z}\,dw \\ &= \int\limits_\gamma \frac{f(w)}{w-z}\,dw - 2\pi i f(z) n(\gamma;z) \\ &= \int\limits_\gamma \frac{f(w)}{w-z}\,dw && (n(\gamma;z) = 0 \text{ as } z \in H). \end{aligned}$$

Hence g is well-defined.

By Lemma 5.1, g is analytic on G (for $m = 1$ with numerator $f(w) - f(z)$) and also by Lemma 5.1 (for $m = 1$ with numerator $f(w)$) g is analytic on H. Hence, g is an entire function (as $H\cup G = \mathbb{C}$).

By Theorem 4.4, H contains the unbounded component of $\mathbb{C}\setminus\{\gamma\}$. So for $z \in \mathbb{C}\setminus\{\gamma\}$ with $|z|$ sufficiently large, we have

$$g(z) = \int\limits_\gamma \frac{f(w)}{w-z}\,dw.$$

Since f is continuous (being analytic) on the compact set $\{\gamma\}$, it is bounded on $\{\gamma\}$. Also,

$$\lim_{z\to\infty} \frac{1}{w-z} = 0$$

uniformly for all w in $\{\gamma\}$

5.5 $$\lim_{z\to\infty} g(z) = \lim_{z\to\infty} \int_\gamma \frac{f(w)}{w-z}\,dw = \int_\gamma f(w) \lim_{z\to\infty} \frac{1}{w-z}\,dw = 0.$$

In particular, there is $R > 0$ such that $|g(z)| \leq 1$ whenever $|z| \geq R$. Since g is continuous on $\overline{B}(a,R)$, g is bounded on $\overline{B}(a,R)$ (i.e. on $|z| \leq R$). Hence, g is bounded on $\mathbb{C}$.

By Liouville's theorem, g is constant (being entire and bounded). Since $\lim_{z\to\infty} g(z) = 0$, it follows that $g \equiv 0$. Then for $a \in G \setminus \{\gamma\}$,

$$\begin{aligned} 0 = g(a) &= \int_\gamma \varphi(a,w)\,dw \\ &= \int_\gamma \frac{f(w)-f(a)}{w-a}\,dw \\ &= \int_\gamma \frac{f(w)}{w-a}\,dw - f(a)\int_\gamma \frac{1}{w-a}\,dw \\ &= \int_\gamma \frac{f(w)}{w-a}\,dw - 2\pi i f(a) n(\gamma;a). \end{aligned}$$

Hence

$$f(a)n(\gamma;a) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}\,dw.$$

□

5.6 Cauchy's Integral Formula (Second Version). *Let G be an open subset of the plane and $f : G \to \mathbb{C}$ an analytic function. If $\gamma_1, \ldots, \gamma_m$ are closed rectifiable curves in G such that $n(\gamma_1;w) + \cdots + n(\gamma_m;w) = 0$ for all w in $\mathbb{C} \setminus G$, then for a in $G \setminus \bigcup_{k=1}^{m} \{\gamma_k\}$*

$$f(a)\sum_{k=1}^{m} n(\gamma_k;a) = \sum_{k=1}^{m} \frac{1}{2\pi i}\int_{\gamma_k} \frac{f(z)}{z-a}\,dz.$$

Proof. Same as above theorem.

Define $\varphi(z,w)$ as in above theorem and let $H = \{w \in \mathbb{C} : n(\gamma_1;w) + \cdots + n(\gamma_m;w) = 0\}$. Define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \sum_{k=1}^{m} \int_{\gamma_k} \varphi(z,w)\,dw & \text{if } z \in G \\ \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w)}{(w-z)}\,dw & \text{if } z \in H. \end{cases}$$

[Complete the proof by mimicking the argument as in the above theorem]. □

5.7 Cauchy's Theorem (First Version). *Let G be an open subset of the plane and $f : G \to \mathbb{C}$ an analytic function. If $\gamma_1, \ldots, \gamma_m$ are closed rectifiable curves in G such that $n(\gamma_1;w) + \cdots + n(\gamma_m;w) = 0$ for all w in $\mathbb{C} \setminus G$ then*

$$\sum_{k=1}^{m} \int_{\gamma_k} f = 0.$$

Proof. Substitute $f(z)(z-a)$ in place of $f(z)$ is the Theorem 5.6. □

Example. Let $G = \{z \in \mathbb{C} : R_1 < |z| < R_2\}$ and define curves γ_1 and γ_2 by

$$\gamma_1(t) = r_1 e^{it}$$
$$\gamma_2(t) = r_2 e^{-it}$$

for $0 \le t \le 2\pi$, where $R_1 < r_1 < r_2 < R_2$.

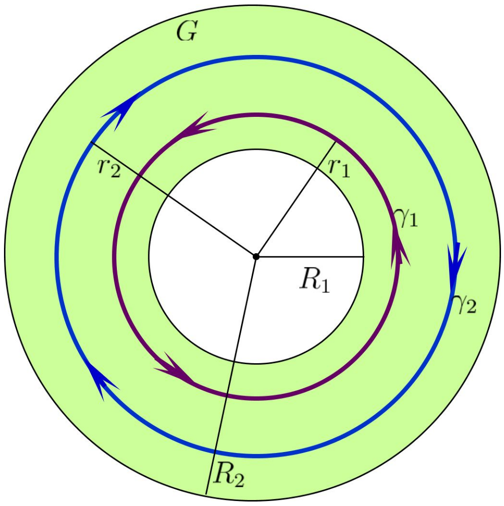


Figure IV.2:

If $|w| \le R_1$, then $n(\gamma_1; w) = 1$ and $n(\gamma_2; w) = -1$. If $|w| \ge R_2$, then $n(\gamma_1; w) = 0 = n(\gamma_2; w)$. Thus for all $w \in \mathbb{C} \setminus G$, we have

$$n(\gamma_1; w) + n(\gamma_2; w) = 0.$$

Hence, by Cauchy's Theorem (5.7), if f is analytic on G then

$$\int_{\gamma_1} f + \int_{\gamma_2} f = 0 \quad \text{or} \quad \int_{\gamma_1} f = -\int_{\gamma_2} f = \int_{-\gamma_2} f.$$

The following result is a generalization of the Cauchy's integral formula, called the *Cauchy's Integral Formula for Derivatives*

5.8 Theorem. *Let G be an open subset of the plane and $f : G \to \mathbb{C}$ an analytic function. If $\gamma_1, \ldots, \gamma_m$ are closed rectifiable curves in G such that $n(\gamma_1; w) + \cdots + n(\gamma_m; w) = 0$ for all w in $\mathbb{C} \setminus G$ then for a in $G \setminus \bigcup_{j=1}^{m} \{\gamma_j\}$ and $k \ge 1$*

$$f^{(k)}(a) \sum_{j=1}^{m} n(\gamma_j; a) = k! \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{(z-a)^{k+1}}\, dz.$$

Proof. This follows immediately by differentiating both sides of the formula in Theorem 5.6 and applying Lemma 5.1.

How it follows?

Since f is continuous on $\{\gamma_j\}$, $1 \le j \le m$, by Lemma 5.1, the integrals $\int\limits_{\gamma_j} \frac{f(z)}{(z-a)}\,dz$ are analytic on $G \setminus \bigcup\limits_{j=1}^{m} \{\gamma_j\}$. By Cauchy's Integral Formula (Second Version, Theorem 5.6), we have

$$f(a)\sum_{j=1}^{m} n(\gamma_j;a) = \sum_{j=1}^{m} \frac{1}{2\pi i}\int_{\gamma_j} \frac{f(z)}{(z-a)}\,dz.$$

Differentiating above with respect to a, by applying Lemma 5.1 repeatedly, and since $\sum\limits_{j=1}^{m} n(\gamma_j;a)$ is constant, we get

$$\begin{aligned}\frac{d^k}{da^k}\left[f(a)\sum_{j=1}^{m} n(\gamma_j;a)\right] &= \frac{d^k}{da^k}\left[\sum_{j=1}^{m} \frac{1}{2\pi i}\int_{\gamma_j} \frac{f(z)}{(z-a)}\,dz\right]\\ \Rightarrow f^{(k)}(a)\sum_{j=1}^{m} n(\gamma_j;a) &= \sum_{j=1}^{m} \frac{k!}{2\pi i}\int_{\gamma_j} \frac{f(z)}{(z-a)^{k+1}}\,dz.\end{aligned}$$

□

5.9 Corollary. *Let G be an open set and $f : G \to \mathbb{C}$ an analytic function. If γ is a closed rectifiable curve in G such that $n(\gamma;w) = 0$ for all w in $\mathbb{C} \setminus G$ then for a in $G \setminus \{\gamma\}$*

$$f^{(k)}n(\gamma;a) = \frac{k!}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-a)^{k+1}}\,dz.$$

Exercise (Exercise IV.5.5)**.** Let γ be a closed rectifiable curve in $\mathbb{C}$ and $a \notin \{\gamma\}$. Show that for $n \ge 2$

$$\int_{\gamma} \frac{1}{(z-a)^n}\,dz = 0.$$

Solution. From Cauchy's Integral Formula for Derivatives (Corollary 5.9), taking $k = n-1$, $f(z) = 1$, for any closed rectifiable curve γ and for any a not in $\{\gamma\}$ we have

$$\begin{aligned} f^{(k)}(a)n(\gamma;a) &= \frac{k!}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-a)^{k+1}}\,dz\\ \Rightarrow f^{(n-1)}(a)n(\gamma;a) &= \frac{(n-1)!}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-a)^{n}}\,dz\\ \Rightarrow 0 &= \frac{(n-1)!}{2\pi i}\int_{\gamma} \frac{1}{(z-a)^{n}}\,dz\\ \Rightarrow \int_{\gamma} \frac{1}{(z-a)^{n}}\,dz &= 0.\end{aligned}$$

□

Exercise (Exercise IV.5.7)**.** Let $\gamma(t) = 1 + e^{it}$ for $0 \le t \le 2\pi$. Find $\int_{\gamma} \left(\frac{z}{z-1}\right)^n dz$ for all positive integers n.

Solution. Fix $n \in \mathbb{N}$. Let $f(z) = z^n$. Then f is analytic on $\{\gamma\}$. Taking $a = 1$, we get $n(\gamma;1) = 1$. Then by Cauchy's integral formula for derivatives (Corollary 5.9) taking $k = n-1$, we have

$$\begin{aligned}
f^{(k)}(a)n(\gamma;a) \quad &= \frac{k!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{k+1}}\,dz \\
\Rightarrow \frac{d^{n-1}}{dz^{n-1}}[z^n]_{z=1}\, n(\gamma;1) &= \frac{(n-1)!}{2\pi i}\int_\gamma \frac{z^n}{(z-1)^n}\,dz \\
\Rightarrow n!(1) \quad &= \frac{(n-1)!}{2\pi i}\int_\gamma \left(\frac{z}{z-1}\right)^n dz \\
\Rightarrow \int_\gamma \left(\frac{z}{z-1}\right)^n dz \quad &= 2\pi i n.
\end{aligned}$$

□

Exercise (L. Ahlfors, page no. 123)**.** Compute $\displaystyle\int_{|z|=1} e^z z^{-n}\,dz$.

Solution. Here we take $f(z) = e^z$, $a = 0$, $k = n-1$, and $\gamma(t) = e^{it}$, $t \in [0,2\pi]$. Then by Corollary 5.9, we have

$$\begin{aligned}
f^{(k)}(a)n(\gamma;a) \quad &= \frac{k!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{k+1}}\,dz \\
\Rightarrow f^{(n-1)}(0)n(\gamma;0) &= \frac{(n-1)!}{2\pi i}\int_\gamma \frac{f(z)}{(z-0)^n}\,dz \\
\Rightarrow (e^0)(1) \quad &= \frac{(n-1)!}{2\pi i}\int_\gamma \frac{e^z}{z^n}\,dz \\
\Rightarrow \int_\gamma e^z z^{-n}\,dz \quad &= \frac{2\pi i}{(n-1)!}.
\end{aligned}$$

□

Definition. A closed polygonal path with three sides is called a *triangular path*.

5.10 Morera's Theorem. *Let G be a region and $f : G \to \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every triangular path T in G; then f is analytic in G.*

Proof. To show that f is analytic in G, it suffices to show that f is analytic on each open disk contained in G (since G, being open, is the union of open disks). Hence, without the loss of generality, we may assume that G is an open disk. Suppose $G = B(a,R)$ for some $a \in G$ and $R > 0$.

Recall that a primitive of f is an analytic function $F : G \to \mathbb{C}$ such that $F' = f$. First we use the hypothesis to show that f has a primitive. We know (by Cauchy's Integral Formula for derivatives) that the derivative of an analytic function is analytic. Hence, it is sufficient to show that f has a primitive in G.

For $z \in G$, define $F : G \to \mathbb{C}$ by

$$F(z) = \int_{[a,z]} f = \int_{[a,z]} f(w)\,dw,$$

where $[a,z]$ is the line-segment joining a and z.

Claim. F is a primitive of f, i.e., $F' = f$.

Fix $z_0 \in G$. Then for any $z \in G$, since a, z, and z_0 form a triangle in G, by hypothesis (i.e. $\int_T f = 0$ for ever triangular path T in G) we have

$$\begin{aligned}
&\int\limits_{[a,z_0]} f + \int\limits_{[z_0,z]} f + \int\limits_{[z,a]} f = 0 \\
&\Rightarrow \int\limits_{[a,z]} f - \int\limits_{[a,z_0]} f = \int\limits_{[z_0,z]} f \\
&\Rightarrow F(z) - F(z_0) = \int\limits_{[z_0,z]} f \\
(*) \qquad &\Rightarrow \frac{F(z)-F(z_0)}{z-z_0} = \frac{1}{z-z_0}\int\limits_{[z_0,z]} f = \frac{1}{z-z_0}\int\limits_{[z_0,z]} f(w)\,dw.
\end{aligned}$$

Since f is continuous, given $\varepsilon > 0$ there is $\delta > 0$ such that $|f(w) - f(z_0)| < \varepsilon$ whenever $|w - z_0| < \delta$. Then by $(*)$ for any $z \in G$, $0 < |z - z_0| < \delta$, we have

$$\begin{aligned}
\frac{F(z)-F(z_0)}{z-z_0} - f(z_0) &= \frac{1}{z-z_0}\int_{[z_0,z]} f(w)\,dw - f(z_0) \\
&= \frac{1}{z-z_0}\int_{[z_0,z]} \big(f(w) - f(z_0)\big)\,dw.
\end{aligned}$$

So,

$$\begin{aligned}
\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| &\le \frac{1}{|z-z_0|}\int_{[z_0,z]} |f(w)-f(z_0)|\,|dw| \\
&\le \frac{|z-z_0|}{|z-z_0|}\varepsilon = \varepsilon. \qquad \text{(by Proposition 1.17 (b))}.
\end{aligned}$$

Therefore,

$$\lim_{z\to z_0} \frac{F(z)-F(z_0)}{z-z_0} = f(z_0),$$

i.e., $F'(z_0) = f(z_0)$. Hence, $F' = f$. Since F is analytic, F' is also analytic, i.e., f is analytic. This completes the proof. □

Remark. It follows from the proof of Morera's theorem that if $f : G \to \mathbb{C}$ has a primitive, then f is analytic in G. Does the converse hold? That is, if $f : G \to \mathbb{C}$ is an analytic function in G, then does it have a primitive?

The answer is **no** in general. For example, $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{z}$$

is analytic in the open set $\mathbb{C} \setminus \{0\}$ but f does not admit a primitive. The reason is the following:
Suppose $F : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is a primitive of f, then by Corollary 1.22, we must have

$$\int_\gamma f = 0$$

for every closed rectifiable curve γ in $\mathbb{C} \setminus \{0\}$.

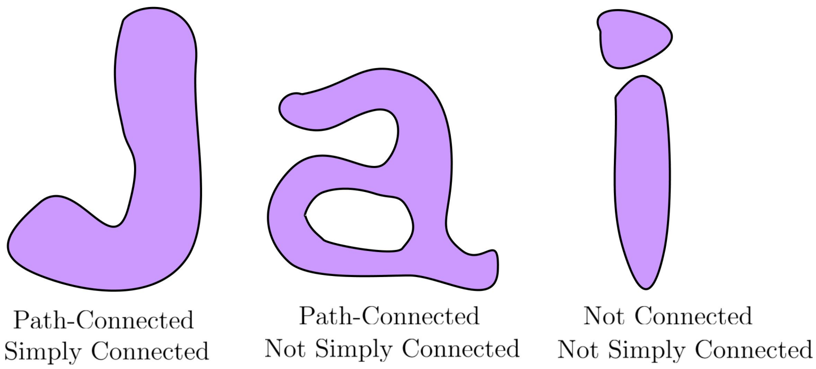
But we have seen earlier that $\int_\gamma \frac{1}{z}\, dz = 2\pi i \neq 0$ for $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$ in $\mathbb{C} \setminus \{0\}$. Hence, f cannot have a primitive. The reason that f does not admit a primitive is that the domain $\mathbb{C} \setminus \{0\}$ (i.e. the punctured plane) is not simply connected.

§6. The homotopic version of Cauchy's Theorem and simple connectivity

Looking at our syllabus, we do not go into algebraic topology to introduce the notion of homotopy (i.e. when two curves are called homotopic to each other) and without which we define vaguely what do we mean by a simply connected set.

Definition. An open set G is *simply connected* if it is connected and every closed curve in G can be continuously shrunk to a point in G.

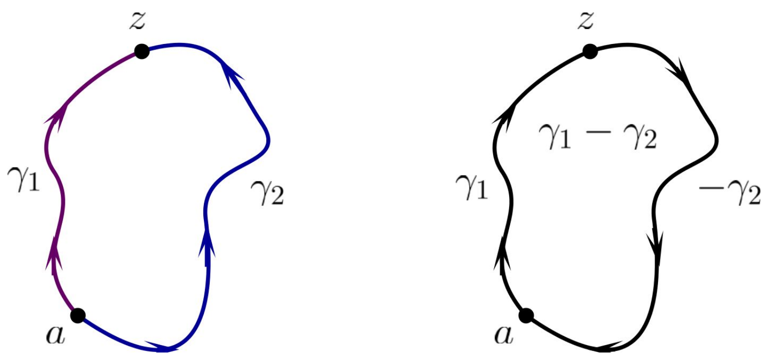
In other words, any curve in G can be continuously transformed in to another curve in G without leaving G.



If the domain G is simply connected, then an analytic function on G will have a primitive. We have the following result.

6.16 Corollary. *If G is simply connected and $f : G \to \mathbb{C}$ is analytic in G then f has a primitive.*

Proof. Fix a point $a \in G$. Let γ_1 and γ_2 be rectifiable curves from a to a point z in G (since G is open and connected, it is path connected and so we can always find such path in G). Then $\gamma_1 - \gamma_2$ is a closed rectifiable curve in G (γ_1 is from a to z and $-\gamma_2$ is from z to a).



Also since G is simply connected $w \in \mathbb{C} \setminus G$ implies that w is in the unbounded component of the closed curve $\gamma_1 - \gamma_2$. Hence, $n(\gamma_1 - \gamma_2; w) = 0$ for all $w \in \mathbb{C} \setminus G$. Then by Cauchy's Theorem (5.7),

$$\int_{\gamma_1 - \gamma_2} f = 0,$$

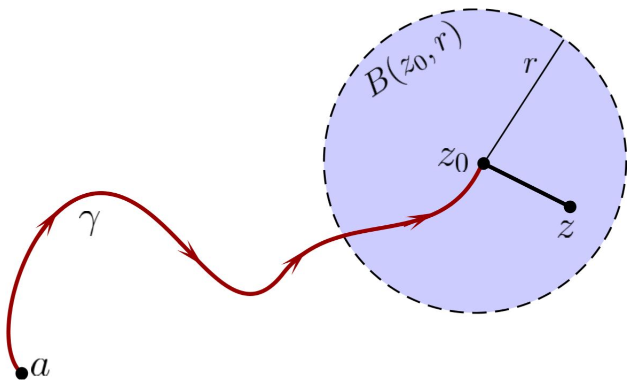
$$\text{i.e., } \int_{\gamma_1} f - \int_{\gamma_2} f = 0 \quad \text{or} \quad \int_{\gamma_1} f = \int_{\gamma_2} f.$$

Therefore if we define $F : G \to \mathbb{C}$ by $F(z) = \int_\gamma f$, where γ is any rectifiable curve from a to z, then F is well-defined (i.e., it is independent of γ).

If $z_0 \in G$, then (since G is open) there is $r > 0$ such that $B(z_0, r) \subset G$. Let γ be a rectifiable curve from a to z_0 in G. For any $z \in B(z_0, r)$, let

$$\gamma_z = \gamma + [z_0, z],$$

i.e., γ_z is the curve γ followed by the line-segment from z_0 to z.



Then

$$F(z) - F(z_0) = \int_{\gamma_z} f - \int_\gamma f$$

$$= \left[\int_\gamma f + \int_{[z_0,z]} f\right] - \int_\gamma f \qquad (\because \gamma_z = \gamma + [z_0, z])$$
$$= \int_{[z_0,z]} f.$$

Therefore,

$$(*) \qquad \frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0}\int_{[z_0,z]} f = \frac{1}{z - z_0}\int_{[z_0,z]} f(w)\,dw.$$

Now, we proceed as in the proof of Morera's Theorem (5.10). Since f is continuous (being analytic) given $\varepsilon > 0$ there is $\delta > 0$ such that

$$|f(w) - f(z_0)| < \varepsilon \text{ whenever } |w - z_0| < \delta.$$

We may choose $\delta < r$. Now for any $z \in G$ with $0 < |z - z_0| < \delta$, by $(*)$ we have

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0}\int_{[z_0,z]} f(w)\,dw - f(z_0)$$
$$= \frac{1}{z - z_0}\int_{[z_0,z]} \big(f(w) - f(z_0)\big)\,dw.$$

So,

$$\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| \le \frac{1}{|z - z_0|}\int_{[z_0,z]} |f(w) - f(z_0)|\,|dw|$$
$$\le \frac{|z - z_0|}{|z - z_0|}\varepsilon = \varepsilon. \qquad \text{(Proposition 1.17 (b))}.$$

Therefore,

$$\lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = f(z_0),$$

i.e., $F'(z_0) = f(z_0)$. Hence, f has a primitive. □

6.17 Corollary. *Let G be simply connected and let $f : G \to \mathbb{C}$ be an analytic function such that $f(z) \neq 0$ for any z in G. Then there is an analytic function $g : G \to \mathbb{C}$ such that $f(z) = \exp g(z)$. If $z_0 \in G$ and $e^{w_0} = f(z_0)$, we may choose g such that $g(z_0) = w_0$.*

Proof. Since $f(z) \neq 0$ for all $z \in G$ the function $\dfrac{f'}{f}$ is analytic on G. Then by Corollary 6.16, it has a primitive, say g_1, i.e.,

$$g_1' = \frac{f'}{f}.$$

Define $h : G \to \mathbb{C}$ by $h(z) = \exp(g_1(z))$. Then h is analytic function in G and $h(z) \neq 0$ for all $z \in G$. So $\frac{f}{h}$ is analytic and its derivative is

$$\left(\frac{f}{h}\right)'(z) = \frac{h(z)f'(z) - f(z)h'(z)}{h(z)^2}.$$

But

$$\begin{aligned} h'(z) &= e^{g_1(z)} g_1'(z) \\ &= h(z)\frac{f'(z)}{f(z)} && \left(\because g_1' = \frac{f'}{f}\right). \end{aligned}$$

Therefore, $\left(\frac{f}{h}\right)'(z) = 0$, i.e., $\frac{f}{h}$ is constant, say c. Then

$$\begin{aligned} f(z) &= c \cdot h(z) \\ &= c\exp(g_1(z)) \\ &= \exp(c')\exp(g_1(z)) && \text{(for some } c'\text{)} \\ &= \exp[g_1(z) + c']. \end{aligned}$$

Define $g : G \to \mathbb{C}$ by $g(z) = g_1(z) + c' + 2\pi ik$ for an appropriate k such that $g(z_0) = w_0$. Then g is analytic and the result is proved. □

The following definition is not in our syllabus, but we state here for the sake of notation used in couple of results of the next section.

6.18 Definition. If G is an open set then γ is *homologous* to zero if $n(\gamma; w) = 0$ for all w in $\mathbb{C} \setminus G$. We denote it by $\gamma \approx 0$.

Remark. By the above definition $\gamma \approx 0$ implies $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$. We shall replace the former notation by the later condition in couple of results of the next section. The reason behind using the condition $n(\gamma; w) = 0$ for all w in $\mathbb{C} \setminus G$ is that, we can apply Cauchy's Theorem, First Version (5.7) and we do not have to use other versions of the same which are not included in our syllabus.

§7. Counting zeros; the Open Mapping Theorem

In Section 3, we saw that if f is an analytic function with zero at $z = a$, then there is an integer m and an analytic function g such that $f(z) = (z-a)^m g(z)$ and $g(a) \neq 0$. Suppose G is a region and f is analytic in G with zeros at $a_1, \ldots, a_m$ (repeated according to multiplicities). Then we can write $f(z) = (z-a_1)(z-a_2)\cdots(z-a_m)g(z)$, where g is analytic on G and $g(z) \neq 0$ for any $z \in G$. Applying the formula for differentiating a product gives

7.1
$$\frac{f'(z)}{f(z)} = \frac{1}{z-a_1} + \frac{1}{z-a_2} + \cdots + \frac{1}{z-a_m} + \frac{g'(z)}{g(z)}$$

for $z \neq a_1, \ldots, a_m$. Once we have this, the proof of the following result, called the *Counting Zeros Principle*, follows easily.

7.2 Theorem. *Let G be a region and let f be an analytic function on G with zeros $a_1,\dots,a_m$ (repeated according to multiplicity). If γ is a closed rectifiable curve in G which does not pass through any point a_k and if $\gamma \approx 0$ then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{m} n(\gamma;a_k).$$

Proof. If $a_1,\dots,a_m$ are zeros of f, then we can factor f as

$$f(z) = (z-a_1)(z-a_2)\cdots(z-a_m)g(z)$$

for all z in G, for some analytic function $g : G \to \mathbb{C}$ such that $g(z) \neq 0$ for all $z \in G$. By product rule of derivatives

$$\frac{f'(z)}{f(z)} = \frac{1}{z-a_1} + \frac{1}{z-a_2} + \cdots + \frac{1}{z-a_m} + \frac{g'(z)}{g(z)}.$$

Since $\frac{g'}{g}$ is analytic in G, γ is closed rectifiable curve in G such that $n(\gamma;w) = 0$ (since $\gamma \approx 0$) for all $w \in \mathbb{C} \setminus G$, by Cauchy's Theorem

$$\int_\gamma \frac{g'(z)}{g(z)} = 0.$$

Therefore

$$\begin{aligned}\int_\gamma \frac{f'(z)}{f(z)} &= \int_\gamma \frac{dz}{z-a_1} + \int_\gamma \frac{dz}{z-a_2} + \cdots + \int_\gamma \frac{dz}{z-a_m} + \int_\gamma \frac{g'(z)}{g(z)}\,dz \\ &= 2\pi i\,[n(\gamma;a_1) + n(\gamma;a_2) + \cdots + n(\gamma;a_m)] \qquad \text{(by def}^{\text{n}}\text{ of index).}\end{aligned}$$

Hence,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{m} n(\gamma;a_k).$$

□

7.3 Corollary. *Let G be a region and $f : G \to \mathbb{C}$ be analytic. Let $a_1,\dots,a_m$ be points in G satisfying $f(z) = \alpha$ for some $\alpha \in C$. Let γ be a closed rectifiable curve in G not passing through any point a_k such that $\gamma \approx 0$ (i.e. $n(\gamma;w) = 0$ for all $w \in \mathbb{C} \setminus G$), then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\alpha}\,dz = \sum_{k=1}^{m} n(\gamma;a_k).$$

Proof. Same as above. Take $h(z) = f(z) - \alpha$ in Theorem 7.2 in place of $f(z)$. □

Example. Compute $\int_\gamma \frac{2z+1}{z^2+z+1}\,dz$ where γ is the circle $|z| = 2$.

***Solution*.** Take $f(z) = z^2 + z + 1$. Then $f'(z) = 2z + 1$. Now,

$$z^3 - 1 = (z-1)(z^2+z+1) = (z-1)(z-w_1)(z-w_2),$$

where $w_1 = \omega$ and $w_2 = \omega^2$ are the non-real cube roots of unity. Thus, f has two zeros w_1, w_2 inside $\gamma : |z| = 2$ and γ does not pass through them. Also, $n(\gamma; w_1) = 1 = n(\gamma; w_2)$. Then by Counting Zero Principle (Theorem 7.2)

$$\frac{1}{2\pi i}\int\limits_{\gamma} \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int\limits_{|z|=2} \frac{2z+1}{z^2+z+1}\,dz = n(\gamma; w_1) + n(\gamma; w_2) = 2.$$

Therefore,

$$\int_{\gamma} \frac{2z+1}{z^2+z+1}\,dz = 4\pi i.$$

□

Example. Evaluate $\int\limits_{\gamma} \frac{f'(z)}{f(z)}\,dz$ where $f(z) = (z^2+1)^3$ and $\gamma(t) = 2e^{it}$, $0 \le t \le 2\pi$.

***Solution*.** Here $f(z) = (z^2+1)^3 = (z-i)^3(z+i)^3$. Then the zeros of f are $a_1 = a_2 = a_3 = i$, $a_4 = a_5 = a_6 = -i$. Moreover, γ does not pass through any of the a_k, and $n(\gamma; a_k) = 1$ for all $k = 1, 2, \ldots, 6$. Therefore by the Counting Zero Principle,

$$\begin{aligned}\int_{\gamma} \frac{f'(z)}{f(z)}\,dz &= \int_{\gamma} \frac{6z(z^2+1)^2}{(z^2+1)^3}\,dz \\ &= \int_{\gamma} \frac{6z}{z^2+1}\,dz \\ &= 2\pi i \sum_{k=1}^{6} n(\gamma; a_k) \\ &= 2\pi i(6) = 12\pi i.\end{aligned}$$

□

Example. Evaluate $\int\limits_{|z|=2} \frac{z^3}{z^4-1}\,dz$.

***Solution*.** Take $f(z) = z^4 - 1 = (z-1)(z+1)(z-i)(z+i)$. Then $f'(z) = 4z^3$. Note that f has four zeros $a_1 = 1$, $a_2 = -1$, $a_3 = i$, $a_4 = -i$ and $\gamma : |z| = 2$ does not pass through any of them. Also, $n(\gamma; a_k) = 1$ for all $k = 1, 2, 3, 4$. Therefore by Counting Zero Principle,

$$\begin{aligned}\int_{\gamma} \frac{f'(z)}{f(z)}\,dz &= \int_{\gamma} \frac{4z^3}{z^4-1}\,dz \\ &= 2\pi i \sum_{k=1}^{4} n(\gamma; a_k) \\ &= 2\pi i(4) = 8\pi i.\end{aligned}$$

Therefore, $\int\limits_{|z|=2} \frac{z^3}{z^4-1}\,dz = 2\pi i.$ □

Exercise (Exercise IV.7.1)**.** Let G be an open set in $\mathbb{C}$ and $f : G \to \mathbb{C}$ be analytic. If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve in G, then $f \circ \gamma$ is also a rectifiable curve.

Solution. Since γ is a curve, it is a continuous function. Also f being analytic, is continuous. Since $\gamma([0,1]) \subset G$ and f is defined on G, it is clear that $f \circ \gamma : [0,1] \to \mathbb{C}$ is well-defined and continuous. That is, $f \circ \gamma$ is a curve.

Define $\varphi : G \times G \to \mathbb{C}$ by $\varphi(z,w) = \frac{f(z)-f(w)}{z-w}$ for $z \neq w$ and $\varphi(z,z) = f'(z)$. Then we know, by Exercise IV.5.1, that φ is continuous.

Observe that $\{\gamma\} \times \{\gamma\}$ is a compact subset of $G \times G$. Since φ is continuous on compact set $\{\gamma\} \times \{\gamma\}$, it is bounded. That is, there is $M > 0$ such that $|\varphi(z,w)| \leq M$ for all $z, w \in \{\gamma\}$. Then

$$|f(z) - f(w)| \leq M|z-w| \quad \forall\, z, w \in \{\gamma\}.$$

If $\{t_0 < t_1 < \cdots < t_m\}$ is any partition of $[0,1]$, then

$$\begin{aligned}\sum_{k=1}^{m} |f \circ \gamma(t_k) - f \circ \gamma(t_{k-1})| &= \sum_{k=1}^{m} |f(\gamma(t_k)) - f(\gamma(t_{k-1}))| \\ &\leq M \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \\ &\leq MV(\gamma).\end{aligned}$$

Therefore $f \circ \gamma$ is of bounded variation. □

Let G be an open subset of $\mathbb{C}$, $f : G \to \mathbb{C}$ be analytic, and $\gamma : [0,1] \to \mathbb{C}$ be a closed rectifiable curve in G. Then $\sigma = f \circ \gamma$ is also a closed rectifiable curve in $\mathbb{C}$.

Now, we compute $n(\sigma;\alpha)$ for $\alpha \in \mathbb{C}$ such that $\alpha \notin \{\sigma\} = \{f \circ \gamma\}$.

$$\begin{aligned} n(\sigma;\alpha) &= \frac{1}{2\pi i} \int_{\sigma} \frac{dw}{w-\alpha} \\ &= \frac{1}{2\pi i} \int_{0}^{1} \frac{d(f \circ \gamma(t))}{(f \circ \gamma)(t) - \alpha} \\ &= \frac{1}{2\pi i} \int_{0}^{1} \frac{f'(\gamma(t))d\gamma(t)}{f(\gamma(t)) - \alpha} && (\because f \text{ is analytic}) \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)-\alpha}\, dz \\ &= \sum_{k=1}^{m} n(\gamma; a_k), \end{aligned}$$

where $a_1, a_2, \ldots, a_m$ are zeros of $f(z) - \alpha$.

The following result is very useful in proving the open mapping theorem. It is called the *Stability Theorem for the Orders of Zeros of Equations*.

7.4 Theorem. *Suppose f is analytic in $B(a,R)$ and let $\alpha = f(a)$. If $f(z)-\alpha$ has a zero of order m at $z=a$ then there is an $\varepsilon > 0$ and $\delta > 0$ such that for $|\zeta - \alpha| < \delta$, the equation $f(z) = \zeta$ has exactly m simple roots in $B(a,\varepsilon)$.*

Note that a simple root of $f(z) = \zeta$ is a zero of $f(z)-\zeta$ of order 1 (i.e. of multiplicity 1). From the above theorem it follows that if $\zeta \in B(\alpha,\delta)$, then there is $z_0 \in B(a,\varepsilon)$ such that $f(z_0) = \zeta$. That is,

$$B(\alpha,\delta) \subset f(B(a,\varepsilon)).$$

Also, $f(z)-\alpha$ has a zero of finite multiplicity guarantees that f is non-constant.

Proof. Since (by Corollary 3.10) zeros of an analytic function are isolated, there is an $\varepsilon > 0$ such that

- $\varepsilon < \frac{R}{2}$ (we can always choose a smaller neighborhood).
- $f(z) = \alpha$ has no solution in $0 < |z-a| < 2\varepsilon$ (since zeros of $f(z)-\alpha$ are isolated).
- $f'(z) \neq 0$ for $0 < |z-a| < 2\varepsilon$.
  (If $m=1$, then $f'(a) \neq 0$ and since f' is continuous, it is non-zero in some neighborhood of a. If $m \geq 2$, then $f'(a) = 0$ and since the analytic function f' has isolated zeros, there is a neighborhood of a in which it is non-zero).

Let $\gamma(t) = a + \varepsilon e^{2\pi i t}$, $0 \leq t \leq 1$, and put $\sigma = f \circ \gamma$. Then, by Exercise IV.7.1, σ is rectifiable and since $f(z)-\alpha \neq 0$ on $\{\gamma\}$, it follows that $\alpha \notin \{\sigma\} = \{f \circ \gamma\}$. Therefore there is $\delta > 0$ such that

$$B(\alpha,\delta) \cap \{\sigma\} = \emptyset.$$

Thus, $B(\alpha,\delta)$ is contained in some component of $\mathbb{C} \setminus \{\sigma\}$. So if $\zeta \in \mathbb{C}$ such that $|\alpha - \zeta| < \delta$, then by Theorem 4.4 (since α and ζ will be in the same component), we have

$$n(\sigma;\alpha) = n(\sigma;\zeta).$$

Now we compute and compare $n(\sigma;\alpha)$ and $n(\sigma;\zeta)$.

$$\begin{aligned} n(\sigma;\alpha) &= \frac{1}{2\pi i} \int_\sigma \frac{dw}{w-\alpha} \\ &= \frac{1}{2\pi i} \int_0^1 \frac{d(f \circ \gamma(t))}{(f \circ \gamma)(t) - \alpha} \\ &= \frac{1}{2\pi i} \int_0^1 \frac{f'(\gamma(t))\, d\gamma(t)}{f(\gamma(t)) - \alpha} \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)-\alpha}\, dz \end{aligned}$$

$$= \sum_{k=1}^{m} n(\gamma; a_k),$$

where $a_1, a_2, \ldots, a_m$ are the points interior to γ in G that satisfy $f(z) = \alpha$. Since γ is the circle $|z-a| = \varepsilon$, $n(\gamma; a_k) = 1$ for all $k = 1, 2, \ldots, m$. Therefore,

$$n(\sigma; \alpha) = m.$$

On the other hand, (computing as above)

$$\begin{aligned} n(\sigma; \zeta) &= \frac{1}{2\pi i} \int_{\sigma} \frac{dw}{w-\zeta} \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)-\zeta} dz \\ &= \sum_{k=1}^{p} n(\gamma; z_k(\zeta)) \end{aligned}$$

for some p, where $z_k(\zeta)$ are the points that satisfy $f(z) = \zeta$. Therefore,

$$m = n(\sigma; \alpha) = n(\sigma; \zeta) = \sum_{k=1}^{p} n(\gamma; z_k(\zeta)).$$

Since γ is positively oriented circle $|z-a| = \varepsilon$,

$$n(\gamma, z_k(\zeta)) = 0 \text{ or } 1.$$

Hence, there are m such points $z_k(\zeta)$ inside γ i.e., in $B(a, \varepsilon)$.

Since $f'(z) \neq 0$ for $0 < |z-a| < 2\varepsilon$, it follows that $z_k(\zeta)$ is a simple root of $f(z) - \zeta$ (since if $z_k(\zeta)$ has order $m \geq 2$, then $f'(z_k(\zeta)) = 0$). □

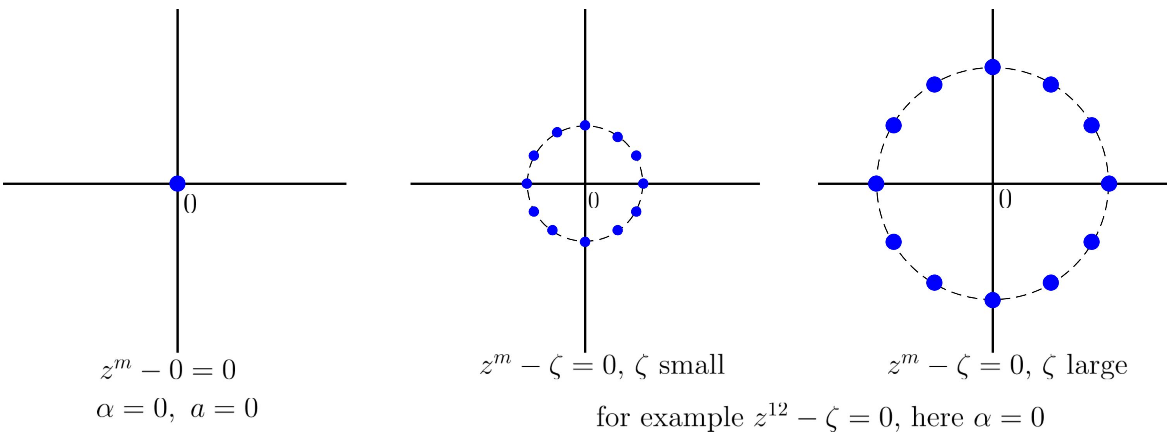


Figure IV.3: Example demonstrating zeros of $f(z) = \alpha$, where $f(z) = z^m$, $a = 0$, $\alpha = 0$ as described in the above theorem.

Now we prove the Open Mapping Theorem which is a consequence of the Stability Theorem (above theorem).

7.5 Open Mapping Theorem. *Let G be a region and suppose that f is a non-constant analytic function on G. Then for any open set U in G, $f(U)$ is open.*

Proof. Let $U \subset G$ be open and $\alpha \in f(U)$. Then there is $a \in U$ such that $f(a) = \alpha$. Since U is open and $a \in U$, there is an $\varepsilon > 0$ such that

$$B(a,\varepsilon) \subset U.$$

This implies,

$$f(B(a,\varepsilon)) \subset f(U).$$

But by the above theorem (7.4), this $\varepsilon > 0$ can be chosen with $\delta > 0$ such that $B(\alpha,\delta) \subset f(B(a,\varepsilon))$. Then

$$B(\alpha,\delta) \subset f(U).$$

Therefore, $f(U)$ is open. □

Remark. Such a map, i.e., a function f which maps open sets to open sets is called an *open map*.

7.6 Corollary. *Suppose $f : G \to \mathbb{C}$ is one-one, analytic and $f(G) = \Omega$. Then $f^{-1} : \Omega \to \mathbb{C}$ is analytic and $(f^{-1})'(w) = [f'(z)]^{-1}$ where $w = f(z)$.*

Chapter V

Singularities

§1. Classification of singularities

1.1 Definition. A function f has an *isolated singularity* at $z = a$ if there is an $R > 0$ such that f is defined and analytic in $B(a,R) \setminus \{a\}$ but not in $B(a,R)$, i.e. not analytic at a.

An isolated singularity a of f is called a *removable singularity* if there is analytic function $g : B(a,R) \to \mathbb{C}$ such that $g(z) = f(z)$ for $0 < |z-a| < R$.

The functions $\frac{\sin z}{z}$, $\frac{1}{z}$, and $\exp \frac{1}{z}$ all have isolated singularities at $z = 0$. However, only $\frac{\sin z}{z}$ has a removable singularity at $z = 0$

How can we determine if an isolated singularity is removable or not? The following result gives a criterion to determine whether an isolated singularity is removable.

1.2 Theorem. *If f has an isolated singularity at a, then the point $z = a$ is a removable singularity if and only if*

$$\lim_{z\to a}(z-a)f(z) = 0.$$

Proof. ($\Rightarrow$) Suppose the isolated singularity $z = a$ of f is a removable singularity. Then by definition, there is an analytic function $g : B(a,R) \to \mathbb{C}$ such that $g(z) = f(z)$ for $0 < |z-a| < R$. Since g is continuous at a (being analytic at a), $\lim_{z\to a} g(z)$ exists and hence the $\lim_{z\to a}(z-a)g(z)$ exists and we have

$$\lim_{z\to a}(z-a)f(z) = \lim_{z\to a}(z-a)g(z) = 0.$$

($\Leftarrow$) Conversely, suppose that f has an isolated singularity at $z = a$ and $\lim_{z\to a}(z-a)f(z) = 0$. By definition of isolated singularity, f is analytic in $B(a,R) \setminus \{a\}$ for some $R > 0$. Define $g : B(a,R) \to \mathbb{C}$ by

$$g(z) = \begin{cases} (z-a)f(z) & \text{for } z \neq a \\ 0 & \text{for } z = a. \end{cases}$$

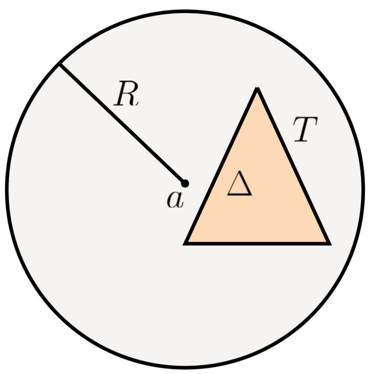
Since $\lim_{z\to a}(z-a)f(z) = 0$, the function g is continuous on $B(a,R)$ and clearly g is analytic in $B(a,R) \setminus \{a\}$ (since f is analytic there).

We shall show that g is analytic in $B(a,R)$ by applying Morera's theorem. Let T be a triangle in $B(a,R)$ and let Δ be the inside of T together with T.

Case-I $a \notin \Delta$.

Then $n(T;w) = 0$ for all $w \in \mathbb{C} \setminus B(a,R)$ and $n(T;a) = 0$. Since g is analytic in open set $G = B(a,R) \setminus \{a\}$ and $n(T;w) = 0$ for all $w \in \mathbb{C} \setminus G$, by Cauchy's Theorem,

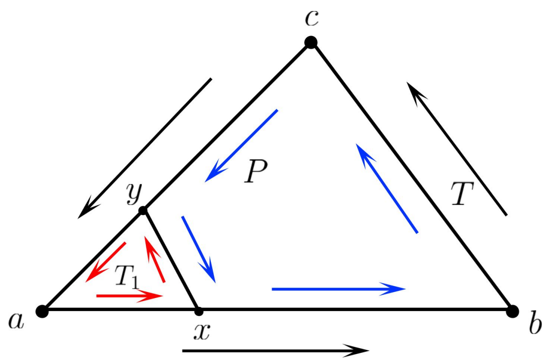
$$\int_T g = 0.$$



Case-II a is a vertex of T.

Then $T = [a,b,c,a]$. Let $x \in [a,b]$ and $y \in [c,a]$ and form the triangle $T_1 = [a,x,y,a]$. If P is the polygon $[x,b,c,y,x]$, then

$$\int_T g = \int_{T_1} g + \int_P g.$$



Also, $n(P;w) = 0$ for all $w \in \mathbb{C} \setminus B(a,R)$ and $n(P;a) = 0$. Since g is analytic in open set $G = B(a,R) \setminus \{a\}$ and $n(P;w) = 0$ for all $w \in \mathbb{C} \setminus G$, by Cauchy's theorem, $\int\limits_P g = 0$. Therefore,

$$\int_T g = \int_{T_1} g.$$

Since g is continuous at a and $g(a) = 0$, given $\varepsilon > 0$ there is $\delta > 0$ such that $|g(z) - g(a)| = |g(z)| < \frac{\varepsilon}{\ell(T)}$ whenever $|z-a| < \delta$, where $\ell(T)$ denotes the length of T. Choose $x \in [a,b]$ and $y \in [a,c]$ such that $|x-a| < \delta$ and $|y-a| < \delta$. Then for any $z \in T_1$, $|z-a| < \delta$ and so

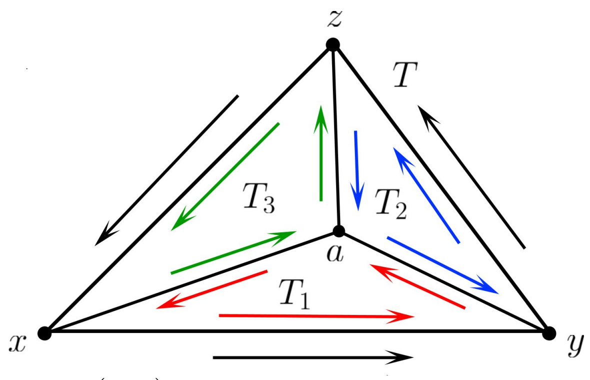
$$\left|\int_T g\right| = \left|\int_{T_1} g(z)\,dz\right| \leq \int_{T_1} |g(z)||dz| \leq \frac{\varepsilon}{\ell(T)}\ell(T_1) < \varepsilon.$$

Hence, $\int\limits_T g = 0$.

Case-III $a \in \Delta$ and $T = [x,y,z,x]$.

Consider the triangles $T_1 = [x,y,a,x]$, $T_2 = [y,z,a,y]$, and $T_3 = [z,x,a,z]$ all having a as their vertex. Then by above case, $\int\limits_{T_1} g = \int\limits_{T_2} g = \int\limits_{T_3} g = 0$. So

$$\int_T g = \int_{T_1} g + \int_{T_2} g + \int_{T_3} g = 0.$$



Hence by Morera's Theorem, g must be analytic in $B(a,R)$.

Since $g(a) = 0$, there is an analytic function $h : B(a,R) \to \mathbb{C}$ such that $g(z) = (z-a)h(z)$ for all $z \in B(a,R)$. But then $f(z) = h(z)$ for all $0 < |z-a| < R$ and hence by definition (of removable singularity), f has a removable singularity at $z = a$. □

1.3 Definition. If $z = a$ is an isolated singularity of f then a is a *pole* of f if $\lim\limits_{z \to a} |f(z)| = \infty$, i.e., for any $M > 0$ there is $\delta > 0$ such that $|f(z)| \geq M$ whenever $0 < |z-a| < \delta$. (This means that $f(z) \neq 0$ in some deleted neighborhood of a).

If an isolated singularity is neither a pole nor a removable singularity, then it is called an *essential singularity*.

For example, $(z-a)^{-m}$ has a pole at $z=a$ for $m \geq 1$ and $\exp(z^{-1})$ has an essential singularity at $z=0$.

1.4 Proposition. *If G is a region with a in G and if f is analytic on $G \setminus \{a\}$ with a pole at $z=a$ then there is a positive integer m and an analytic function $g : G \to \mathbb{C}$ such that*

1.5 $$f(z) = \frac{g(z)}{(z-a)^m}.$$

Proof. First we show that if f has a pole at $z=a$, then $\frac{1}{f}$ has a removable singularity at $z=a$ (by applying above theorem).

Since a is a pole of f, by definition, $\lim\limits_{z\to a} |f(z)| = \infty$. This implies that $\lim\limits_{z\to a} \frac{1}{f(z)} = 0$ and hence $\lim\limits_{z\to a} (z-a)\frac{1}{f(z)} = 0$. Since $z=a$ is a pole (and hence isolated singularity) of f, f is analytic in some deleted neighborhood of a. Also $\lim\limits_{z\to a} |f(z)| = \infty$ implies $f(z) \neq 0$ in $B(a,R) \setminus \{a\}$ for some $R>0$. Hence, $\frac{1}{f}$ is analytic in $B(a,R) \setminus \{a\}$. Since $\lim\limits_{z\to a} (z-a)\frac{1}{f(z)} = 0$, by Theorem 1.2, $\frac{1}{f}$ has a removable singularity at $z=a$. Then by definition of removable singularity, there is an analytic function $h : B(a,R) \to \mathbb{C}$ such that $h(z) = \frac{1}{f(z)}$ for $0 < |z-a| < R$. Since h is continuous at a,

$$h(a) = \lim_{z\to a} h(z) = \lim_{z\to a} \frac{1}{f(z)} = 0.$$

Since $h(a) = 0$, by Corollary IV.3.9, there is a integer $m \geq 1$ and an analytic function h_1 in $B(a,R)$ such that $h(z) = (z-a)^m h_1(z)$ and $h_1(a) \neq 0$. Then for $z \neq a$,

$$\frac{1}{h_1(z)} = (z-a)^m \frac{1}{h(z)} = (z-a)^m f(z).$$

Since $h_1(a) \neq 0$ and h_1 is continuous (being analytic) at a, there is a $0 < \delta < R$ such that $h_1(z) \neq 0$ for all $z \in B(a,\delta)$. Therefore, $f(z) = \frac{\frac{1}{h_1(z)}}{(z-a)^m}$ for all $z \in B(a,\delta) \setminus \{a\}$. Define $g : G \to \mathbb{C}$ by

$$g(z) = \begin{cases} \frac{1}{h_1(z)} & z \in B(a,\delta) \\ (z-a)^m f(z) & z \in G \setminus B(a,\delta). \end{cases}$$

Then g is analytic on G (since $\frac{1}{h_1}$ is analytic in $B(a,\delta)$ because h_1 is non-zero there) and $g(z) = (z-a)^m f(z)$ for all $z \in G \setminus \{a\}$. Therefore for all $z \in G \setminus \{a\}$,

$$f(z) = \frac{g(z)}{(z-a)^m}.$$

□

1.6 Definition. If f has a pole at $z=a$ and m is the smallest positive integers such that $f(z)(z-a)^m$ has a removable singularity at $z=a$, then f has a *pole of order m* at $z=a$.

Note that if m is the order of the pole at $z = a$ and g is an analytic function satisfying (1.5), then $g(a) \neq 0$ (since m is the smallest integer that satisfies (1.5)).

Let f have a pole of order m at $z = a$ and put $f(z) = g(z)(z-a)^{-m}$. Since g is analytic in a disk $B(a,R)$, it has a power series expansion about a. For $z \in B(a,R)$, let

$$\begin{aligned} g(z) &= \sum_{k=0}^{\infty} b_k (z-a)^k \\ &= b_0 + b_1(z-a) + \cdots + b_{m-1}(z-a)^{m-1} + b_m(z-a)^m + b_{m+1}(z-a)^{m+1} \cdots, \end{aligned}$$

where $b_0 \neq 0$ (as $g(a) \neq 0$). Relabeling the coefficients ($b_0 = A_m, \dots b_{m-1} = A_1$, $b_m = a_0$, $b_{m+1} = a_1, \dots$) , we write

$$g(z) = A_m + A_{m-1}(z-a) + \cdots + A_1(z-a)^{m-1} + (z-a)^m \sum_{k=0}^{\infty} a_k (z-a)^k.$$

Hence,

1.7 $$f(z) = \frac{A_m}{(z-a)^m} + \frac{A_{m-1}}{(z-a)^{m-1}} + \cdots + \frac{A_1}{(z-a)} + g_1(z),$$

where $g_1(z)$ is an analytic in $B(a,R)$ given by the power series $\sum_{k=0}^{\infty} a_k(z-a)^k$ and $A_m \neq 0$.

Thus, if m is the order of the pole of f at $z = a$, then the coefficient of $(z-a)^{-m}$ is nonzero and all other coefficients of $(z-a)^{-n}$ for $n > m$ are zero. An expansion of f given in (1.7) is called Laurent series expansion of f. We state the following result about the Laurent Expansion of f about a when f has a pole of order m at $z = a$ but before that we fix a notation.

Notation. If $0 \leq R_1 < R_2 \leq \infty$ and a is a complex number, define

$$\text{ann}\,(a; R_1, R_2) = \{z \in \mathbb{C} \mid R_1 < |z-a| < R_2\}.$$

Notice that ann $(a; 0, R_2)$ is a punctured disk, i.e. deleted neighborhood of 0.

Also if A is any set, then A^- denotes $\overline{A}$, the closure of A.

1.18 Corollary. *Let $z = a$ be an isolated singularity of f and let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n$ be its Laurent expansion in* ann$\,(a; 0, R)$*. Then:*

(a)

(b) $z = a$ is a pole of order m if and only if $a_{-m} \neq 0$ and $a_n = 0$ for $n \leq -(m+1)$.

(c)

We know that f has an essential singularity at $z = a$ when $\lim_{z \to a} f(z)$ fails to exists (where existing includes the possibility that limit is infinity). This means that as z approaches a, the values of $f(z)$ wanders through $\mathbb{C}$. The following result says that not only $f(z)$ wanders through $\mathbb{C}$, as z approaches a, but $f(z)$ comes arbitrarily close to every complex number. A result due to Picard (Picard's great theorem) says that if f has an essential singularity at $z=a$, then on a deleted neighborhood of a, f takes on all possible complex values, with at most one exception, infinitely many times.

1.21 Casorati-Weierstrass Theorem. *If f has an essential singularity at $z=a$, then for every $\delta>0$, $\{f[\operatorname{ann}(a;0,\delta)]\}^- = \mathbb{C}$.*

In other words, an isolated singularity $z=a$ of f is an essential singularity, then the set $\{f(z) \mid 0<|z-a|<\delta\}$ is dense in $\mathbb{C}$, i.e.

$$\overline{f(B(a,\delta)\smallsetminus\{a\})} = \mathbb{C}.$$

Proof. Suppose f is analytic in ann $(a;0,R)$. We have to show that given any $c\in\mathbb{C}$ and any $\varepsilon>0$, for each $\delta>0$, we can find a $z\in B(a,\delta)\smallsetminus\{a\}$ such that $|f(z)-c|<\varepsilon$.

Suppose this is not the case. Then there is a $c\in\mathbb{C}$ and an $\varepsilon>0$ such that $|f(z)-c|\geq\varepsilon$ for all $z\in G=$ ann $(a;0,\delta)$, for some $\delta>0$. Thus,

$$\lim_{z\to a}\left|\frac{f(z)-c}{z-a}\right| \geq \lim_{z\to a}\frac{\varepsilon}{|z-a|} = \infty,$$

which implies that the function $\frac{f(z)-c}{z-a}$ has a pole at $z=a$. If $m\geq 1$ is the order of the pole (which we know exists by Proposition 1.4), then (by (1.7) and Corollary 1.18 (b))

$$\begin{aligned} &\lim_{z\to a}|z-a|^{m+1}\left|\frac{f(z)-c}{z-a}\right| = 0 \\ \Rightarrow\ &\lim_{z\to a}|z-a|^{m}|f(z)-c| = 0. \\ \Rightarrow\ &\lim_{z\to a}|z-a|^{m+1}|f(z)-c| = 0. \end{aligned}$$

Now,

$$|z-a|^{m+1}|f(z)-c| \geq |z-a|^{m+1}|f(z)| - |z-a|^{m+1}|c|$$

or

$$|z-a|^{m+1}|f(z)| \leq |z-a|^{m+1}|f(z)-c| + |z-a|^{m+1}|c|.$$

Therefore,

$$\begin{aligned} \lim_{z\to a}|z-a|^{m+1}|f(z)| &\leq \lim_{z\to a}|z-a|^{m+1}|f(z)-c| + \lim_{z\to a}|z-a|^{m+1}|c| \\ &= 0+0=0. \end{aligned}$$

Since $z=a$ is an isolated singularity of f and $\lim\limits_{z\to a}(z-a)^{m+1}f(z)=0$ $(m\geq 1)$, by Theorem 1.2 it follows that $f(z)(z-a)^m$ has a removable singularity at $z=a$. This is contradiction to the hypothesis that $z=a$ is an essential singularity of f (as $z=a$ will either be a removable singularity or a pole of f). □

3

Unit 3

§3. The Argument Principle

Suppose that f is analytic and has a zero of order m at $z=a$. So $f(z)=(z-a)^m g(z)$, where g is analytic and $g(a)\neq 0$. Then $f'(z)=m(z-a)^{m-1}g(z)+(z-a)^m g'(z)$. Hence,

3.1 $$\frac{f'(z)}{f(z)}=\frac{m}{z-a}+\frac{g'(z)}{g(z)}$$

and $\frac{g'}{g}$ is analytic near $z=a$ since $g(a)\neq 0$. Now suppose that f has a pole of order m at $z=a$, i.e., $f(z)=(z-a)^{-m}g(z)$, where g is analytic and $g(a)\neq 0$. Then $f'(z)=-m(z-a)^{-m-1}g(z)+(z-a)^{-m}g'(z)$ and so

3.2 $$\frac{f'(z)}{f(z)}=\frac{-m}{z-a}+\frac{g'(z)}{g(z)}$$

and $\frac{g'}{g}$ is analytic near $z=a$ as $g(a)\neq 0$.

3.3 Definition. Let G be an open subset of $\mathbb{C}$. A function f defined and analytic in G except for poles is called a *meromorphic function* on G.

The following result is an extension of the Counting Zero Principle for meromorphic functions (i.e. for the functions which are analytic except for the poles in their domain).

3.4 Argument Principle. *Let f be meromorphic in G with poles $p_1,p_2,\dots,p_m$ and zeros $z_1,z_2,\dots,z_n$ counted according to multiplicity. If γ is a closed rectifiable curve in G with $n(\gamma;w)=0$ for all $w\in\mathbb{C}\setminus G$ and not passing through $p_1,\dots,p_m$; $z_1,\dots,z_n$; then*

3.5 $$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz=\sum_{k=1}^{n} n(\gamma;z_k)-\sum_{j=1}^{m} n(\gamma;p_j).$$

Proof. Since $p_1,\dots,p_m$ are the poles of f and $z_1,\dots,z_n$ are the zeros of f counted according to their multiplicities, there is an analytic function $g : G \to \mathbb{C}$ such that

$$f(z) = \frac{(z-z_1)\cdots(z-z_n)}{(z-p_1)\cdots(z-p_m)}g(z)$$

for all $z \in G \smallsetminus \{p_1,\dots,p_m\}$ and $g(z) \neq 0$ for all $z \in G$. Observe that (by applying (3.1) and (3.2))

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{n}\frac{1}{z-a_k} - \sum_{j=1}^{m}\frac{1}{z-p_j} + \frac{g'(z)}{g(z)} \qquad (\forall\, z \in \{\gamma\}).$$

Since g is analytic and non-vanishing in G, the function $\frac{g'}{g}$ is analytic in G. Since $n(\gamma;w) = 0$ for all $w \in \mathbb{C} \smallsetminus G$, by Cauchy's theorem $\int\limits_{\gamma} \frac{g'}{g} = 0$. Therefore, by the definition of index, we have

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f'(z)}{f(z)}dz &= \frac{1}{2\pi i}\int\limits_{\gamma}\left(\sum_{k=1}^{n}\frac{1}{z-a_k} - \sum_{j=1}^{m}\frac{1}{z-p_j}\right)dz \\ &= \sum_{k=1}^{n} n(\gamma;z_k) - \sum_{j=1}^{m} n(\gamma;p_j).\end{aligned}$$

□

Why is it called the "Argument Principle"?

Since no zero or pole of f lies on $\{\gamma\}$, for each $a \in \{\gamma\}$ there is an open disk $B(a,r)$ such that $f(z) \neq 0$ or ∞ in $B(a,r)$. Then a branch of $\log f(z)$ can be defined on $B(a,r)$. These balls form an open cover of $\{\gamma\}$.

Recall that Lebesgue's Covering Lemma says that:

"If the metric space (X,d) is compact and an open cover of X is given, then there exists a number $\delta > 0$ such that every subset of X having diameter less than δ is contained in some member of the cover".

Thus, by Lebesgue's Covering Lemma there is $\varepsilon > 0$ such that $B(a,\varepsilon) \subset B(a,r)$ for some member $B(a,r)$ of the open cover of $\{\gamma\}$. Hence, for each a in $\{\gamma\}$ we can define a branch of $\log f(z)$ on $B(a,\varepsilon)$. Suppose γ is defined on $[0,1]$. Then γ is uniformly continuous and so for above $\varepsilon > 0$ there is $\delta > 0$ such that for all $s,t \in [0,1]$,

$$|\gamma(s) - \gamma(t)| < \varepsilon \text{ whenever } |s-t| < \delta.$$

Thus there is a partition $P = \{0 = t_0 < t_1 < \cdots < t_k = 1\}$ with $\|P\| < \delta$ such that for all $1 \le j \le k$,

$$\gamma(t) \in B(\gamma(t_{j-1}),\varepsilon) \text{ for } t_{j-1} \le t \le t_j.$$

Let ℓ_j be the branch of $\log f$ on $B(\gamma(t_{j-1}),\varepsilon)$ for $1 \le j \le k$. Note that

$$\gamma(t_j) \in B(\gamma(t_{j-1}),\varepsilon) \cap B(\gamma(t_j),\varepsilon).$$

Then we can choose $\ell_1, \ldots, \ell_k$ so that

$$\ell_1(\gamma(t_1)) = \ell_2(\gamma(t_1));\ \ell_2(\gamma(t_2)) = \ell_3(\gamma(t_2));\ \ldots;\ \ell_{k-1}(\gamma(t_{k-1})) = \ell_k(\gamma(t_{k-1})).$$

If γ_j is the path γ restricted to $[t_{j-1}, t_j]$ then, since ℓ_j is branch of $\log f$ we have $\ell_j' = \frac{f'}{f}$ and so

$$\int_{\gamma_j} \frac{f'}{f} = \ell_j(\gamma(t_j)) - \ell_j(\gamma(t_{j-1}))$$

for $1 \le j \le k$. Summing both sides of the above equation and simplifying the right hand side, we get

$$\int_{\gamma} \frac{f'}{f} = \ell_k(a) - \ell_1(a),$$

where $a = \gamma(0) = \gamma(1)$. That is, $\ell_k(a) - \ell_1(a) = 2\pi i K$. Because $2\pi i K$ is purely imaginary, we get $\operatorname{Im}(\ell_k(a)) - \operatorname{Im}(\ell_1(a)) = 2\pi K$. We know that $\log f(z) = \log|f(z)| + i(\arg f(z))$.

Thus, Argument principle says that as z traces out γ, $\arg f(z)$ (which is imaginary part of $\log f(z)$) changes by $2\pi K$. This K is the integer on the right hand side of (3.5).

The following result is a generalization of the Argument Principle.

3.6 Theorem. *Let f be meromorphic in the region G with zeros $z_1, z_2, \ldots, z_n$ and poles $p_1, p_2, \ldots, p_m$ counted according to multiplicity. If g is analytic in G and γ is a closed rectifiable curve in G with $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$ and not passing through any z_i or p_j; then*

$$\frac{1}{2\pi i} \int_{\gamma} g \frac{f'}{f} dz = \sum_{i=1}^{n} g(z_i) n(\gamma; z_i) - \sum_{j=1}^{m} g(p_j) n(\gamma; p_j).$$

Proof. Since $z_1, \ldots, z_n$ are the zeros of f and $p_1, \ldots, p_m$ are the poles of f counted according to their multiplicities, there is an analytic function $h : G \to \mathbb{C}$ such that

$$f(z) = \frac{(z - z_1) \cdots (z - z_n)}{(z - p_1) \cdots (z - p_m)} h(z)$$

for all $z \in G \setminus \{p_1, \ldots, p_m\}$ and $h(z) \neq 0$ for all $z \in G$. Observe that

$$\frac{f'(z)}{f(z)} = \sum_{i=1}^{n} \frac{1}{z - z_i} - \sum_{j=1}^{m} \frac{1}{z - p_j} + \frac{h'(z)}{h(z)} \qquad (\forall\, z \in \{\gamma\})$$

$$\Rightarrow g(z) \frac{f'(z)}{f(z)} = \sum_{i=1}^{n} \frac{g(z)}{z - z_i} - \sum_{j=1}^{m} \frac{g(z)}{z - p_j} + g(z) \frac{h'(z)}{h(z)} \qquad (\forall\, z \in \{\gamma\}).$$

Since h is analytic and non-vanishing in G, the function $\frac{h'}{h}$ is analytic in G. Since g is analytic in G, $g \frac{h'}{h}$ is also analytic in G and since $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$, by Cauchy's theorem

$\int\limits_\gamma g\frac{h'}{h} = 0$. Therefore, by the definition of index and Cauchy's integral formula, we have

$$\frac{1}{2\pi i}\int\limits_\gamma g(z)\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int\limits_\gamma \left(\sum_{i=1}^{n}\frac{g(z)}{z-z_i} - \sum_{j=1}^{m}\frac{g(z)}{z-p_j}\right)dz$$
$$= \sum_{i=1}^{n} g(z_i)n(\gamma;z_i) - \sum_{j=1}^{m} g(p_j)n(\gamma;p_j).$$

□

Example. Evaluate $\int_{|z|=3}\frac{f'(z)}{f(z)}\,dz$ if $f(z) = \frac{(z^2+1)^2}{(z^2+3z+2)^3}$.

Solution. Note that the zeros of f are $i, i, -i, -i$ and the poles of f are $-1, -1, -1, -2, -2, -2$ counted according to their multiplicities all of which are inside $\gamma : |z| = 3$. Also $n(\gamma; z) = 1$ for every zero and pole of f. Thus from argument principle it follows that $\int_{|z|=3}\frac{f'}{f} = -4\pi i$. □

As an application of the above theorem, we get a formula for calculating the inverse of an one-one analytic function.

3.7 Proposition. *Let f be analytic on an open set containing $\bar{B}(a,R)$ and suppose that f is one-one on $B(a,R)$. If $\Omega = f[B(a,R)]$ and γ is the circle $|z-a| = R$ then $f^{-1}(w)$ is defined for each $w \in \Omega$ by the formula*

$$f^{-1}(w) = \frac{1}{2\pi i}\int\limits_\gamma \frac{zf'(z)}{f(z)-w}dz.$$

Proof. Let $w \in \Omega = f[B(a,R)]$. Then there is $z_0 \in B(a,R)$ such that $f(z_0) = w$. Since f is one-one in $B(a,R)$, the function $f(z)-w$ is also one-one in $B(a,R)$ and so has one and only one zero in $B(a,R)$ which is z_0. Since z_0 is the unique zero of $f(z)-w$ and $f(z_0) = w$, we can denote z_0 by $f^{-1}(w)$. Also note that $n(\gamma; f^{-1}(w)) = 1$. Taking $g(z) = z$ and $f(z)-w$ in place of $f(z)$ by Theorem 3.6, we get

$$\frac{1}{2\pi i}\int\limits_\gamma g(z)\frac{(f(z)-w)'}{f(z)-w}dz = g(f^{-1}(w))n(\gamma; f^{-1}(w))$$

or

$$f^{-1}(w) = \frac{1}{2\pi i}\int\limits_\gamma \frac{zf'(z)}{f(z)-w}dz.$$

□

3.8 Rouché's Theorem. *Suppose f and g are meromorphic in a neighborhood of $\bar{B}(a,R)$ with no zeros or poles on the circle $\gamma = \{z : |z-a| = R\}$. If Z_f, Z_g (P_f, P_g) are the number of zeros (poles) of f and g inside γ counted according to their multiplicities and if*

$$|f(z)+g(z)| < |f(z)|+|g(z)|$$

on γ, then

$$Z_f - P_f = Z_g - P_g.$$

Proof. Since g has no zeros on γ, by hypothesis, we can write

$$\left|\frac{f(z)}{g(z)}+1\right| < \left|\frac{f(z)}{g(z)}\right|+1$$

on γ. If $\lambda = \frac{f(z)}{g(z)}$ for some z on γ and if λ is a positive real number then the above inequality becomes $\lambda+1<\lambda+1$ which is a contradiction. Also since $f(z)\neq 0$ on γ, $\lambda\neq 0$. Therefore the meromorphic function f/g maps γ onto $\Omega=\mathbb{C}\setminus[0,\infty)$. So there is a branch of the logarithm defined on Ω, say ℓ. Then $\ell(f/g)$ is a well-defined primitive for $(f/g)'(f/g)^{-1}$ in a neighborhood of γ. Thus,

$$\begin{aligned}
0 &= \frac{1}{2\pi i}\int_\gamma \frac{(f/g)'}{f/g} && \text{(by Corollary IV.1.22)}\\
&= \frac{1}{2\pi i}\int_\gamma \frac{g}{f}\,\frac{f'g-fg'}{g^2}\\
&= \frac{1}{2\pi i}\int_\gamma \left[\frac{f'}{f}-\frac{g'}{g}\right]\\
&= \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} - \frac{1}{2\pi i}\int_\gamma \frac{g'}{g}\\
&= (Z_f-P_f)-(Z_g-P_g) && \text{(by the Argument Principle).}
\end{aligned}$$

□

This statement of Rouché's Theorem was discovered by Irving Glicksberg (*A*mer. Math. Monthly, **83** (1976), 186-187). In the more classical statements of the theorem, f and g are assumed to satisfy $|f+g|<|g|$ on γ. As an application of a weaker version we can deduce Fundamental Theorem of Algebra which is given below.

Corollary. *Fundamental Theorem of Algebra can be deduced from Rouché's theorem.*

Proof. Let $p(z)$ be a polynomial of degree $n\geq 1$. We may assume that (dividing all other coefficients by non-zero leading coefficient) $p(z)=z^n+a_1z^{n-1}+\cdots+a_n$. Then

$$\frac{p(z)}{z^n}=1+\frac{a_1}{z}+\cdots+\frac{a_n}{z^n}$$

and this approaches 1 as z goes to infinity. So (for $\varepsilon=1$) there is a sufficiently large r with

$$\left|\frac{p(z)}{z^n}-1\right|<1 \quad\text{whenever}\quad |z|\geq r.$$

So for any $R>r$, we have

$$\left|\frac{p(z)}{z^n}-1\right|<1$$

for $|z|=R$, i.e., $|p(z)-z^n|<|z|^n$ for $|z|=R$. Taking $f=p(z)$ and $g(z)=z^n$, by Rouché's theorem, we have

$$Z_f-P_f=Z_g-P_g.$$

Since $p(z)$ and z^n are analytic, they have no poles and so $P_f = P_g = 0$. Then we have $Z_f = Z_g$, i.e. the number of zeros of $p(z)$ and z^n inside $|z| = R$ are same. Since z^n has exactly n zeros inside $|z| = R$, it follows that $p(z)$ has n zeros inside $|z| = R$. □

Chapter VI

The Maximum Modulus Theorem

§1. The Maximum Principle

1.1 Maximum Modulus Theorem — First Version. *If f is analytic in a region G and a is a point in G with $|f(a)| \geq |f(z)|$ all z in G then f must be a constant function.*

1.2 Maximum Modulus Theorem — Second Version. *Let G be a bounded open set in $\mathbb{C}$ and suppose f is continuous function on G^- which is analytic in G. Then*

$$\max\{|f(z)| : z \in G^-\} = \max\{|f(z)| : z \in \partial G\}.$$

1.3 Definition. If $f : G \to \mathbb{R}$ and $a \in G^-$ or $a = \infty$, then the *limit superior* of $f(z)$ as z approaches a, denoted by $\limsup_{z\to a} f(z)$, is defined by

$$\limsup_{z\to a} f(z) = \lim_{r\to 0+} \sup\{f(z) : z \in G \cap B(a,r)\}.$$

(If $a = \infty$, then $B(a,r)$ is the ball in the metric $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$). Similarly, the *limit inferior* of $f(z)$ as z approaches a, denoted by $\liminf_{z\to a} f(z)$, is defined as

$$\liminf_{z\to a} f(z) = \lim_{r\to 0+} \inf\{f(z) : z \in G \cap B(a,r)\}.$$

It is easy to see that $\lim_{z\to a} f(z)$ exists and equals α if and only if $\alpha = \limsup\limits_{z\to a} f(z) = \liminf\limits_{z\to a} f(z)$.

If $G \subset \mathbb{C}$ then let $\partial_\infty G$ denote the boundary of G in $\mathbb{C}_\infty$ and call it the *extended boundary* of G. Clearly $\partial_\infty G = \partial G$ if G is bounded and $\partial_\infty G = \partial G \cup \{\infty\}$ if G is unbounded.

1.4 Maximum Modulus Theorem — Third Version. *Let G be a region in $\mathbb{C}$ and f an analytic function on G. Suppose there is a constant M such that $\limsup\limits_{z\to a} |f(z)| \leq M$ for all a in $\partial_\infty G$. Then $|f(z)| \leq M$ for all z in G.*

§2. Schwarz's Lemma

2.1 Schwarz's Lemma. *Let $D = \{z : |z| < 1\}$ and suppose f is analytic on D with*

(a) $|f(z)| \le 1$ *for z in D,*

(b) $f(0) = 0$.

Then $|f'(0)| \le 1$ and $|f(z)| \le |z|$ for all z in the disk D. Moreover if $|f'(0)| = 1$ or if $|f(z)| = |z|$ for some $z \ne 0$ then there is a constant c, $|c| = 1$, such that $f(w) = cw$ for all w in D.

Proof. Define $g : D \to \mathbb{C}$ by

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \ne 0 \\ f'(0) & z = 0. \end{cases}$$

Then g is analytic on D.

Why is g analytic on D?

Since $f(0) = 0$, (by Corollary IV.3.9) there exists an analytic function h on D such that $f(z) = zh(z)$. Then $g(z) = \frac{f(z)}{z} = h(z)$ for $z \ne 0$. Thus, g is analytic for $z \in D$, $z \ne 0$. Also,

$$\lim_{z\to 0} g(z) = \lim_{z\to 0} \frac{f(z)}{z} = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = f'(0) = g(0).$$

Thus, g is continuous at $z = 0$. Since h is analytic on D, it is continuous on D and so

$$g(0) = \lim_{z\to 0} g(z) = \lim_{h\to 0} h(z) = h(0).$$

Therefore g is analytic on D ($\because g = h$).

Note: Alternatively, g can be shown analytic on D using the power series argument.

Let $0 < r < 1$. Then by Maximum Modulus Theorem (Second Version),

$$\max\{|g(z)| : z \in \overline{B(a,r)}\} = \max\{|g(z)| : |z| = r\}.$$

Therefore, for $|z| \le r$ (or equivalently for $|z| = r$),

$$|g(z)| = \left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{r} \le \frac{1}{r} \qquad (\because |f(z)| \le 1 \text{ on } D).$$

Letting r approach 1 gives $|g(z)| \le 1$ for all $z \in D$. That is, $|f(z)| \le |z|$ for all $z \in D$, $z \ne 0$. Since $|f(0)| = 0$, we have $|f(z)| \le |z|$ for all $z \in D$. Also, $|f'(0)| = |g(0)| \le 1$.

If $|f(z_0)| = |z_0|$ for some $z_0 \in D$, $z_0 \ne 0$ or $|f'(0)| = 1$, then $|g|$ assumes maximum value inside D. Thus, by Maximum Modulus Theorem (First Version), $g(z) = c$ for some constant c and for all $z \in D$. Now, $|c| = |g(0)| = |f'(0)| = 1$ or $|c| = |g(z_0)| = \frac{|f(z_0)|}{|z_0|} = 1$. Hence, $f(z) = cz$ for all $z \in D$ with $|c| = 1$. □

We will apply Schwarz's Lemma to characterize the conformal (angle preserving) maps of the open unit disk onto itself. First we introduce a class of such maps.

Definition. For $|a| < 1$ define the Mobius transformation

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}.$$

Note that φ_a is analytic for $|z| < \frac{1}{|\bar{a}|} = |a|^{-1}$. Since $|a| < 1$ (i.e. $\frac{1}{|a|} > 1$), φ_a is analytic in an open disk containing closure of the unit disk $D = \{z : |z| < 1\}$. In particular, φ_a is analytic on D.

2.2 Proposition. *If $|a| < 1$, then φ_a is one-one map of $D = \{z : |z| < 1\}$ onto itself; the inverse of φ_a is φ_{-a}. Furthermore, φ_a maps ∂D onto ∂D, $\varphi_a(a) = 0$, $\varphi_a'(0) = 1 - |a|^2$, and $\varphi_a'(a) = (1-|a|^2)^{-1}$.*

Proof. First we show that φ_a maps D to D. Note that

$$\begin{aligned}
|\varphi_a(z)| < 1 &\Leftrightarrow \left|\frac{z-a}{1-\bar{a}z}\right| < 1 \\
&\Leftrightarrow |z-a|^2 < |1-\bar{a}z|^2 \\
&\Leftrightarrow (z-a)(\bar{z}-\bar{a}) < (1-\bar{a}z)(1-a\bar{z}) \\
&\Leftrightarrow |z|^2 - z\bar{a} - a\bar{z} + |a|^2 < 1 - \bar{a}z - a\bar{z} + |a|^2|z|^2 \\
&\Leftrightarrow |z|^2(1-|a|^2) < 1 - |a|^2 \\
&\Leftrightarrow |z| < 1.
\end{aligned}$$

Thus, if $z \in D$, then $|z| < 1$ and so $|\varphi_a(z)| < 1$, i.e. $\varphi_a(z) \in D$. Hence, φ_a maps D to D.

For $z \in D$, if we show that $\varphi_a(\varphi_{-a}(z)) = z = \varphi_{-a}(\varphi_a(z))$, then φ_a is one-one and onto on D, and inverse of φ_a will be φ_{-a}. The reason is shown below.

- $\varphi_a(z) = \varphi_a(w) \Rightarrow \varphi_{-a}(\varphi_a(z)) = \varphi_{-a}(\varphi_a(w)) \Rightarrow z = w$. Thus, φ_a is one-one.
- Let $w \in D$. Then $|w| < 1$. Take $z = \frac{w+a}{1+\bar{a}w} = \varphi_{-a}(w)$. Then $\varphi_a(z) = \varphi_a(\varphi_{-a}(w)) = w$. Since $|w| < 1$, by a similar argument as above $|z| = |\varphi_{-a}(w)| < 1$, i.e. $\varphi_a(z) \in D$. Thus, φ_a is onto.

So it suffices to show that $\varphi_a(\varphi_{-a}(z)) = z = \varphi_{-a}(\varphi_a(z))$ for all $z \in D$. Now, for $z \in D$,

$$\begin{aligned}
\varphi_a(\varphi_{-a}(z)) &= \varphi_a\left(\frac{z+a}{1+\bar{a}z}\right) \\
&= \frac{(z+a)/(1+\bar{a}z) - a}{1 - \bar{a}(z+a)/(1+\bar{a}z)} \\
&= \frac{(z+a) - a(1+\bar{a}z)}{(1+\bar{a}z) - \bar{a}(z+a)} \\
&= \frac{z(1-|a|^2)}{1-|a|^2} = z.
\end{aligned}$$

Similarly, $\varphi_{-a}(\varphi_a(z)) = z$. Hence, $(\varphi_a)^{-1} = \varphi_{-a}$.

Now, we show that φ_a maps ∂D onto ∂D. Let $z \in \partial D$. Then $z = e^{i\theta}$ and

$$\begin{aligned} |\varphi_a(z)| = |\varphi_a(e^{i\theta})| &= \left| \frac{e^{i\theta} - a}{1 - \bar{a}e^{i\theta}} \right| \\ &= \left| \frac{e^{i\theta} - a}{1 - \bar{a}e^{i\theta}} \right| \frac{1}{|e^{-i\theta}|} \\ &= \left| \frac{e^{i\theta} - a}{e^{-i\theta} - \bar{a}} \right| = 1 \qquad (\because |z| = |\bar{z}|). \end{aligned}$$

Thus, $\varphi_a(z) \in \partial D$ and we have $\varphi_a(\partial D) \subset \partial D$. For the reverse inclusion, let $w = e^{i\theta} \in \partial D$, then $z = \frac{e^{i\theta}+a}{1+\bar{a}e^{i\theta}} \in \partial D$, then $|z| = 1$ (by same argument as above), and $\varphi_a(z) = e^{i\theta} = w$. Hence, $\varphi_a(\partial D) = \partial D$.

Clearly, $\varphi_a(a) = \frac{a-a}{1-|a|^2} = 0$. Now,

$$\begin{aligned} \varphi_a'(z) &= \frac{(z-a)'(1-\bar{a}z) - (z-a)(1-\bar{a}z)'}{(1-\bar{a}z)^2} \\ &= \frac{(1-\bar{a}z) + \bar{a}(z-a)}{(1-\bar{a}z)^2} \\ &= \frac{1-|a|^2}{(1-\bar{a}z)^2}. \end{aligned}$$

Therefore, $\varphi_a'(0) = 1 - |a|^2$ and $\varphi_a'(a) = (1-|a|^2)^{-1}$. □

Thus, what we saw in Proposition 2.2 is that, for $|a| < 1$, the maps φ_a is one-one analytic which maps D onto itself, and $\varphi_a(a) = 0$. Our goal is to show that any map $f : D \to D$ which is one-one, analytic and maps D onto itself with $f(a) = 0$ is a constant times φ_a, where the constant has absolute value 1.

Lemma. Let f be analytic on $D = \{z : |z| < 1\}$ with $|f(z)| \le 1$ for all $z \in D$. Let $f(a) = \alpha$ for some $a \in \mathbb{C}$. Then

2.3 $$|f'(a)| \le \frac{1-|\alpha|^2}{1-|a|^2}.$$

Moreover, the equality holds in the above inequality exactly when $f(z) = \varphi_{-\alpha}(c\varphi_a(z))$ for some $c \in \mathbb{C}$ with $|c| = 1$.

Proof. Let $a \in D$. Then $|a| < 1$. If $|\alpha| = 1$, then by Maximum Modulus Theorem (First Version), f is constant and the inequality holds trivially. So $|\alpha| < 1$ whenever $|a| < 1$, i.e. f maps D into D.

Let $g = \varphi_\alpha \circ f \circ \varphi_{-a}$. Then g is analytic and maps D into D. Also $|g(z)| \le 1$ for all $z \in D$ and $g(0) = \varphi_\alpha(f(\varphi_{-a}(0))) = \varphi_\alpha(f(a)) = \varphi_\alpha(\alpha) = 0$. Then by applying Schwarz's Lemma to g, we get $|g'(0)| \le 1$. Applying the Chain rule, we get

$$g'(0) = (\varphi_\alpha \circ f)'(\varphi_{-a}(0))\, \varphi_{-a}'(0)$$

$$
\begin{aligned}
&= (\varphi_\alpha \circ f)'(a)\,(1-|a|^2) \\
&= \varphi_\alpha'(f(a))\,f'(a)(1-|a|^2) \\
&= \varphi_\alpha'(\alpha)\,f'(a)(1-|a|^2) \\
&= \frac{1-|a|^2}{1-|\alpha|^2} f'(a).
\end{aligned}
$$

Since $|g'(0)| \le 1$, we have

$$|f'(a)| \le \frac{1-|\alpha|^2}{1-|a|^2} \text{ i.e. } |f'(a)| \le \frac{1-|f(a)|^2}{1-|a|^2}.$$

The equality occurs in the above inequality exactly when $|g'(0)| = 1$. Then by Schwarz's Lemma, there is a constant c with $|c| = 1$ such that $g(z) = cz$ for all $z \in D$. Then for all $z \in D$,

$$
\begin{aligned}
& g(z) = cz \\
\Rightarrow\ & \varphi_\alpha \circ f \circ \varphi_{-a}(z) = cz \\
\Rightarrow\ & f(\varphi_{-a}(z)) = \varphi_{-\alpha}(cz) \qquad \text{(applying } \varphi_{-\alpha} = (\varphi_\alpha)^{-1}\text{)}.
\end{aligned}
$$

Since φ_a is one-one and onto on D, there is one-one correspondence $z \leftrightarrow \varphi_a(z)$ on D. Therefore, replacing z by $\varphi_a(z)$ in above equation, we get

2.4
$$f(z) = \varphi_{-\alpha}(c\varphi_a(z))$$

for $|z| < 1$. □

Note that if $|c| = 1$ and $|a| < 1$, then $f = c\varphi_a$ defines a one-one analytic map of the open disk D onto itself. The following result, which is one of the main consequence of the Schwarz's Lemma, says that the converse is also true.

2.5 Theorem. *Let $f : D \to D$ be a one-one analytic map of D onto itself and suppose $f(a) = 0$. Then there is a complex number c with $|c| = 1$ such that $f = c\varphi_a$.*

Proof. Since f is one-one and onto, (by Corollary IV.7.6) there is an analytic function $g : D \to D$ such that $g(f(z)) = z$ for $|z| < 1$. Since $f(a) = 0$, $g(f(a)) = g(0)$, i.e. $g(0) = a$. Applying inequality (2.3) to both f and g, we get

(∗)
$$
\begin{aligned}
|f'(a)| &\le \frac{1-|f(a)|^2}{1-|a|^2} = (1-|a|^2)^{-1} && (\because f(a) = 0) \\
|g'(0)| &\le \frac{1-|g(0)|^2}{1-|0|^2} = 1-|a|^2 && (\because g(0) = a).
\end{aligned}
$$

Since $g \circ f(z) = z$, by chain rule $g'(f(z))f'(z) = 1$ for all $z \in D$. In particular, for $z = a$, we have

$$1 = g'(f(a))f'(a) = g'(0)f'(a).$$

So $|f'(a)| = \frac{1}{|g'(0)|} \geq (1-|a|^2)^{-1}$ and hence $|f'(a)| = (1-|a|^2)^{-1}$.

This means that equality holds in the inequality $(*)$ and hence by the above Lemma and (2.4), we get

$$\begin{aligned} f(z) &= \varphi_{-\alpha}(c\varphi_a(z)) \\ &= \varphi_0(c\varphi_a(z)) && (\because \alpha = f(a) = 0) \\ &= c\varphi_a(z) && (\because \varphi_0(z) = z), \end{aligned}$$

for some c with $|c| = 1$. □

Chapter VII

Compactness and Convergence in the Space of Analytic Functions

§1. The space of continuous functions $C(G,\Omega)$

In this chapter (Ω, d) denotes a complete metric space.

1.1 Definition. If G is an open set in $\mathbb{C}$ and (Ω, d) is a complete metric space then denote by $C(G,\Omega)$ the set of all continuous functions from G to Ω.

The set $C(G,\Omega)$ is never empty since it always contains the constant functions. However, it is possible that $C(G,\Omega)$ contains only the constant functions. For example, suppose that G is connected and $\Omega = \mathbb{N}$. If $f \in C(G,\Omega)$, then $f(G)$ must be connected in Ω and hence it must be one point set, i.e. f is constant.

Our main concern are the cases when $\Omega = \mathbb{C}$ or $\Omega = \mathbb{C}_\infty$ in which $C(G,\Omega)$ has many non-constant elements. In fact, each analytic function on G is in $C(G,\mathbb{C})$ and each meromorphic function is in $C(G,\mathbb{C}_\infty)$.

We want to give a metric on $C(G,\Omega)$. Before we do this, we prove a fact about open subsets of $\mathbb{C}$.

1.2 Proposition. *If G is open in $\mathbb{C}$ then there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \bigcup_{n=1}^{\infty} K_n$. Moreover, the sets K_n can be chosen to satisfy the following conditions:*

(a) $K_n \subset \text{int}(K_{n+1})$*;*

(b) $K \subset G$ *and* K *compact implies* $K \subset K_n$ *for some* n*;*

(c) Every component of $\mathbb{C}_\infty \setminus K_n$ *contains a component of* $\mathbb{C}_\infty \setminus G$*.*

Proof. For each positive integer n, let

$$K_n = \{z : |z| \leq n\} \cap \left\{z : d(z, \mathbb{C} \setminus G) \geq \frac{1}{n}\right\}.$$

Since K_n is bounded (being subset of $\bar{B}(0,n)$) and closed (intersection of two closed subsets of $\mathbb{C}$), it follows (by Heine-Borel theorem) that K_n is compact, for each n.

Let $z \notin G$, i.e., $z \in \mathbb{C} \setminus G$ Then $d(z, \mathbb{C} \setminus G) = 0$ and so $z \notin K_n$. Hence, $K_n \subset G$ for all n.

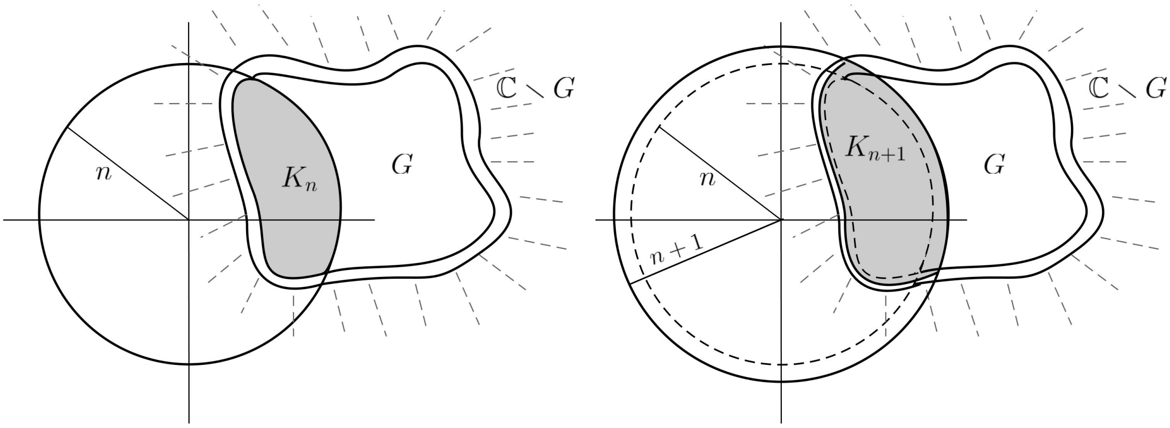


Figure VII.1: Construction of K_n. Here G is shown to be bounded and connected (which may not be the case always). Similarly K_n also need not to be connected as it appears in the above figure. This is just for demonstration.

Since $K_n \subset G, \forall\, n$, we have $\bigcup_{n=1}^{\infty} K_n \subset G$. For the reverse inclusion, let $z \in G$. Then there is $n_1 \in \mathbb{N}$ such that $|z| \leq n_1$. Since G is open, there is $r > 0$ such that $B(z,r) \subset G$. This implies, $d(z, \mathbb{C} \setminus G) \geq r$. Since $r > 0$, there is $n_2 \in \mathbb{N}$ such that $\frac{1}{n_2} < r$. Take $n_0 = \max\{n_1, n_2\}$. Then $|z| \leq n_1 \leq n_0$ and $d(z, \mathbb{C} \setminus G) \geq r > \frac{1}{n_2} \geq \frac{1}{n_0}$. This implies $z \in K_{n_0}$ and so $G \subset \bigcup_{n=1}^{\infty} K_n$. Hence, $G = \bigcup_{n=1}^{\infty} K_n$.

Observe that the set

$$\{z : |z| < n+1\} \cap \left\{z : d(z, \mathbb{C} \setminus G) > \frac{1}{n+1}\right\}$$

is open (intersection of two open sets), contains K_n (as the parts of K_n are subsets of the corresponding parts of this set), and is contained in K_{n+1} (parts of this set are subsets of corresponding parts of K_{n+1}). This set is nothing but $\text{int}(K_{n+1})$ (as intersection of interior is same as interior of intersection). Thus we have

$$K_n \subset \text{int}(K_{n+1}) \subset K_{n+1}.$$

This gives (a) and shows that K_n is a nested sequence.

Now,

$$\bigcup_{n=1}^{\infty} K_n \subset \bigcup_{n=1}^{\infty} \operatorname{int}(K_{n+1}) \subset \bigcup_{n=1}^{\infty} \operatorname{int}(K_n) \subset \bigcup_{n=1}^{\infty} K_n.$$

Thus, $G = \bigcup_{n=1}^{\infty} K_n = \bigcup_{n=1}^{\infty} \operatorname{int}(K_n)$. If K is a compact subset of G, then $\{\operatorname{int}(K_n) : n \in \mathbb{N}\}$ forms an open cover of K. Since K is compact, it is contained in union of finitely many members of the open cover. Since $\{K_n\}$ is a nested sequence, it follows that $K \subset \operatorname{int}(K_n) \subset K_n$ for some n which gives (b).

Since K_n is bounded, $\mathbb{C}_\infty \setminus K_n$ has an unbounded component which must contain ∞. Since $K_n \subset G$, we have $\mathbb{C}_\infty \setminus G \subset \mathbb{C}_\infty \setminus K_n$. Hence, the component of $\mathbb{C}_\infty \setminus G$ containing ∞ (i.e. the unbounded component of $\mathbb{C}_\infty \setminus G$) is contained in the unbounded component of $\mathbb{C}_\infty \setminus K_n$. Thus (c) holds for the unbounded component.

Since $K_n \subset \{z : |z| \leq n\}$, the set $\{z : |z| > n\}$ (being unbounded) is contained in the unbounded component of $\mathbb{C}_\infty \setminus K_n$. Let D be a bounded component of $\mathbb{C}_\infty \setminus K_n$ and let $z \in D$. Then $|z| \leq n$. But $z \notin K_n$ and so $d(z, \mathbb{C} \setminus G) < \frac{1}{n}$. By the definition of $d(z, \mathbb{C} \setminus G)$ in terms of infimum, there is a $w \in \mathbb{C} \setminus G$ such that $|w - z| < \frac{1}{n}$. So $z \in B\left(w, \frac{1}{n}\right)$. Let $\alpha \in B\left(w, \frac{1}{n}\right)$. Then $d(\alpha, \mathbb{C} \setminus G) \leq |\alpha - w| < \frac{1}{n}$. Then $\alpha \notin K_n$. Thus, we have $z \in B\left(w, \frac{1}{n}\right) \subset \mathbb{C}_\infty \setminus K_n$. Since the disk $B\left(w, \frac{1}{n}\right)$ is connected, it is entirely contained in exactly one component of $\mathbb{C}_\infty \setminus K_n$. Since $z \in B\left(w, \frac{1}{n}\right) \cap D$, and D is a component of $\mathbb{C}_\infty \setminus K_n$, we have $B\left(w, \frac{1}{n}\right) \subset D$. If D_1 is a component of $\mathbb{C}_\infty \setminus G$ containing w, then $w \in D_1 \subset \mathbb{C}_\infty \setminus G \subset \mathbb{C}_\infty \setminus K_n$. Since D_1 is connected, it is entirely contained in a component of $\mathbb{C}_\infty \setminus K_n$. Since $B\left(w, \frac{1}{n}\right) \subset D$, it follows that $D_1 \subset D$. Thus (c) holds. □

If $G = \bigcup_{n=1}^{\infty} K_n$, where each K_n is compact and $K_n \subset \operatorname{int}(K_{n+1})$, define

1.3
$$\rho_n(f,g) = \sup\{d(f(z), g(z)) : z \in K_n\}$$

for all functions f and g in $C(G,\Omega)$, where d is the metric on Ω.

Also define

1.4
$$\rho(f,g) = \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)}.$$

Since $\frac{t}{1+t} \leq 1$ for all $t \geq 0$, the series in (1.4) is dominated by $\sum \left(\frac{1}{2}\right)^n$ and hence it must converge. We shall show that ρ is a metric for $C(G,\Omega)$. For this we see the following lemma, the proof of which is left as an exercise.

1.5 Lemma. *If (S,d) is a metric space then*

$$\mu(s,t) = \frac{d(s,t)}{1+d(s,t)}$$

is also a metric on S. A set if open in (S,d) if and only if it is open in (S,μ); a sequence is a Cauchy sequence in (S,d) if and only if it is a Cauchy sequence in (S,μ).

Proof. Exercise. $\square$

1.6 Proposition. *$(C(G,\Omega),\rho)$ is a metric space.*

Proof. We verify the properties of metric for ρ.

- Let $f,g \in C(G,\Omega)$. By definition, $\rho_n(f,g) \geq 0$ for all n and so $\rho(f,g) \geq 0$.

  Suppose $\rho(f,g) = 0$. Then $\rho_n(f,g) = 0$ for all n. This implies $f(z) = g(z)$ for all $z \in K_n$. Since $G = \bigcup\limits_{n=1}^{\infty} K_n$, it follows that $f(z) = g(z)$ for all $z \in G$. Conversely, if $f(z) = g(z)$ for all $z \in G$, then $\rho(f,g) = 0$. Thus,

$$\rho(f,g) = 0 \Leftrightarrow f = g.$$

- Since d is a metric, $d(f(z),g(z)) = d(g(z),f(z))$ and so for all n,

$$\rho_n(f,g) = \sup\{d(f(z),g(z)) : z \in K_n\} = \sup\{d(g(z),f(z)) : z \in K_n\} = \rho_n(g,f).$$

  Therefore,

$$\rho(f,g) = \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n \frac{\rho_n(g,f)}{1+\rho_n(g,f)} = \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)} = \rho(g,f).$$

- Let $f,g,h \in C(G,\Omega)$. Since d is a metric, for each $z \in K_n$

$$d(f(z),g(z)) \leq d(f(z),h(z)) + d(h(z),g(z)).$$

  So

$$\sup\{d(f(z),g(z)) : z \in K_n\} \leq \sup\{d(f(z),h(z)) : z \in K_n\} + \sup\{d(h(z),g(z)) : z \in K_n\}$$

  or $\rho_n(f,g) \leq \rho_n(f,h) + \rho_n(h,g)$ for all n. Thus, ρ_n is a metric for each n. Then by previous lemma, $\frac{\rho_n(f,g)}{1+\rho_n(f,g)}$ is also a metric and hence

$$\frac{\rho_n(f,g)}{1+\rho_n(f,g)} \leq \frac{\rho_n(f,h)}{1+\rho_n(f,h)} + \frac{\rho_n(h,g)}{1+\rho_n(h,g)}.$$

  Therefore,

$$\begin{aligned}
\rho(f,g) &= \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)} \\
&\leq \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n \left(\frac{\rho_n(f,h)}{1+\rho_n(f,h)} + \frac{\rho_n(h,g)}{1+\rho_n(h,g)} \right) \\
&= \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n \frac{\rho_n(f,h)}{1+\rho_n(f,h)} + \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n \frac{\rho_n(h,g)}{1+\rho_n(h,g)} \quad \text{(by absolute convergence)} \\
&= \rho(f,h) + \rho(h,g).
\end{aligned}$$

Hence, ρ is a metric on $C(G,\Omega)$. □

1.7 Lemma. *Let the metric ρ be defined as in (1.4). If $\varepsilon > 0$ is given then there is a $\delta > 0$ and a compact set $K \subset G$ such that for f and g in $C(G,\Omega)$,*

1.8 $$\sup\{d(f(z),g(z)) : z \in K\} < \delta \Rightarrow \rho(f,g) < \varepsilon.$$

Conversely, if $\delta > 0$ and a compact set K are given, there is an $\varepsilon > 0$ such that for f and g in $C(G,\Omega)$,

1.9 $$\rho(f,g) < \varepsilon \Rightarrow \sup\{d(f(z),g(z)) : z \in K\} < \delta.$$

Proof. Let $\varepsilon > 0$ be fixed.

- Since $\sum\limits_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ is convergent, there is a positive integer p such that $\sum\limits_{n=p+1}^{\infty}\left(\frac{1}{2}\right)^n < \frac{\varepsilon}{2}$. Put $K = K_p$.
- Since $\lim\limits_{t\to 0}\frac{t}{1+t} = 0$, there is $\delta > 0$ such that for $0 \le t < \delta$, we have $\frac{t}{1+t} < \frac{\varepsilon}{2}$.

Suppose $f,g \in C(G,\Omega)$ satisfy $\sup\{d(f(z),g(z)) : z \in K\} < \delta$. Since $K_n \subset K_p = K$ for $1 \le n \le p$, $\rho_n(f,g) = \sup\{d(f(z),g(z)) : z \in K_n\} < \delta$ for $1 \le n \le p$. This (by the second point above) gives

$$\frac{\rho_n(f,g)}{1+\rho_n(f,g)} < \frac{1}{2}\varepsilon$$

for $1 \le n \le p$. Therefore,

$$\begin{aligned}
\rho(f,g) &= \sum_{n=1}^{\infty}\left(\tfrac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)} \\
&= \sum_{n=1}^{p}\left(\tfrac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)} + \sum_{n=p+1}^{\infty}\left(\tfrac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)} \\
&< \sum_{n=1}^{p}\left(\tfrac{1}{2}\right)^n \tfrac{\varepsilon}{2} + \sum_{n=p+1}^{\infty}\left(\tfrac{1}{2}\right)^n && \left(\because \tfrac{\rho_n(f,g)}{1+\rho_n(f,g)} < 1 \text{ and by above}\right) \\
&< \sum_{n=1}^{\infty}\left(\tfrac{1}{2}\right)^n \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} && \text{(by first point above)} \\
&= \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. && \left(\because \sum_{n=1}^{\infty}\left(\tfrac{1}{2}\right)^n = 1\right)
\end{aligned}$$

Conversely, suppose that compact set $K \subset G$ and $\delta > 0$ are given. Since $G = \bigcup\limits_{n=1}^{\infty} K_n = \bigcup\limits_{n=1}^{\infty} \operatorname{int}(K_n)$ and K is compact, by Proposition 1.2 (b), there is an integer $p \ge 1$ such that $K \subset K_p$. This implies

$(*)$ $$\rho_p(f,g) = \sup\{d(f(z),g(z)) : z \in K_p\} \ge \sup\{d(f(z),g(z)) : z \in K\}.$$

Since $\lim\limits_{s\to 0}\frac{s}{1-s}=0$, $\varepsilon>0$ be chosen so that $0\le s<2^p\varepsilon$ implies $\frac{s}{1-s}<\delta$. Then $\frac{t}{1+t}<2^p\varepsilon$ implies (with $s=\frac{t}{1+t}$) that

$$\frac{t/(1+t)}{1-t/(1+t)}=\frac{t}{(1+t)-t}=t<\delta.$$

So if $\rho(f,g)=\sum\limits_{n=1}^{\infty}\left(\frac{1}{2}\right)^n\frac{\rho_n(f,g)}{1+\rho_n(f,g)}<\varepsilon$, then for $n=p$, we have

$$\left(\frac{1}{2}\right)^p\frac{\rho_p(f,g)}{1+\rho_p(f,g)}<\varepsilon \text{ or } \frac{\rho_p(f,g)}{1+\rho_p(f,g)}<2^p\varepsilon.$$

Then (with $t=\rho_p(f,g)$) by above $\rho_p(f,g)<\delta$. By (∗), this gives

$$\sup\{d(f(z),g(z)):z\in K\}\le\rho_p(f,g)<\delta.$$

□

1.10 Proposition.

(a) *A set $\mathcal{O}\subset(C(G,\Omega),\rho)$ is open if and only if for each f in $\mathcal{O}$ there is a compact set K and a $\delta>0$ such that*

$$\mathcal{O}\supset\{g:d(f(z),g(z))<\delta,\ z\in K\}.$$

(b) *A sequence $\{f_n\}$ in $(C(G,\Omega),\rho)$ converges to f if and only if $\{f_n\}$ converges to f uniformly on all compact subsets of G.*

Proof.

(a) Let $\mathcal{O}\subset(C(G,\Omega),\rho)$ be open and $f\in\mathcal{O}$. Then there is $\varepsilon>0$ such that

$$B_\rho(f,\varepsilon)\subset\mathcal{O}.$$

That is,

$$\{g\in C(G,\Omega):\rho(f,g)<\varepsilon\}\subset\mathcal{O}.$$

By first part of the above lemma (i.e. by (1.8)), there is $\delta>0$ and a compact set K such that

$$\sup\{d(f(z),g(z)):z\in K\}<\delta\Rightarrow\rho(f,g)<\varepsilon.$$

Now for given $f\in C(G,\Omega)$, if $g\in C(G,\Omega)$ satisfies $d(f(z),g(z))<\delta$ for all $z\in K$, then $\sup\{d(f(z),g(z)):z\in K\}<\delta$ and so by above $\rho(f,g)<\varepsilon$. Hence, we have

$$\{g:d(f(z),g(z))<\delta,\ z\in K\}\subset\{g\in C(G,\Omega):\rho(f,g)<\varepsilon\}\subset\mathcal{O}.$$

Conversely, suppose that $\mathcal{O}\subset C(G,\Omega)$ and for each $f\in\mathcal{O}$ there is $\delta>0$ and a compact set K such that

(∗) $$\{g:d(f(z),g(z))<\delta,\ z\in K\}\subset\mathcal{O}.$$

Then $\sup\{d(f(z),g(z)) : z \in K\} < \delta$ and so by the above lemma (second part, i.e. (1.9)), there is $\varepsilon > 0$ such that

$$\rho(f,g) < \varepsilon \Rightarrow \sup\{d(f(z),g(z)) : z \in K\} < \delta.$$

Thus if $g \in \{g : \rho(f,g) < \varepsilon\}$ then by above $\sup\{d(f(z),g(z)) : z \in K\} < \delta$. Then $d(f(z),g(z)) < \delta$ for all $z \in K$ and so $g \in \mathcal{O}$, by $(*)$. That is, $\{g \in C(G,\Omega)) : \rho(f,g) < \varepsilon\} \subset \mathcal{O}$. Hence, $\mathcal{O}$ is open.

(b) Exercise.

□

1.11 Corollary. *The collection of open sets is independent of the choice of the sets* $\{K_n\}$. *That is, if* $G = \bigcup_{n=1}^{\infty} K_n'$ *where each* K_n' *is compact and* $K_n' \subset \operatorname{int}(K_{n+1}')$ *and if* μ *is the metric defined by the sets* $\{K_n'\}$ *then a set is open in* $(C(G,\Omega),\mu)$ *if and only if it is open in* $(C(G,\Omega),\rho)$.

Proof. This follows from part (a) of the above proposition in which the open sets of $(C(G,\Omega),\rho)$ are classified in terms of compact sets and the characterization does not depend on the choice of the sets $\{K_n\}$. □

So from now on, whenever we consider $C(G,\Omega)$ as a metric space, we may assume that the metric is ρ given in (1.4) for some sequence $\{K_n\}$ of compact subsets of G such that $K_n \subset \operatorname{int}(K_{n+1})$ and $G = \bigcup_{n=1}^{\infty} K_n$.

So far we have not used the fact that (Ω,d) is a complete metric space. If Ω is not complete, then $C(G,\Omega)$ is not complete. So we assume that Ω is complete and we have the following result.

1.12 Proposition. $C(G,\Omega)$ *is a complete metric space.*

Proof. Suppose $\{f_n\}$ is a Cauchy sequence in $C(G,\Omega)$. Then for each compact set $K \subset G$, the restrictions of the functions f_n to K gives a Cauchy sequence in $C(K,\Omega)$. We prove the convergence of $\{f_n\}$ on compact subsets of G.

Suppose $\delta > 0$ and a compact set $K \subset G$ are given. By Lemma 1.7 (second part), there is $\varepsilon > 0$ such that

$$\rho(f,g) < \varepsilon \Rightarrow \sup\{d(f(z),g(z)) : z \in K\} < \delta.$$

Since $\{f_n\}$ is a Cauchy sequence in $C(G,\Omega)$, (for above) $\varepsilon > 0$ there is an integer $N = N(\varepsilon)$ such that $\rho(f_n,f_m) < \varepsilon$ for all $n,m \geq N$. So from above, we have

1.13 $$\sup\{d(f_n(z),f_m(z)) : z \in K\} < \delta$$

for $n,m \geq N$. So $d(f_n(z),f_m(z)) < \delta$ for all $z \in K$, whenever $n,m \geq N$. Thus, in particular for each fixed $z \in K$, $\{f_n(z)\}$ is a Cauchy sequence in (Ω,d). Since Ω is complete, $\{f_n(z)\}$

converges to some point in Ω. Define $f : K \to \Omega$ by $f(z) = \lim f_n(z)$. Since $G = \bigcup_{n=1}^{\infty} K_n$ for K_n compact, this gives a function $f : G \to \Omega$. Now we show that $\rho(f_n, f) \to 0$, i.e. $f_n \to f$.

Let K be compact and fix $\delta > 0$. Choose $N \in \mathbb{N}$ such that (1.13) holds for $n,m \geq N$. Fix some $z \in K$. Since $f_n(z) \to f(z)$, there is some $m \in \mathbb{N}$ with $m \geq N$ such that $d(f(z), f_m(z)) < \delta$. But then for all $n \geq N$,

$$d(f(z), f_n(z)) \leq d(f(z), f_m(z)) + d(f_m(z), f_n(z)) < \delta + \delta = 2\delta.$$

Since N does not depend on z (note that N depends on ε which depends on K), this gives

$$\sup\{d(f(z), f_n(z)) : z \in K\} \to 0$$

as $n \to \infty$. That is, $\{f_n\}$ converges to f uniformly on every compact set in G. Then f is continuous on every compact subset of G and hence $f \in C(G,\Omega)$. Also, by Proposition 1.10 (b), $f_n \to f$, i.e. $\rho(f_n, f) \to 0$. □

1.14 Definition. A set $\mathscr{F} \subset C(G,\Omega)$ is *normal* if each sequence in $\mathscr{F}$ has a subsequence which converges to a function f in $C(G,\Omega)$.

Recall that a set $\mathscr{F}$ is *sequentially compact* if every sequence in $\mathscr{F}$ has a subsequence converging to an element of $\mathscr{F}$. Thus, the definition of normal is different from sequentially compact as in the former case the limit of the subsequence need not be in the set $\mathscr{F}$.

1.15 Proposition. *A set $\mathscr{F} \subset C(G,\Omega)$ is normal if and only if its closure is compact.*

Proof. Exercise. □

Before we proceed, we note the following theorem which will be used in the subsequent results.

4.9 Theorem (Chapter II, page no. 22 in Conway). *Let (X,d) be a metric space; then the following are equivalent statements:*

(a) X is compact;
(b) Every infinite set in X has a limit point;
(c) X is sequentially compact;
(d) X is complete and for every $\varepsilon > 0$ there are a finite number of points $x_1, \ldots, x_n$ in X such that

$$X = \bigcup_{k=1}^{n} B(x_k, \varepsilon).$$

The property (b) is called *limit point compactness* and the property mentioned in (d) is called *total boundedness*.

1.16 Proposition. *A set $\mathscr{F} \subset C(G,\Omega)$ is normal if and only if every compact set $K \subset G$ and $\delta > 0$ there are functions $f_1,\dots,f_n$ in $\mathscr{F}$ such that for f in $\mathscr{F}$ there is at least one k, $1 \le k \le n$, with*

$$\sup\{d(f(z),f_k(z)) : z \in K\} < \delta.$$

Proof. Suppose $\mathscr{F}$ is normal. Let a compact set K and $\delta > 0$ be given. By Lemma 1.7 (part 2, i.e. (1.8)), there is an $\varepsilon > 0$ such that for f and g in $C(G,\Omega)$,

$$(*) \qquad \rho(f,g) < \varepsilon \Rightarrow \sup\{d(f(z),g(z)) : z \in K\} < \delta.$$

Since $\mathscr{F}$ is normal, by Proposition 1.15, $\overline{\mathscr{F}}$ is compact, $\mathscr{F}$ is totally bounded.

Why is $\mathscr{F}$ totally bounded if $\overline{\mathscr{F}}$ is compact ?

Let $\varepsilon > 0$ be given. Since $\overline{\mathscr{F}}$ is compact by Theorem II.4.9, $\overline{\mathscr{F}}$ is totally bounded. Then (for $\frac{\varepsilon}{2} > 0$) there are $g_1,\dots,g_n$ in $\overline{\mathscr{F}}$ such that

$$\overline{\mathscr{F}} \subset \bigcup_{k=1}^{n} B_\rho\left(g_k, \tfrac{\varepsilon}{2}\right).$$

Since $g_k \in \overline{\mathscr{F}}$, for $1 \le k \le n$, (for $\frac{\varepsilon}{2} > 0$) there is $f_k \in \mathscr{F}$ such that $f_k \in B_\rho\left(g_k, \frac{\varepsilon}{2}\right)$, i.e., $\rho(f_k,g_k) < \frac{\varepsilon}{2}$. Thus, we have $f_1,\dots,f_n$ in $\mathscr{F}$ corresponding to $g_1,\dots,g_n$ in $\overline{\mathscr{F}}$.
Claim. $\mathscr{F} \subset \bigcup_{k=1}^{n} B_\rho(f_k,\varepsilon)$, i.e., $\mathscr{F} \subset \bigcup_{k=1}^{n} \{f : \rho(f,f_k) < \varepsilon\}$.
Let $f \in \mathscr{F}$. Then $f \in \overline{\mathscr{F}}$ and so there is $g_k \in \overline{\mathscr{F}}$ such that $\rho(f,g_k) < \frac{\varepsilon}{2}$. But then as argued above, there is $f_k \in \mathscr{F}$ such that $\rho(f_k,g_k) < \frac{\varepsilon}{2}$. Then by triangle inequality,

$$\rho(f,f_k) \le \rho(f,g_k) + \rho(g_k,f_k) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

i.e., $f \in B_\rho(f_k,\varepsilon)$. Hence, the claim which shows that $\mathscr{F}$ is totally bounded.

Since $\mathscr{F}$ is totally bounded, there are $f_1,\dots,f_n$ in $\mathscr{F}$ such that

$$\mathscr{F} \subset \bigcup_{k=1}^{n} \{f : \rho(f,f_k) < \varepsilon\}.$$

But then by $(*)$, this choice of ε gives

$$\mathscr{F} \subset \bigcup_{k=1}^{n} \{f : d(f(z),f_k(z)) < \delta,\ z \in K\}.$$

This means given any $f \in \mathscr{F}$ there is at least one k, $1 \le k \le n$, such that

$$\sup\{d(f(z),f_k(z)) : z \in K\} < \delta.$$

Conversely, assume that $\mathscr{F}$ has the above stated property. Then it follows that $\overline{\mathscr{F}}$ also satisfies this condition.

If $\mathscr{F}$ satisfies the given property (for $\frac{\delta}{2}$), then so does $\overline{\mathscr{F}}$ (for δ).

Let K be a compact set and $\delta > 0$. Assume that $\mathscr{F}$ satisfies given property i.e., given compact set K and $\frac{\delta}{2} > 0$ there are $f_1, \dots, f_n \in \mathscr{F} \subset \overline{\mathscr{F}}$ such that for $g \in \mathscr{F}$ there is at least one k, $1 \le k \le n$, with

$$\sup\{d(g(z), f_k(z)) : z \in K\} < \tfrac{\delta}{2}.$$

Since compact set K and $\frac{\delta}{2} > 0$ are given, by Lemma 1.7 there is $\varepsilon > 0$ such that for any $f, g \in C(G,\Omega)$,

$$\rho(f,g) < \varepsilon \Rightarrow \sup\{d(f(z), g(z)) : z \in K\} < \tfrac{\delta}{2}.$$

Now, let $f \in \overline{\mathscr{F}}$. Then by definition of closure, (given above $\varepsilon > 0$) there is $g \in \mathscr{F}$ such that $g \in B_\rho(f, \varepsilon)$, i.e. $\rho(f,g) < \varepsilon$. Then by above, $\sup\{d(f(z), g(z)) : z \in K\} < \frac{\delta}{2}$. Now,

$$\begin{aligned}
&d(f(z), f_k(z)) \le d(f(z), g(z)) + d(g(z), f_k(z)) \\
\Rightarrow\ &\sup\{d(f(z), f_k(z)) : z \in K\} \le \sup\{d(f(z), g(z)) : z \in K\} + \sup\{d(g(z), f_k(z)) : z \in K\} \\
\Rightarrow\ &\sup\{d(f(z), f_k(z)) : z \in K\} \le \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\end{aligned}$$

That is, for every compact set $K \subset G$ and $\delta > 0$ there are functions $f_1, \dots, f_n$ in $\overline{\mathscr{F}}$ such that for f in $\overline{\mathscr{F}}$ there is at least one k, $1 \le k \le n$, with

$$\sup\{d(f(z), f_k(z)) : z \in K\} < \delta.$$

But given any compact set K and $\delta > 0$, by Lemma 1.7 there is $\varepsilon > 0$ such that for f, f_k in $C(G,\Omega)$,

$$\sup\{d(f(z), f_k(z)) : z \in K\} < \delta \Rightarrow \rho(f, f_k) < \varepsilon.$$

This means $f \in B_\rho(f_k, \varepsilon)$ for some k, $1 \le k \le n$. Thus, $\overline{\mathscr{F}} \subset \bigcup_{k=1}^{n} B_\rho(f_k, \varepsilon)$, i.e. $\overline{\mathscr{F}}$ is totally bounded.

Since $\overline{\mathscr{F}}$ is closed subspace of complete metric space $C(G,\Omega)$, it is also complete. Being complete and totally bounded, by Theorem II.4.9, $\overline{\mathscr{F}}$ is compact. Then by Proposition 1.15, $\mathscr{F}$ is normal. □

Let (X_n, d_n) be a metric space for each $n \ge 1$ and let $X = \prod_{n=1}^{\infty} X_n$ be their Cartesian product. That is, $X = \{\xi = \{x_n\} : x_n \in X_n \text{ for each } n \ge 1\}$. For $\xi = \{x_n\}$ and $\eta = \{y_n\}$ in X define

1.17
$$d(\xi, \eta) = \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}.$$

1.18 Proposition. *$\left(\prod_{n=1}^{\infty} X_n, d\right)$, where d is defined by (1.17), is a metric space. If $\xi^k = \{x_n^k\}_{n=1}^{\infty}$ is in $X = \prod_{n=1}^{\infty} X_n$ then $\xi^k \to \xi = \{x_n\}$ if and only if $x_n^k \to x_n$ for each n. Also if each (X_n, d_n) is compact then X is compact.*

1.21 Definition. A set $\mathscr{F} \subset C(G,\Omega)$ is *equicontinuous* at a point z_0 in G if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|z - z_0| < \delta$,

$$d(f(z), f(z_0)) < \varepsilon$$

for all f in $\mathscr{F}$.

A set $\mathscr{F} \subset C(G,\Omega)$ is *equicontinuous* over a set $E \subset G$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for z and z' in E and $|z - z'| < \delta$,

$$d(f(z), f(z')) < \varepsilon$$

for all f in $\mathscr{F}$.

Note that if $\mathscr{F}$ is a singleton set $\{f\}$, then $\mathscr{F}$ is equicontinuous at z_0 is same as saying that f is continuous at z_0. The important thing about equicontinuity is that the same δ will work for all the functions in $\mathscr{F}$. Also, $\mathscr{F} = \{f\}$ is equicontinuous over E is same as saying that f is uniformly continuous on E.

Due to this analogy with continuity and uniform continuity, we have the following result.

1.22 Proposition. *Suppose $\mathscr{F} \subset C(G,\Omega)$ is equicontinuous at each point of G then $\mathscr{F}$ is equicontinuous over each compact subset of G.*

Proof. Suppose $\mathscr{F}$ is equicontinuous at each point of G. Let K be a compact subset of G and $\varepsilon > 0$ be given. Then by the definition of equicontinuous, for each w in K there is a $\delta_w > 0$ such that

$$(*) \qquad |w - w'| < \delta_w \Rightarrow d(f(w'), f(w)) < \tfrac{1}{2}\varepsilon$$

for all f in $\mathscr{F}$.

Now, $\{B(w, \delta_w) : w \in K\}$ forms an open cover of K and since K is compact, it is sequentially compact (Theorem II.4.9). By Lebesgue's Covering Lemma (Lemma II.4.8), there is $\delta > 0$ such that for each z in K, $B(z, \delta)$ is contained in one of the sets of this cover. So if z and z' are in K and $|z - z'| < \delta$, there is $w \in K$ with

$$z' \in B(z, \delta) \subset B(w, \delta_w).$$

Then by $(*)$, for all $f \in \mathscr{F}$, we have

$$d(f(z), f(w)) < \frac{\varepsilon}{2} \text{ and } d(f(z'), f(w)) < \frac{\varepsilon}{2}.$$

By triangular inequality,

$$d(f(z),f(z')) \le d(f(z),f(w)) + d(f(z'),f(w)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $|z-z'| < \delta$ and for all $f \in \mathscr{F}$. So $\mathscr{F}$ is equicontinuous over K. □

1.23 Arzela-Ascoli Theorem. *A set $\mathscr{F} \subset C(G,\Omega)$ is normal if and only if the following two conditions are satisfied:*

(a) for each z in G, $\{f(z) : f \in \mathscr{F}\}$ has compact closure in Ω;
(b) $\mathscr{F}$ is equicontinuous at each point of G.

Proof. First assume that $\mathscr{F}$ is normal. Notice that for each z in G the map of $C(G,\Omega) \to \Omega$ defined by $f \rightsquigarrow f(z)$ is continuous.

For each $z \in G$, the map $\varphi_z : C(G,\Omega) \to \Omega$ defined by $\varphi_z(f) = f(z)$ is continuous.

Given $\varepsilon > 0$, we have to show that there is $\delta > 0$ such that for any $f,g \in C(G,\Omega)$,

$$\rho(f,g) < \delta \Rightarrow d(\varphi_z(f),\varphi_z(g)) < \varepsilon, \text{ i.e., } d(f(z),g(z)) < \varepsilon.$$

Let $K = \{z\}$ be a compact subset of G. By Lemma 1.7 (second part, i.e. (1.8)), given $\varepsilon > 0$ and a compact set K there is $\delta > 0$ (note the interchange of δ and ε here) such that for any $f,g \in C(G,\Omega)$,

$$\rho(f,g) < \delta \Rightarrow \sup\{d(f(z),g(z)) : z \in K\} < \varepsilon.$$

Since $K = \{z\}$, equivalently we have $\rho(f,g) < \delta \Rightarrow d(f(z),g(z)) < \varepsilon$ and hence φ_z is continuous for each $z \in G$.

Since $\mathscr{F}$ is normal, by Proposition 1.15, $\overline{\mathscr{F}}$ is compact. Since continuous image of compact set is compact, its image is compact in Ω and (a) follows.

Why (a) follows? What is the image of $\overline{\mathscr{F}}$ under φ_z?

Since $\overline{\mathscr{F}}$ is compact and φ_z is continuous, $\varphi_z(\overline{\mathscr{F}}) = \{\varphi_z(f) : f \in \overline{\mathscr{F}}\} = \{f(z) : f \in \overline{\mathscr{F}}\}$ is compact and hence closed (as it is compact subset of metric space (Ω,d)). Now,

$$\begin{aligned} &\{f(z) : f \in \mathscr{F}\} \subset \{f(z) : f \in \overline{\mathscr{F}}\} \\ \Rightarrow\ &\overline{\{f(z) : f \in \mathscr{F}\}} \subset \overline{\{f(z) : f \in \overline{\mathscr{F}}\}} \\ \Rightarrow\ &\overline{\{f(z) : f \in \mathscr{F}\}} \subset \{f(z) : f \in \overline{\mathscr{F}}\} \qquad (\because \{f(z) : f \in \overline{\mathscr{F}}\} \text{ is closed}) \end{aligned}$$

Since $\overline{\{f(z) : f \in \mathscr{F}\}}$ is a closed subset of the compact set $\{f(z) : f \in \overline{\mathscr{F}}\}$, it is compact. Thus the set $\{f(z) : f \in \mathscr{F}\}$ has a compact closure and (a) follows.

Now we show (b), i.e. $\mathscr{F}$ is equicontinuous at each point of G. For this fix some z_0 in G and let $\varepsilon > 0$ be given. We aim to find a $\delta > 0$ such that for all $f \in \mathscr{F}$,

$$|z - z_0| < \delta \Rightarrow d(f(z), f(z_0)) < \varepsilon.$$

Choose $R > 0$ such that $K = \overline{B}(z_0, R) \subset G$.

Why such an $R > 0$ exists?

Since $G \subset \mathbb{C}$ is open and $z_0 \in G$, there is $r > 0$ such that $B(z_0, r) \subset G$. Take $R = \frac{r}{2}$ (or anything less than r). Then $\overline{B}(z_0, R) \subset B(z_0, r) \subset G$.

Then K is a compact subset of G (by Heine-Borel theorem). Since $\mathscr{F}$ is normal, by Proposition 1.16, given a compact set K and $\frac{\varepsilon}{3} > 0$ (taking $\delta = \frac{\varepsilon}{3}$ here), there are functions $f_1, \ldots, f_k$ in $\mathscr{F}$ such that for each f in $\mathscr{F}$ there is at least one f_k with

1.24
$$\sup\{d(f(z), f_k(z) : z \in K\} < \frac{\varepsilon}{3}.$$

But since each $f_k \in \mathscr{F} \subset C(G, \Omega)$, i.e. each f_k is continuous (given $\frac{\varepsilon}{3} > 0$), there is $0 < \delta < R$ (if $\delta > R$, then by choosing a smaller neighborhood around z_0 we may assume that $\delta < R$) such that $|z - z_0| < \delta$ implies that

$$d(f_k(z), f_k(z_0)) < \frac{\varepsilon}{3}$$

for $1 \le k \le n$. Therefore if $|z - z_0| < \delta$, $f \in \mathscr{F}$ and k is chosen so that (1.24) holds, then by triangle inequality

$$\begin{aligned} d(f(z), f(z_0)) &\le d(f(z), f_k(z)) + d(f_k(z), f_k(z_0)) + d(f_k(z_0), f(z_0)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

(Note that the first and the last inequality follows from (1.24) as both $z, z_0 \in K$ and the middle inequality follows from above fact that each f_k is continuous).
So $\mathscr{F}$ is equicontinuous at z_0. Since z_0 is arbitrary, it follows that $\mathscr{F}$ is equicontinuous at each point of G and hence (b) holds.

Conversely, suppose that $\mathscr{F}$ satisfies conditions (a) and (b). To show that $\mathscr{F}$ is normal, we have to show that every sequence in $\mathscr{F}$ has a subsequence that converges in $C(G, \Omega)$. Since $C(G, \Omega)$ is complete, it suffices to show that every sequence in $\mathscr{F}$ has a Cauchy subsequence.

Let $\{z_n\}$ be the sequence of all points in G with rational real and imaginary parts (then $\{z_n\}$ is dense in G and so for z in G and $\delta > 0$ there is a z_n with $|z - z_n| < \delta$). For each $n \geq 1$ let

$$X_n = \overline{\{f(z_n) : f \in \mathscr{F}\}} \subset \Omega.$$

From (a), it follows that each (X_n, d) is compact metric space. By Proposition 1.18, $X = \prod_{n=1}^{\infty} X_n$ is a compact metric space. For $f \in \mathscr{F}$, define $\tilde{f}$ in X by

$$\tilde{f} = \{f(z_1), f(z_2), \ldots\}.$$

Let $\{f_k\}$ be a sequence in $\mathscr{F}$. Then $\{\tilde{f}_k\}$ is a sequence in the compact metric space X. By Theorem II.4.9, X is sequentially compact and so the sequence $\{\tilde{f}_k\}$ has a convergent subsequence which converges to some $\xi = \{w_n\}$ in X. For the sake of convenient notation (to avoid using subscripts for subsequence), we denote this subsequence of $\{\tilde{f}_k\}$ by $\{\tilde{f}_k\}$ itself, i.e. we assume that $\lim_{k\to\infty} \tilde{f}_k = \xi$. Again by Proposition 1.18,

1.25
$$\lim_{k\to\infty} f_k(z_n) = w_n$$

for all n. We will show that the sequence $\{f_k\}$ is Cauchy in $C(G,\Omega)$ and since $C(G,\Omega)$ is complete, it will converge to some f in $C(G,\Omega)$. By Proposition 1.10 (b) (since convergence implies uniform convergence on compact sets), we have to show that $\{f_k\}$ converges uniformly on compact subsets of G. Hence, it suffices to show that $\{f_k\}$ is uniformly Cauchy, i.e., given any compact set $K \subset G$ and a $\varepsilon > 0$ we can find an integer J such that for $k, j \geq J$,

1.26
$$\sup\{d(f_k(z), f_j(z)) : z \in K\} < \varepsilon.$$

So let K be a compact subset of G and let $\varepsilon > 0$. Since K is compact, $R = d(K, \partial G) > 0$.

Why $d(K, \partial G) > 0$?

This follows from the Theorem II.5.17 which states that if A and B are disjoint sets in X with B closed and A compact then $d(A,B) > 0$.

Note that here $A = K$ is compact and $B = \partial G = \overline{G} \cap \overline{\mathbb{C} \setminus G}$ is closed.

Now, we show that K and ∂G are disjoint. Note that $\partial G \subset \overline{\mathbb{C} \setminus G} = \mathbb{C} \setminus G$ (since $\mathbb{C} \setminus G$ is closed for G being open). But $K \subset G$. This implies

$$K \cap \partial G \subset G \cap (\mathbb{C} \setminus G) = \emptyset.$$

Hence, $K \cap \partial G = \emptyset$.

Let $K_1 = \left\{z : d(z,K) \leq \frac{1}{2}R\right\}$. Then K_1 is compact (being closed and bounded subset of $\mathbb{C}$) and $K \subset \operatorname{int}(K_1) \subset K_1 \subset G$.

Why $K \subset \text{int}(K_1) \subset K_1 \subset G$?

$z \in K \Rightarrow d(z,K) = 0 < \frac{R}{2} \Rightarrow z \in \{z : d(z,K) < \frac{R}{2}\} = \text{int}(K_1)$. Hence $K \subset \text{int}(K_1)$.
Let $z \notin G$, i.e. $z \in \mathbb{C} \smallsetminus G$. Then since $K \subset G$, $d(z,K) \geq d(K, \partial G) = R$. So by definition of K_1, $z \notin K_1$. Hence $K_1 \subset G$.

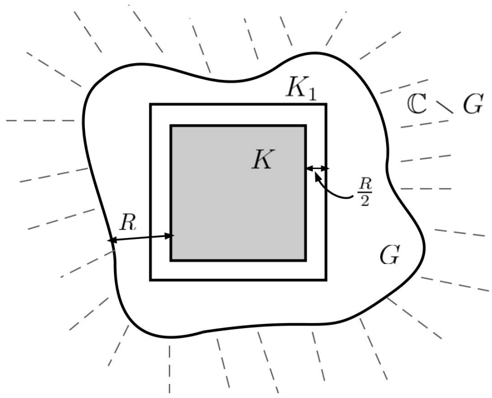


Figure VII.2: Here G is shown bounded and connected which may not be the case. The figure is for demonstration purpose only.

By assumption (b), since $\mathscr{F}$ is equicontinuous at each point of G, by Proposition 1.22, it follows that $\mathscr{F}$ is equicontinuous on K_1. So (given $\frac{\varepsilon}{3} > 0$) there is $0 < \delta < \frac{R}{2}$ (we can always choose a smaller δ) such that for $z, z' \in K_1$ with $|z - z'| < \delta$,

1.27 $$d(f(z), f(z')) < \frac{\varepsilon}{3}$$

for all $f \in \mathscr{F}$. Now let $D = \{z_n\} \cap K_1$, i.e.,

$$D = \{z_n : z_n \in K_1\}.$$

If $z \in K \subset G$, then (since $\{z_n\}$ is dense in G) there is a z_n with $|z - z_n| < \delta$. But since $\delta < \frac{R}{2}$ this gives $d(z_n, K) \leq |z_n - z| < \delta < \frac{R}{2}$ and so $z_n \in K_1$ (by the definition of K_1), or $z_n \in D$ (by the definition of D). This shows that given any $z \in K$ there is a $z_n \in D$ such that $z \in B(z_n, \delta)$. Hence $\{B(w, \delta) : w \in D\}$ forms an open cover of K. Since K is compact, there are $w_1, w_2, \ldots, w_n \in D$ such that

(∗) $$K \subset \bigcup_{i=1}^{n} B(w_i, \delta).$$

By (1.25), $\lim_{k\to\infty} f_k(z_n)$ exists for all z_n and since $D \subset \{z_n\}$, it follows that $\lim_{k\to\infty} f_k(w)$ exists for all $w \in D$. Hence, $\lim_{k\to\infty} f_k(w_i)$ exists for $1 \leq i \leq n$. That is, the sequence $\{f_k(w_i)\}$ is convergent, for $1 \leq i \leq n$, and hence Cauchy. So (given $\frac{\varepsilon}{3} > 0$) there is an integer J such that for $j, k \geq J$

1.28 $$d(f_k(w_i), f_j(w_i)) < \frac{\varepsilon}{3}$$

for $i = 1, \dots, n$.

Let z be an arbitrary point in K. Then by $(*)$, there is w_i such that $|z - w_i| < \delta$. For $k, j \geq J$, by (1.27) and (1.28), applying triangle inequality we get

$$\begin{aligned} d(f_k(z), f_j(z)) &\leq d(f_k(z), f_k(w_i)) + d(f_k(w_i), f_j(w_i)) + d(f_j(w_i), f_j(z)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

(Note that the first and the third inequality follows from (1.27) as it holds for all $f \in \mathscr{F}$ and $f_k, f_j \in \mathscr{F}$ while the middle inequality follows from (1.28)).
Since $z \in K$ was arbitrary, $\sup\{d(f_k(z), f_j(z)) : z \in K\} < \varepsilon$ for all $k, j \geq J$. Thus, (1.26) is established and hence the result. □

4

Unit 4

§2. Spaces of analytic functions

Let G be an open subset of the complex plane. Let $H(G)$ be the collection of analytic functions on G. Then $H(G) \subset C(G, \mathbb{C})$.

The notation $A(G)$ is universally used to denote the collection of continuous function $f : \overline{G} \to \mathbb{C}$ that are analytic in G. So we denote by $H(G)$, the collection of analytic functions; the letter H is used as they are also called holomorphic functions.

The first question is that: Is $H(G)$ closed in $C(G, \mathbb{C})$? The next result affirms this and also says that the function $f \mapsto f'$ is continuous from $H(G)$ into $H(G)$.

2.1 Theorem. *If $\{f_n\}$ is a sequence in $H(G)$ and f belongs to $C(G, \mathbb{C})$ such that $f_n \to f$ then f is analytic and $f_n^{(k)} \to f^{(k)}$ for each integer $k \geq 1$.*

Proof. We will show that f is analytic by applying Morera's Theorem (Theorem IV.5.10). To prove that f is analytic in G, we shall prove that f is analytic in every disk $D \subset G$. Since f is given to be continuous, it suffices to show that $\int_T f = 0$ for every triangle T in D.

So let T be a triangle contained inside a disk $D \subset G$. Since T is compact, $\{f_n\}$ converges to f uniformly on T (by Proposition 1.10 (b)). Since each f_n is analytic on D and T is closed, by Cauchy's Theorem $\int_T f_n = 0$. Since the convergence $f_n \to f$ is uniform on T, (by Lemma IV.2.7)

$$0 = \lim_{n \to \infty} \int_T f_n = \int_T \lim_{n \to \infty} f_n = \int_T f.$$

So by Morera's theorem f is analytic in D.

Now we show that $f_n^{(k)} \to f^{(k)}$ for all $k \geq 1$. Let $D = \overline{B}(a,r) \subset G$. Then there is a number $R > r$ such that $\overline{B}(a,R) \subset G$. If γ is the circle $|z-a| = R$, then by Cauchy's Integral Formula (Corollary IV.5.9) for all z in D we have

$$f_n^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f_n(w)}{(w-z)^{k+1}}\,dw, \quad f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{k+1}}\,dw$$

and so

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f_n(w)-f(w)}{(w-z)^{k+1}}\,dw$$

for all $z \in D$. Let $M_n = \sup\{|f_n(w) - f(w)| : |w-a| = R\}$. Then by Proposition IV.1.17 (b),

$$\begin{aligned} |f_n^{(k)}(z) - f^{(k)}(z)| &= \frac{k!}{2\pi}\left|\int_\gamma \frac{f_n(w)-f(w)}{(w-z)^{k+1}}dw\right| \\ &\leq \frac{k!}{2\pi}\int_\gamma \frac{|f_n(w)-f(w)|}{|w-z|^{k+1}}|dw| \\ &\leq \frac{k!}{2\pi}\frac{M_n 2\pi R}{(R-r)^{k+1}} \qquad \text{for } |z-a| \leq r. \end{aligned}$$

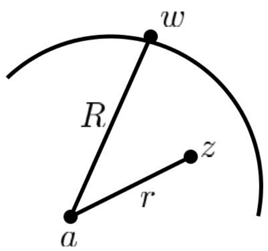
Why $\frac{1}{|w-z|} \leq \frac{1}{(R-r)}$ for $|z-a| \leq r$?

For $|z-a| \leq r$ and $|w-a| = R$, we have

$$|w-z| = |(w-a)-(z-a)| \geq |w-a| - |z-a| \geq R-r,$$

or

$$\frac{1}{|w-z|} \leq \frac{1}{R-r}.$$



Therefore,

2.2 $$|f_n^{(k)}(z) - f^{(k)}(z)| \leq \frac{k! M_n R}{(R-r)^{k+1}} \quad \text{for } |z-a| \leq r.$$

Since $f_n \to f$ in $C(G,\mathbb{C})$, by Proposition 1.10 (b), $f_n \to f$ uniformly on compact set $\overline{B}(a,R)$. That is, $\sup\{|f_n(z) - f(z)| : z \in \overline{B}(a,R)\} \to 0$ and so $\lim M_n = 0$. Hence from (2.2), it follows that $f_n^{(k)} \to f^{(k)}$ uniformly on $\overline{B}(a,r)$ and hence on $B(a,r)$.

Now if K is an arbitrary compact subset of G and $0 < r < d(K,\partial G)$, then there are finitely many $a_1, a_2, \ldots, a_n$ in K such that $K \subset \bigcup_{j=1}^{n} B(a_j, r)$ (since K is totally bounded). Since $f_n^{(k)} \to f^{(k)}$ uniformly on each $B(a_j,r) \subset \overline{B}(a_j,r)$, $1 \leq j \leq n$, the convergence is uniform on K. But by Proposition 1.10 (b), uniform convergence on compact sets implies convergence with respect to the metric ρ and so $f_n^{(k)} \to f^{(k)}$ for all $k \geq 1$. ☐

We will assume that the metric on $H(G)$ is the metric which it inherits as a subset of $C(G,\mathbb{C})$. The next result follows by the completeness of $C(G,\mathbb{C})$.

2.3 Corollary. *$H(G)$ is a complete metric space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $H(G)$. Then $\{f_n\}$ is Cauchy in $C(G,\mathbb{C})$ and since $C(G,\mathbb{C})$ is complete, $f_n \to f$ for some $f \in C(G,\mathbb{C})$. Then by above theorem, f is analytic, i.e., $f \in H(G)$ and hence $H(G)$ is complete. □

2.4 Corollary. *If $f_n : G \to \mathbb{C}$ is analytic and $\sum\limits_{n=1}^{\infty} f_n(z)$ converges uniformly on compact sets to $f(z)$ then*

$$f^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z).$$

Proof. For each n, let $\{S_n(z)\}$ be the sequence of partial sum of the series $\sum\limits_{k=1}^{\infty} f_k(z)$, i.e., $S_n(z) = \sum\limits_{k=1}^{n} f_n(z)$. Since $\sum\limits_{k=1}^{\infty} f_k(z)$ converges uniformly on compact sets of G to $f(z)$, $S_n \to f$ uniformly on compact subsets of G. By Proposition 1.10 (b), $S_n \to f$. Then by above theorem, $S_n^{(k)} \to f^{(k)}$, i.e.,

$$f^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z).$$

□

Remark. Note that Theorem 2.1 does not hold for the functions of a real variable. For example, the absolute value function can be obtained as the uniform limit of a sequence of differentiable functions. Also, it can be shown (by a result of Weierstrass) that a continuous nowhere differentiable function on $[0,1]$ is the limit of a sequence of polynomials.

A contradiction in another direction is given by the following example. Let $f_n(x) = \frac{1}{n}x^n$ for $0 \le x \le 1$. Then $f_n \to 0$ uniformly on $[0,1]$ but the sequence of its derivatives $\{f_n'\} = \{x^{n-1}\}$ does not converge uniformly on $[0,1]$.

Conway therefore calls the analytic functions "special" and justifies his claim by illustrating the following result due to A. Hurwitz's.

2.5 Hurwitz's Theorem. *Let G be a region and suppose the sequence $\{f_n\}$ in $H(G)$ converges to f. If $f \not\equiv 0$, $\overline{B}(a,R) \subset G$, and $f(z) \neq 0$ for $|z-a| = R$ then there is an integer N such that for $n \ge N$, f and f_n have the same number of zeros in $B(a,R)$.*

Proof. Since $f(z) \neq 0$ for $|z-a| = R$,

$$\delta = \inf\{|f(z)| : |z-a| = R\} > 0.$$

Why $\inf\{|f(z)| : |z-a| = R\} > 0$ **?**

Since f is continuous and $\{z : |z-a| = R\}$ is compact, the set $A = \{f(z) : |z-a| = R\}$ be the compact set and $B = \{0\}$ is closed. Since $f(z) \neq 0$ for $|z-a| = R$, A and B are disjoint. Then by Theorem II.5.17, $d(A,B) > 0$. But

$$\begin{aligned} d(A,B) &= \inf\{d(a,b) : a \in A,\ b \in B\} \\ &= \inf\{|f(z) - 0| : |f(z)| \in A,\ 0 \in B\} \\ &= \inf\{|f(z)| : |z-a| = R\}. \end{aligned}$$

Since $f_n \to f$ and the set $\{z : |z-a| = R\}$ is compact, by Proposition 1.10 (b), $f_n \to f$ uniformly on $\{z : |z-a| = R\}$. So (given $\frac{\delta}{2} > 0$) there is an integer N such that if $n \geq N$ and $z \in \mathbb{C}$ with $|z-a| = R$ then

$$|f(z) - f_n(z)| < \frac{1}{2}\delta < \delta < |f(z)| \leq |f(z)| + |f_n(z)|.$$

So by Rouché's Theorem (V.3.8), f_n and f have the same number of zeros in $B(a,R)$. □

2.6 Corollary. *If $\{f_n\} \subset H(G)$ converges to f in $H(G)$ and each f_n never vanishes on G then either $f \equiv 0$ or f never vanishes.*

Proof. If $f \equiv 0$, then we are done. So assume $f \not\equiv 0$ and suppose f vanishes at some point of G, say a, i.e. $f(a) = 0$. Since zeros of analytic function are isolated, there is $R > 0$ such that $f(z) \neq 0$ for all z in $\overline{B}(0,R) \subset G$. In particular, $f(z) \neq 0$ for $|z-a| = R$. Since $f_n \to f$, by Hurwitz's theorem, there is an integer N such that for $n \geq N$, f and f_n have the same number of zeros in $B(a,R)$. But f_n never vanishes and f has a zero at $z = a$ in $B(a,R)$. □

In order to classify normal families in $H(G)$ we need the following terminology.

2.7 Definition. A set $\mathscr{F} \subset H(G)$ is *locally bounded* if for each point a in G there are constants $M > 0$ and $r > 0$ such that for all f in $\mathscr{F}$,

$$|f(z)| \leq M, \quad \text{for } |z-a| < r.$$

Alternately, $\mathscr{F}$ is locally bounded if there is $r > 0$ such that

$$\sup\{|f(z)| : |z-a| < r,\ f \in \mathscr{F}\} < \infty.$$

Thus, $\mathscr{F}$ is locally bounded if about each point a in G there is a disk containing a on which $\mathscr{F}$ is uniformly bounded.

This immediately extends the condition to $\mathscr{F}$ being uniformly bounded on compact subsets of G and we have the following lemma.

2.8 Lemma. *A set $\mathscr{F}$ in $H(G)$ is locally bounded if and only if for each compact set $K \subset G$ there is a constant M such that*

$$|f(z)| \leq M$$

for all f in $\mathscr{F}$ and z in K.

Proof. Exercise. □

Alternately, above lemma can be stated as: a set $\mathscr{F}$ in $H(G)$ is locally bounded if and only if for each compact set $K \subset G$

$$\sup\{|f(z)| : z \in K,\ f \in \mathscr{F}\} < \infty.$$

Now, we classify the normal subsets of $H(G)$.

2.9 Montel's Theorem. *A family $\mathscr{F}$ in $H(G)$ is normal if and only if $\mathscr{F}$ is locally bounded.*

Proof. Let $\mathscr{F}$ be normal. Suppose, if possible, $\mathscr{F}$ is not locally bounded. Then by Lemma 2.8 there is a compact set $K \subset G$ such that $\sup\{|f(z)| : z \in K,\ f \in \mathscr{F}\} = \infty$. That is, there is a sequence $\{f_n\}$ in $\mathscr{F}$ such that

$$(*) \qquad \sup\{|f_n(z)| : z \in K\} \geq n.$$

Explanation (for understanding only; do not write in exam)

- If $\sup\{|f(z)| : z \in K\} < 1$ for all $f \in \mathscr{F}$, then by definition, $\mathscr{F}$ is locally bounded. We have assumed it is not locally bounded. So $\sup\{|f(z)| : z \in K\} \not< 1$ for all $f \in \mathscr{F}$, i.e., there is $f_1 \in \mathscr{F}$ such that $\sup\{|f_1(z)| : z \in K\} \geq 1$.

- Again if $\sup\{|f(z)| : z \in K\} < 2$ for all $f \in \mathscr{F}$, then $\mathscr{F}$ is locally bounded. So $\sup\{|f(z)| : z \in K\} \not< 2$ for all $f \in \mathscr{F}$, i.e., there is $f_2 \in \mathscr{F}$ such that $\sup\{|f_2(z)| : z \in K\} \geq 2$.
- Thus, for each n there is $f_n \in \mathscr{F}$ such that $\sup\{|f_n(z)| : z \in K\} \geq n$.

Since $\mathscr{F}$ is normal, by the definition of normal, there is a function f in $H(G)$ and a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$. By Proposition 1.10 (b), $f_{n_k} \to f$ uniformly on compact set K, i.e., $\sup\{|f_{n_k}(z) - f(z)| : z \in K\} \to 0$ as $k \to \infty$. Since f is continuous on K (being analytic on G) and K is compact, there is $M > 0$ such that $|f(z)| \leq M$ for z in K. Then by $(*)$ we have

$$\begin{aligned} n_k &\leq \sup\{|f_{n_k}(z)| : z \in K\} \\ &\leq \sup\{|f_{n_k}(z) - f(z)| : z \in K\} + \sup\{|f(z)| : z \in K\} \end{aligned}$$

$$\leq \sup\{|f_{n_k}(z)-f(z)| : z \in K\} + M.$$

Now $n_k \to \infty$ as $k \to \infty$ but the right hand side in the above inequality converges to M as $k \to \infty$. This is a contradiction and hence $\mathscr{F}$ must be locally bounded.

Conversely, suppose that $\mathscr{F}$ is locally bounded. We shall show that $\mathscr{F}$ is normal using the Arzela-Ascoli Theorem (1.23). Since condition (a) of Theorem 1.23 is clearly satisfied, we must show that $\mathscr{F}$ is equicontinuous at each point of G.

Why condition (a) of Theorem 1.23 is clearly satisfied?

Since $\mathscr{F}$ is locally bounded, for each point $a \in G$, there are constants $M > 0$ and $r > 0$ such that $|f(z)| \leq M$ for $|z-a| < r$ for all $f \in \mathscr{F}$. So in particular, for each $z \in G$, the set $\{f(z) : f \in \mathscr{F}\}$ is a bounded subset of $\mathbb{C}$ and hence has a compact closure (by Heine-Borel Theorem).

Now we show equicontinuity of $\mathscr{F}$ at each point a in G. Fix $a \in G$ and $\varepsilon > 0$. Since $\mathscr{F}$ is locally bounded, there is $r > 0$ and $M > 0$ such that $\overline{B}(a,r) \subset G$ and $|f(z)| \leq M$ for all $z \in \overline{B}(a,r)$ and for all $f \in \mathscr{F}$.

Why can we get such $r > 0$?

Fix $a \in G$. Since G is open, there is $R_1 > 0$ such that $B(a,R_1) \subset G$.
Since $\mathscr{F}$ is locally bounded, there is $R_2 > 0$ and $M > 0$ such that $|f(z)| \leq M$ for all $z \in B(a,R_2)$ and for all $f \in \mathscr{F}$.
Let $R = \min\{R_1,R_2\}$. Then $B(a,R) \subset G$ and $|f(z)| \leq M$ for all $z \in B(a,R)$ and for all $f \in \mathscr{F}$. Taking $r = \frac{R}{2}$ or any positive number smaller than R, we get $\overline{B}(a,r) \subset B(a,R) \subset G$ and $|f(z)| \leq M$ for all $z \in \overline{B}(a,r)$ and for all $f \in \mathscr{F}$.

Let $|z-a| < \frac{1}{2}r$ and $f \in \mathscr{F}$. Let $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$. If $w \in \{\gamma\}$, then $|w-a| = r \geq \frac{r}{2}$. Since $|z-a| < \frac{r}{2}$, we have $-|z-a| > -\frac{r}{2}$ and so

$$|w-z| = |w-a+a-z| \geq |w-a| + |a-z| > r - \frac{r}{2} = \frac{r}{2}.$$

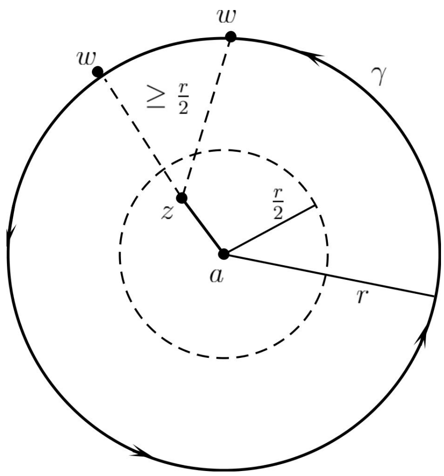


Figure VII.3:

By Cauchy's Integral Formula (First Version, Theorem IV.5.4) applied to $f(a)$ and $f(z)$, we get

$$\begin{aligned}
|f(a)-f(z)| &= \left|\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}\,dw - \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw\right| \\
&= \frac{1}{2\pi}\left|\int_\gamma \frac{f(w)(a-z)}{(w-a)(w-z)}\,dw\right| \\
&\leq \frac{1}{2\pi}\int_\gamma \frac{|f(w)||(a-z)|}{|w-a||w-z|}\,|dw| \\
&\leq \frac{1}{2\pi}\frac{M|z-a|}{\frac{r}{2}\frac{r}{2}}\int_\gamma |dw| && \left(\because \begin{smallmatrix}|w-a|\geq \frac{r}{2}\\ |w-z|\geq\frac{r}{2}\end{smallmatrix}, |f(z)|\leq M\right) \\
&= \frac{2M|z-a|}{\pi r^2}\cdot 2\pi r && (\because \gamma(t)=a+re^{it}) \\
&= \frac{4M}{r}|z-a|.
\end{aligned}$$

Let $\delta < \min\left\{\frac{1}{2}r, \frac{r}{4M}\varepsilon\right\}$. Then $|z-a|<\delta$ implies $|f(z)-f(a)|<\varepsilon$ for all $f\in\mathscr{F}$, i.e., $\mathscr{F}$ is equicontinuous at a in G. Since a is arbitrary, $\mathscr{F}$ is equicontinuous at each point of G. Hence by Arzela-Ascoli Theorem, it follows that $\mathscr{F}$ is normal. □

2.10 Corollary. *A set $\mathscr{F}\subset H(G)$ is compact if and only if it is closed and locally bounded.*

Proof. Suppose $\mathscr{F}$ is compact. Since compact subset of a Hausdorff space (in particular, any metric space) is closed, it follows that So $\overline{\mathscr{F}}=\mathscr{F}$. Since $\mathscr{F}$ has a compact closure, by Proposition 1.15, $\mathscr{F}$ is normal. Hence, by Theorem 2.9, $\mathscr{F}$ is locally bounded.

Conversely assume that $\mathscr{F}$ is closed and locally bounded. Since $\mathscr{F}$ is locally bounded, by above theorem (2.9), $\mathscr{F}$ is normal. Then by Proposition 1.15, $\overline{\mathscr{F}}$ is compact. But since $\mathscr{F}$ is closed, it follows that $\mathscr{F}$ is compact. □

§5. The Weierstrass Factorization Theorem

5.1 Definition. If $\{z_n\}$ is a sequence of complex numbers and if $z = \lim_{n\to\infty}\left(\prod_{k=1}^{n} z_k\right)$ exists, then z is the *infinite product* of the number z_n and it is denoted by

$$z = \prod_{n=1}^{\infty} z_n.$$

5.A Remark. Suppose that none of the numbers z_n is zero, and that $z = \prod_{n=1}^{\infty} z_n$ exists and is also not zero. Let $p_n = \prod_{k=1}^{n} z_k$ for $n \geq 1$; then no p_n is zero and $\frac{p_n}{p_{n-1}} = z_n$. Since $z \neq 0$ and $p_n \to z$ we have that $\lim_{n\to\infty} z_n = 1$. So that except for the cases where zero appears, a necessary condition for the convergence of an infinite product is that the n-th term must go to 1.

On the other hand, note that for $z_n = a$ for all n and $|a| < 1$, the product $\prod z_n = 0$ although $\lim_{n\to\infty} z_n = a \neq 0$. So even if none of z_n is zero, the infinite product can still be zero.

Since the exponential of a sum is the product of the exponentials of the individual terms, it is possible to discuss the convergence of the infinite product $\prod z_n$ (when zero is not involved) by discussing the convergence of the series $\sum \log z_n$, where log is the principal branch of the logarithm.

5.2 Proposition. *Let* $\operatorname{Re} z_n > 0$ *for all* $n \geq 1$. *Then* $\prod_{n=1}^{\infty} z_n$ *converges to a nonzero number if and only if the series* $\sum_{n=1}^{\infty} \log z_n$ *converges.*

Proof. Suppose $\prod_{n=1}^{\infty} z_n$ converges to z $(z \neq 0)$, where $z = re^{i\theta}$, $-\pi < \theta \leq \pi$. Let $p_n = (z_1 \cdots z_n) = \prod_{k=1}^{n} z_k$. Then $\lim_{n\to\infty} p_n = z$.

Define a branch of logarithm ℓ which is continuous at z (i.e. $\arg_\ell$ is continuous at $z = re^{i\theta}$). Let $\ell(p_n) = \log|p_n| + i\theta_n$, where $\theta - \pi < \theta_n \leq \theta + \pi$. Since $p_n \to z$, we have $\lim_{n\to\infty} |p_n| = |z| = r$ and $\lim_{n\to\infty} \theta_n = \theta$. Hence,

$$(*) \qquad \lim_{n\to\infty} \ell(p_n) = \lim_{n\to\infty} (\log|p_n| + i\theta_n) = \log|z| + i\theta = \ell(z).$$

Let $s_n = \log z_1 + \cdots + \log z_n$. Then $\exp(s_n) = p_n$ and so $s_n = \ell(p_n) + 2\pi i k_n$ for some integer k_n. But

$$s_n - s_{n-1} = \sum_{k=1}^{n} \log z_k - \sum_{k=1}^{n-1} \log z_k = \log z_n.$$

Since $\prod z_n$ converges, by Remark 5.A, $\lim z_n \to 1$ and so

$$\lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} \log z_n = \log \lim_{n\to\infty} z_n = \log 1 = 0.$$

Consequently,

$$\lim_{n\to\infty}\left([\ell(p_n)-\ell(p_{n-1})]+2\pi i\,[k_n-k_{n-1}]\right)=0.$$

But by ((∗)), $\ell(p_n)-\ell(p_{n-1})\to\ell(z)-\ell(z)=0$. Hence (by above) $(k_n-k_{n-1})\to 0$ as $n\to\infty$. Since each k_n is an integer, this gives that there is an integer n_0 and a k such that $k_m=k_n=k$ for $m,n\geq n_0$ (i.e. the sequence $\{k_n\}$ is eventually constant). So,

$$\lim_{n\to\infty}s_n=\lim_{n\to\infty}(\ell(p_n)+2\pi ik_n)=\ell(z)+2\pi ik.$$

Hence, $\sum\limits_{n=1}^{\infty}\log z_n$ converges (as $\lim s_n$ exists, where $\{s_n\}$ is sequence of partial sum of the series).

Conversely, assume that $\sum\limits_{n=1}^{\infty}\log z_n$ converges. If $s_n=\sum\limits_{k=1}^{n}\log z_k$, then $s_n\to s$ for some $s\in\mathbb{C}$. Then $\exp s_n\to\exp s\neq 0$. But $\exp s_n=\prod\limits_{k=1}^{n}z_k=p_n$. Thus, $p_n\to e^s\neq 0$, i.e., the infinite product $\prod\limits_{n=1}^{\infty}z_n$ converges to $z=e^s\neq 0$. □

5.B Lemma. *If* $|z|<\frac{1}{2}$ *then* $\frac{1}{2}|z|\leq|\log(1+z)|\leq\frac{3}{2}|z|$.

Proof. The power series expansion of $\log(1+z)$ about $z=0$ is given by

$$\log(1+z)=\sum_{n=2}^{\infty}(-1)^{n-1}\frac{z^n}{n}=z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\cdots,$$

which has radius of convergence 1. So if $|z|<1$, then

$$\begin{aligned}\left|1-\frac{\log(1+z)}{z}\right| &= |\tfrac{1}{2}z-\tfrac{1}{3}z^2+\cdots|\\ &\leq \tfrac{1}{2}|z|+\tfrac{1}{3}|z|^2+\cdots\\ &\leq \tfrac{1}{2}\left(|z|+|z|^2+\cdots\right)\\ &= \tfrac{1}{2}\tfrac{|z|}{1-|z|}.\end{aligned}$$

If $|z|<\frac{1}{2}$, then $\frac{|z|}{1-|z|}<1$ and so

$$\left|1-\frac{\log(1+z)}{z}\right|\leq\frac{1}{2}\quad\text{or}\quad|z-\log(1+z)|\leq\tfrac{|z|}{2}.$$

We know that $\big||z|-|w|\big|\leq|z-w|$, i.e., $|z|-|w|\leq|z-w|$ and $|w|-|z|\leq|z-w|$. Hence from the above inequality, we have

(∗) $$|\log(1+z)|-|z|\leq\tfrac{|z|}{2}\Rightarrow|\log(1+z)|\leq\tfrac{3}{2}|z|$$

(∗∗) $$|z|-|\log(1+z)|\leq\tfrac{|z|}{2}\Rightarrow\tfrac{1}{2}|z|\leq|\log(1+z)|.$$

Combining (∗) and (∗∗), we have

5.3 $$\frac{1}{2}|z|\leq|\log(1+z)|\leq\frac{3}{2}|z|.$$

□

This inequality will be useful in proving the following result.

5.4 Proposition. *Let* $\operatorname{Re} z_n > -1$*; then the series* $\sum \log(1+z_n)$ *converges absolutely if and only if the series* $\sum z_n$ *converges absolutely.*

Proof. Suppose $\sum_{n=1}^{\infty} z_n$ converges absolutely, i.e., $\sum_{n=1}^{\infty} |z_n|$ converges. Then $|z_n| \to 0$ and so $z_n \to 0$ as $n \to \infty$. So (by definition of convergence, given $\varepsilon = \frac{1}{2}$) there is $n_0 \in \mathbb{N}$ such that $|z_n| < \frac{1}{2}$ for all $n \geq n_0$. By (5.3), for all $n \geq n_0$, $|\log(1+z_n)| \leq \frac{3}{2}|z_n|$. Therefore,

$$\sum_{n\geq n_0} |\log(1+z_n)| \leq \frac{3}{2} \sum_{n\geq n_0} |z_n| < \infty.$$

Hence, the series $\sum \log(1+z_n)$ converges absolutely.

Conversely, suppose that $\sum_{n=1}^{\infty} \log(1+z_n)$ converges absolutely, i.e., $\sum_{n=1}^{\infty} |\log(1+z_n)|$ converges. Then $\lim_{n\to\infty} |\log(1+z_n)| = 0$ and so $\lim_{n\to\infty} z_n = 0$. So there is $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have $|z| < \frac{1}{2}$. Then by (5.3) for all $n \geq n_1$, we have $\frac{1}{2}|z| \leq |\log(1+z_n)|$. Hence,

$$\frac{1}{2} \sum_{n\geq n_1} |z| \leq \sum_{n\geq n_1} |\log(1+z_n)| < \infty.$$

Therefore, the series $\sum_{n=1}^{\infty} |z_n|$ converges. □

We wish to define the absolute convergence of an infinite product. We may be tempted to define it as follows:

"If $\prod |z_n|$ converges, then we say that $\prod z_n$ converges absolutely."

However, this is not an appropriate definition as absolute convergence does not imply convergence. For example, let $z_n = -1$ for all n. Then $|z_n| = 1$ for all n and so $\prod |z_n|$ converges to 1. But $\prod_{k=1}^{n} z_k$ is ± 1 depending on whether n is even or odd and so $\prod z_n$ does not converge.

The following definition of absolute convergence is based on Proposition 5.2 and is justified.

5.5 Definition. If $\operatorname{Re} z_n > 0$ for all n then the infinite product $\prod z_n$ is said to *converge absolutely* if the series $\sum \log z_n$ converges absolutely.

5.C Remark.

- From Proposition 5.2 and the fact that absolute convergence of a series implies convergence, we have that absolute convergence of an infinite product implies the convergence of the product.
- If a product converges absolutely, then any rearrangement of the terms of the product results in a product which is still absolutely convergent.

Combining Proposition 5.2 and Proposition 5.4 with Definition 5.5, we have the following fundamental criterion for the absolute convergence of an infinite product.

5.6 Corollary. *If* Re $z_n > 0$ *then the product* $\prod z_n$ *converges absolutely if and only if the series* $\sum(z_n - 1)$ *converges absolutely.*

Proof. We have,

$$\begin{aligned} & \prod z_n \text{ coverges absolutely} & \\ \Leftrightarrow & \sum \log z_n \text{ converges absolutely} & (\text{by definition as } \mathrm{Re}\, z_n > 0) \\ \Leftrightarrow & \sum \log(1+(z_n-1)) \text{ converges absolutely} & \\ & \text{i.e.,} \sum \log(1+z_m) \text{ converges absolutely} & (\mathrm{Re}\, z_m = \mathrm{Re}\,(z_n-1) > -1) \\ \Leftrightarrow & \sum z_m \text{ converges absolutely} & (\text{where } z_m = z_n - 1) \\ & \text{i.e.,} \sum (z_n-1) \text{ converges absolutely} & (\text{by Proposition 5.4}). \end{aligned}$$

□

Now we apply these results to the convergence of products of functions.

5.7 Lemma. *Let X be a set and let $f, f_1, f_2, \ldots$ be functions from X in to $\mathbb{C}$ such that $f_n(x) \to f(x)$ uniformly for x in X. If there is a constant a such that* Re $f(x) \le a$ *for all x in X then* $\exp f_n(x) \to \exp f(x)$ *uniformly for x in X.*

Proof. Let $\varepsilon > 0$ be given. Since e^z is continuous at $z = 0$, there is $\delta > 0$ such that $|e^z - 1| < \varepsilon e^{-a}$ whenever $|z| < \delta$. Since $f_n(x) \to f(x)$ uniformly for x in X, (given $\varepsilon = \delta > 0$) there is $n_0 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \delta$ for all x in X whenever $n \ge n_0$. Then for $n \ge n_0$, for all x in X (taking $z = f_n(x) - f(x)$) we have

$$\begin{aligned} & |\exp[f_n(x) - f(x)] - 1| < \varepsilon e^{-a} \\ \Rightarrow & \left| \frac{\exp f_n(x)}{\exp f(x)} - 1 \right| < \varepsilon e^{-a}. \end{aligned}$$

So for all $n \ge n_0$ for all $x \in X$,

$$\begin{aligned} |\exp f_n(x) - \exp f(x)| &< \varepsilon e^{-a} |\exp f(x)| \\ &= \varepsilon e^{-a} \exp[\mathrm{Re}\,(f(x))] \qquad (\because |\exp f(x)| = \exp[\mathrm{Re}\,(f(x))]) \\ &= \varepsilon \exp[\mathrm{Re}\,(f(x)) - a] \\ &\le \varepsilon \qquad (\because \mathrm{Re}\,(f(x)) \le a). \end{aligned}$$

That is, $\exp f_n(x)$ converges to $\exp f(x)$ uniformly on X. □

5.8 Lemma. *Let (X,d) be a compact metric space and let $\{g_n\}$ be a sequence of continuous functions from X into $\mathbb{C}$ such that $\sum g_n(x)$ converges absolutely and uniformly for x in X. Then the product*

$$f(x) = \prod_{n=1}^{\infty}(1+g_n(x))$$

converges absolutely and uniformly for x in X. Also there is an integer n_0 such that $f(x)=0$ if and only if $g_n(x)=-1$ for some n, $1 \le n \le n_0$.

Proof. Since $\sum g_n(x)$ converges uniformly for x in X, there is an integer n_0 such that $|g_n(x)| < \frac{1}{2}$ for all x in X and $n > n_0$.

Why?

Since $\sum g_n(x)$ converges uniformly for all $x \in X$, $g_n(x) \to 0$ uniformly for all $x \in X$. Then given $\varepsilon = \frac{1}{2}$ there is $n_0 \in \mathbb{N}$ such that $|g_n(x)| < \frac{1}{2}$ whenever $n > n_0$, for all $x \in X$.

This implies that Re $[1+g_n(x)] > 0$ for all $n > n_0$ and x in X and also by inequality (5.3), $|\log(1+g_n(x))| \le \frac{3}{2}|g_n(x)|$ for all $n > n_0$ and x in X.

Why Re $[1+g_n(x)] > 0$ **?**

We have $|g_n(x)| < \frac{1}{2}$. Since $|\text{Re } g(x)| \le |g_n(x)| < \frac{1}{2}$, it follows that $-\frac{1}{2} < \text{Re } g(x) < \frac{1}{2}$. This implies

$$\text{Re }[1+g_n(x)] = 1 + \text{Re } g_n(x) > 1 - \frac{1}{2} = \frac{1}{2} > 0.$$

Since the series $\sum_{n=1}^{\infty} \frac{3}{2}|g_n(x)|$ converges uniformly for x in X and for $n > n_0$,

$$h(x) = \sum_{n=n_0+1}^{\infty} \log(1+g_n(x))$$

converges uniformly and absolutely for x in X. Since each g_n is continuous, $\log(1+g_n)$ is continuous and hence h is continuous. Since X is compact, $h(X)$ is a compact subset of $\mathbb{C}$ and so it bounded. It follows that h must be bounded; in particular, there is a constant a such that $|h(x)| < a$ and so Re $h(x) < a$ for all x in X. By Lemma 5.7

$$\exp h(x) = \prod_{n=n_0+1}^{\infty}(1+g_n(x))$$

converges uniformly for x in X. Also, since $\sum_{n=n_0+1}^{\infty} \log(1+g_n(x))$ converges absolutely, by definition, $\prod_{n=n_0+1}^{\infty}(1+g_n(x))$ converges absolutely. Hence,

$$f(x) = [1+g_1(x)]\cdots[1+g_{n_0}(x)]\exp h(x) = \prod_{n=1}^{\infty}(1+g_n(x))$$

converges absolutely and uniformly for x in X.

Finally, since $\exp h(x) \neq 0$ for any x in X, $f(x) = 0$ if and only if $1 + g_n(x) = 0$ for some n with $1 \leq n \leq n_0$, i.e. $g_n(x) = -1$ for some n with $1 \leq n \leq n_0$. □

5.9 Theorem. *Let G be a region in $\mathbb{C}$ and let $\{f_n\}$ be a sequence in $H(G)$ such that no f_n is identically zero. If $\sum [f_n(z) - 1]$ converges absolutely and uniformly on compact subsets of G then $\prod_{n=1}^{\infty} f_n(z)$ converges in $H(G)$ to an analytic function $f(z)$. If a is a zero of f then a is a zero of only a finite number of functions f_n, and the multiplicity of the zero of f at a is the sum of the multiplicities of the zeros of the functions f_n at a.*

Proof. Since $\sum [f_n(z) - 1]$ converges uniformly and absolutely on compact subsets of G, by previous lemma, $f(z) = \prod f_n(z)$ converges uniformly and absolutely on compact subsets of G. Then by Proposition 1.10 (b), the infinite product $\prod_{n=1}^{\infty} f_n(z)$ converges in $H(G)$ (with respect to the metric ρ on $H(G)$).

Suppose a is a zero of f, i.e., $f(a) = 0$. Let $r > 0$ be chosen such that $\overline{B}(a, r) \subset G$. Since $\overline{B}(a, r)$ is compact, by hypothesis, $\sum [f_n(z) - 1]$ converges uniformly on the compact set $\overline{B}(a, r)$. By Lemma 5.8 (as seen in the proof), there is an integer n such that

$$f(z) = f_1(z) \cdots f_n(z) g(z),$$

where g does not vanish in $\overline{B}(a, r)$. So a is a zero of only a finite number of functions f_n and the multiplicity of the zero of f at a is the sum of multiplicities of the zeros of the functions f_n at a. □

In the beginning of the section, Conway starts by discussing the main problem, which is recalled again here. If $\{a_n\}$ is a sequence in a region G with no zeros in G (but possibly some point may be repeated in the sequence a finite number of times), then consider the functions $(z - a_n)$. By Theorem 5.9, if we can find functions which are analytic on G, have no zeros in G such that $\sum |(z - a_n) g_n(z) - 1|$ converges uniformly on compact subsets of G; then $f(z) = \prod (z - a_n) g_n(z)$ is analytic and has its zeros only at the points $z = a_n$. The safest way to guarantee that $g_n(z)$ never vanishes is to express it as $g_n(z) = \exp h_n(z)$ for some analytic function $h_n(z)$. In fact, if G is simply connected, it follows (from Corollary IV.6.17) that $g_n(z)$ must be of this form. The functions we are going to see were introduced by Weierstrass.

5.10 Definition. An *elementary factor* is one of the following functions $E_p(z)$ for $p = 0, 1, \ldots$:

$$E_0(z) = 1 - z,$$
$$E_p(z) = (1 - z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right), \; p \geq 1.$$

The function $E_p : \mathbb{C} \to \mathbb{C}$ defined above is also called *Weierstrass elementary factor*. Note that, the function $E_p(z/a)$ has a simple zero at $z = a$ and no other zero. Also if b is a point in $\mathbb{C} \setminus G$ then $E_p\left(\frac{a-b}{z-b}\right)$ has a simple zero at $z = a$ and is analytic in G.

5.11 Lemma. *If* $|z| \le 1$ *and* $p \ge 0$ *then* $|1 - E_p(z)| \le |z|^{p+1}$.

Proof. For $p = 0$,

$$|1 - E_p(z)| = |1 - E_0(z)| = |1 - (1 - z)| = |z| \le |z|^{0+1} = |z|^{p+1}.$$

For a fixed $p \ge 1$, $E_p(z)$ is an entire function. Let

$$E_p(z) = \sum_{k=0}^{\infty} a_k z^k$$

be the power series expansion of $E_p(z)$ about $z = 0$. Since $E_p(0) = 1$ (by definition of $E_p(z)$), we have $a_0 = 1$. So

$$E_p(z) = 1 + \sum_{k=1}^{\infty} a_k z^k.$$

Then

$$(*) \qquad E_p'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}.$$

On the other hand, from the definition of $E_p(z)$,

$$\begin{aligned} E_p'(z) &= (-1)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) \\ &\qquad + (1-z)(1+z+z^2+\cdots+z^{p-1})\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) \\ &= -(1-(1-z^p))\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) \\ (**) \qquad &= -z^p \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right). \end{aligned}$$

Comparing the two expressions $(*)$ and $(**)$,

Comparing $(*)$ and $(**)$

$$\begin{aligned} \sum_{k=1}^{\infty} k a_k z^{k-1} &= -z^p \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) \\ &= -z^p\left[1 + \left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) + \frac{1}{2!}\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right)^2 + \cdots\right] \end{aligned}$$

$$= -\left[z^p + \left(z^{p+1} + \frac{z^{p+2}}{2} + \cdots + \frac{z^{2p}}{p}\right) + \frac{z^p}{2!}\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right)^2 + \cdots\right]$$

we deduce the following two things about the coefficients a_k.

- First, $a_1 = a_2 = \cdots = a_p = 0$; and second
- $a_k \leq 0$ for $k \geq p+1$.

Thus, $|a_k| = -a_k$ for $k \geq p+1$. For $z = 1$, since $a_1 = \cdots = a_p = 0$, this gives

$$0 = E_p(1) = 1 + \sum_{k=1}^{\infty} a_k = 1 + \sum_{k=p+1}^{\infty} a_k \qquad (\because a_1 = \cdots = a_p = 0)$$

or

$$(***) \qquad \sum_{k=p+1}^{\infty} |a_k| = -\sum_{k=p+1}^{\infty} a_k = 1.$$

So for $|z| \leq 1$,

$$\begin{aligned}
|1 - E_p(z)| = |E_p(z) - 1| &= \left|\left(1 + \sum_{k=p+1}^{\infty} a_k z^k\right) - 1\right| \\
&= \left|\sum_{k=p+1}^{\infty} a_k z^k\right| \\
&= |z|^{p+1}\left|\sum_{k=p+1}^{\infty} a_k z^{k-p-1}\right| \\
&\leq |z|^{p+1} \sum_{k=p+1}^{\infty} |a_k||z|^{k-p-1} \\
&\leq |z|^{p+1} \sum_{k=p+1}^{\infty} |a_k| && (\because |z| \leq 1) \\
&= |z|^{p+1} && (\text{by } (***))
\end{aligned}$$

which is the desired inequality. □

In the following result we apply Weierstrass M-test and so we recall it now.

Weierstrass M-test

Let $\{f_n\}$ be a sequence of (real or) complex-valued functions on a set S and $\{M_n\}$ be a sequence of positive numbers satisfying $|f_n(z)| \leq M_n$ for all $z \in S$ and for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} M_n < \infty$, then the series $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely and uniformly on S.

5.12 Theorem. *Let $\{a_n\}$ be a sequence in $\mathbb{C}$ such that $\lim |a_n| = \infty$ and $a_n \neq 0$ for all $n \geq 1$. (This is not a sequence of distinct points; but, by hypothesis, no point is repeated an infinite number of times.) If $\{p_n\}$ is any sequence of integers such that*

5.13
$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^{p_n+1} < \infty$$

for all $r > 0$ then

$$f(z) = \prod_{n=1}^{\infty} E_{p_n}(z/a_n)$$

converges in $H(\mathbb{C})$. The function f is an entire function with zeros only at the points a_n. If z_0 occurs in the sequence $\{a_n\}$ exactly m times then f has a zero at $z = z_0$ of multiplicity m. Furthermore, if $p_n = n-1$ then (5.13) will be satisfied.

Proof. Suppose $\{p_n\}$ is a sequence of integers satisfying (5.13). Then by Lemma 5.11,

$$|1 - E_{p_n}(z/a_n)| \leq \left| \frac{z}{a_n} \right|^{p_n+1} \leq \left(\frac{r}{|a_n|} \right)^{p_n+1}$$

whenever $|z| \leq r$ and $r \leq |a_n|$ (so that $|z/a_n| \leq r/|a_n| \leq 1$). Since $\lim |a_n| = \infty$, for a fixed $r > 0$ there is an integer N such that $|a_n| \geq r$ for all $n \geq N$. Thus, for each $r > 0$

$$\sum_{n=1}^{\infty} |1 - E_{p_n}(z/a_n)| \leq \sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^{p_n+1} \qquad \text{for } z \in \overline{B}(0,r).$$

By (5.13), the right hand side of the above inequality is finite and so $\sum_{n=1}^{\infty} [1 - E_{p_n}(z/a_n)]$ converges absolutely on $\overline{B}(0,r)$. Also by Weierstrass M-test, the series converges uniformly on $\overline{B}(0,r)$. Since r is arbitrary, the series $\sum_{n=1}^{\infty} [1 - E_{p_n}(z/a_n)]$ converges absolutely and uniformly on compact subsets of $\mathbb{C}$ and hence it converges absolutely in $H(\mathbb{C})$. Then by Theorem 5.9, the finite product $\prod_{n=1}^{\infty} E_{p_n}(z/a_n)$ converges (absolutely) in $H(\mathbb{C})$.

Thus, $f(z) = \prod_{n=1}^{\infty} E_{p_n}(z/a_n)$, where f is an entire function. By the definition of elementary factor (Definition 5.10), it is clear that the zeros of f are only at the points a_n.

Now suppose z_0 occurs in the sequence $\{a_n\}$ exactly m times, say $a_{\alpha_1} = \cdots a_{\alpha_m} = z_0$ and $a_n \neq z_0$ for all $n \neq \alpha_i$, $i = 1, \ldots, m$. Then again by the definition of elementary factor $E_{p_{\alpha_i}}(z/|a_{\alpha_i}|)$ is zero only at $z = z_0$ for $i = 1, \ldots, m$ and $E_{p_n}(z_0/|a_n|) \neq 0$ for all $n \neq \alpha_i$, $i = 1, \ldots, m$.

Finally we show that if $p_n = n-1$, then (5.13) is satisfied. Since $|a_n| \to \infty$, for any r there is an integer N such that $|a_n| > 2r$ for all $n \geq N$. This gives, $\left(\frac{r}{|a_n|} \right) < \frac{1}{2}$ for all $n \geq N$. Hence if $p_n = n-1$ for all n, then

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^{p_n+1} = \sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^{n}$$

$$= \sum_{n=1}^{N} \left(\frac{r}{|a_n|}\right)^n + \sum_{n=N+1}^{\infty} \left(\frac{r}{|a_n|}\right)^n$$
$$\leq \sum_{n=1}^{N} \left(\frac{r}{|a_n|}\right)^n + \sum_{n=N+1}^{\infty} \left(\frac{1}{2}\right)^n < \infty.$$

□

5.14 The Weierstrass Factorization Theorem. *Let f be an entire function and let $\{a_n\}$ be the non-zero zeros of f repeated according to multiplicity; suppose f has a zero at $z = 0$ of order $m \geq 0$ (a zero of order $m = 0$ at $z = 0$ means $f(0) \neq 0$). Then there is an entire function g and a sequence of integers $\{p_n\}$ such that*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

Proof. Finite Case. If f has finitely many zeros, then the result is immediate.

Finite Case.

Suppose f has finitely many non-zero zeros $a_1, \ldots, a_n$ (i.e., $a_n \neq 0$) counted according to their multiplicities and $z = 0$ is a zero of f of order $m \geq 0$. Since f is an entire function, (by Corollary IV.3.9) there is an entire function h such that

$$f(z) = z^m (z - a_1) \cdots (z - a_n) h(z) = z^m (a_1 \cdots a_n) h(z) \prod_{k=1}^{n} (1 - z/a_k).$$

and $h(z) \neq 0$ for all $z \in \mathbb{C}$. Since $(a_1 \cdots a_n)h(z)$ is a non-vanishing entire function and $\mathbb{C}$ is simply connected, (by Corollary IV.6.17) there is an entire function g such that $(a_1 \cdots a_n)h(z) = \exp(g(z))$ for all $z \in \mathbb{C}$. Hence,

$$f(z) = z^m \exp(g(z)) \prod_{k=1}^{n} (1 - z/a_k) = z^m e^{g(z)} \prod_{k=1}^{n} E_0\left(\frac{z}{a_k}\right).$$

General Case. Assume that there are infinitely many non-zero zeros a_n of f counted according to their multiplicities, i.e. the sequence $\{a_n\}$ is infinite, and suppose $z = 0$ is a zero of f of order $m \geq 0$. Then note that $\lim |a_n| = \infty$.

Why $\lim |a_n| = \infty$ **?**

Since the set of zeros $\{a_n\}$ of f is countable, it follows that $f \not\equiv 0$.
Suppose, if possible, $\lim |a_n| \neq \infty$. Then
Case 1. $\{a_n\}$ is a bounded sequence, i.e., $|a_n| \leq M$ for some M. By Bolzano-Weierstrass property, it has a convergent subsequence. By Identity Theorem (IV.3.7), $f \equiv 0$ which is a contradiction.
Case 2. $\{a_n\}$ is not bounded. Since $\lim_{n\to\infty} |a_n| \neq \infty$, there exists $M > 0$ such that for all $N \in \mathbb{N}$, $|a_n| < M$ for some $n \geq N$.

for example

To understand it better consider the sequence $\{a_n\}$ to be

$$1,2,1,3,-1,4,i,5,-1,-7,1,8,-i,9,-1,10,\dots.$$

The subsequence of odd terms in the above sequence is bounded.

Then $\{a_n\}$ has a subsequence which is bounded. Hence, by Case 1, it further has a subsequence which is convergent and so $f \equiv 0$.
Hence, $|a_n| \to \infty$ as $n \to \infty$.

According to the previous theorem (5.12), there is a sequence $\{p_n\}$ of integers such that

$$h(z) = z^m \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

has the same zeros as f with the same multiplicities. So $f(z)/h(z)$ has removable singularities at $z = 0, a_1, a_2, \dots$. Thus, f/h is an entire function and has no zeros. Since $\mathbb{C}$ is simply connected, (by Corollary IV.6.17) there is an entire function g such that

$$\frac{f(z)}{h(z)} = e^{g(z)}.$$

Hence,

$$f(z) = h(z)e^{g(z)} = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

□

§6. Factorization of the sine function

In this section, we apply the Weierstrass Factorization Theorem to express $\sin \pi z$ as an infinite product using the fact that its zeros are precisely the integers. Since the definition of the elementary factor $E_0(z)$ is different from $E_p(z)$, $p \geq 1$, we treat the zero of f at $z = 0$ differently from the other zeros of f. In this regard, we introduce the following notations for the sum and product without the case $n = 0$.

If an infinite sum or product is followed by a prime (apostrophe), i.e., $\sum'$ or $\prod'$, then the sum or product is to be taken over all the indicated indices n except $n = 0$. For example,

$$\sum_{n=-\infty}^{\infty}{}' a_n = \sum_{n=1}^{\infty} a_{-n} + \sum_{n=1}^{\infty} a_n \quad \text{and} \quad \prod_{n=-\infty}^{\infty}{}' a_n = \prod_{n=1}^{\infty} a_{-n} \prod_{n=1}^{\infty} a_n.$$

We state the following exercise, without proof, which will be used in expressing $\sin \pi z$ as an infinite product.

Exercise. Suppose G is an open set and $\{f_n\}$ is a sequence in $H(G)$ converging to f in $H(G)$. If $f(z) \neq 0$ for all z in some compact set $K \subset G$, then $\frac{f_n'}{f_n} \to \frac{f_n}{f}$ uniformly on K.

6.A Theorem. *For all* $z \in \mathbb{C}$*,*

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

and the convergence is uniform over all compact subsets of $\mathbb{C}$*.*

Proof. The zeros of $\sin \pi z = \frac{1}{2i}(e^{inz} - e^{-inz})$ are precisely the integers. Moreover, each zero of $\sin \pi z$ is a simple zero.

Why each zero of $\sin \pi z$ is simple?

Let $f(z) = \sin \pi z$. Then $f'(z) = \pi \cos \pi z$ and $\pi \cos \pi n = \pm\pi \neq 0$ for all $n \in \mathbb{Z}$. So zeros of $f(z) = \sin \pi z$ are of multiplicity one.

Now for all $r > 0$,

$$\sum_{n=-\infty}^{\infty}{}' \left(\frac{r}{n}\right)^2 = 2\sum_{n=1}^{\infty} \left(\frac{r}{n}\right)^2 = 2r^2 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Thus, (5.13) in the hypothesis of Theorem 5.12 is satisfied by choosing $p_n = 1$ for all n. So by Weierstrass Factorization Theorem (5.14) (with $a_n = n$ and $p_n = 1$ for all $n \in \mathbb{Z}$), we get

$$\sin \pi z = z\,[\exp g(z)] \prod_{n=-\infty}^{\infty}{}' E_{p_n}\left(\frac{z}{a_n}\right) = z\,[\exp g(z)] \prod_{n=-\infty}^{\infty}{}' \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}$$

for all $z \in \mathbb{C}$ and for some entire function $g(z)$. Now the infinite product converges absolutely in $H(\mathbb{C})$ (see the proof of Theorem 5.12) and so the terms can be rearranged to obtain

6.1 $$\sin \pi z = [\exp g(z)]\, z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

How?

Since $\left(1 - \frac{z}{n}\right)\left(1 - \frac{z}{-n}\right) = \left(1 - \frac{z^2}{n^2}\right)$.

Now we determine the entire function $g(z)$. If $f(z) = \sin \pi z$, then $f'(z) = \pi \cos \pi z$ and so $\pi \cot \pi z = \frac{\pi \cos \pi z}{\sin \pi z} = \frac{f'(z)}{f(z)}$. Let $f_n(z) = [\exp g(z)]\, z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$. Then by (6.1), $f_n(z) \to f(z) = \sin \pi z$. By Theorem 2.1, $f_n'(z) \to f'(z) = \pi \cos \pi z$. So

$$\pi \cos \pi z = f'(z) = [\exp g(z)] \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) + z\,[g'(z) \exp g(z)] \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

$$+z\,[\exp g(z)]\left(\sum_{j=1}^{\infty}\frac{-2z}{j^2}\prod_{\substack{n=1\\n\neq j}}^{\infty}\left(1-\frac{z^2}{n^2}\right)\right).$$

Then

$$\pi\cot\pi z=\frac{f'(z)}{f(z)}=\frac{1}{z}+g'(z)+\sum_{j=1}^{\infty}\frac{-2z}{j^2}\frac{1}{1-z^2/j^2}$$
$$=\frac{1}{z}+g'(z)+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$$

and by the above exercise, the convergence is uniform over the compact subsets of $\mathbb{C}$ that does not contain any integers. By another exercise (Exercise V.2.8 on page no. 122 of Conway), $\pi\cot\pi z=\frac{1}{z}+\sum\limits_{n=1}^{\infty}\frac{2z}{z^2-n^2}$ for $z\notin\mathbb{Z}$. So we must have $g'(z)=0$ and hence $g(z)=a$ for some constant $a\in\mathbb{C}$ on compact subsets of $\mathbb{C}\setminus\mathbb{Z}$. Since g is an entire function, by Liouville's theorem, $g(z)=a$ for all $z\in\mathbb{C}$. Thus, from (6.1) for $0<|z|<1$

$$\frac{\sin\pi z}{\pi z}=\frac{e^a}{\pi}\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right).$$

So

$$1=\lim_{z\to 0}\frac{\sin\pi z}{\pi z}=\lim_{z\to 0}\frac{e^a}{\pi}\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)=\frac{e^a}{\pi}.$$

Thus letting z approach zero gives $e^a=\pi$ and therefore

6.2
$$\sin\pi z=\pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

and the convergence is uniform over compact subset of $\mathbb{C}$ (**Why?**). □

Using (6.2), we can obtain infinite product expression of $\cos\pi z$ (Exercise 1). Replacing z by iz, we get $\sinh\pi z=\pi z\prod\limits_{n=1}^{\infty}\left(1+\frac{z^2}{n^2}\right)$ (Exercise 2). Also, for $z=\frac{1}{2}$, (6.2) reduces to Walli's formula (Exercise 4)

$$\frac{\pi}{2}=\prod_{n=1}^{\infty}\frac{(2n)^2}{(2n-1)(2n+1)}=\frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdots}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdots}.$$

Exercises

1 . Show that $\cos\pi z=\prod\limits_{n=1}^{\infty}\left[1-\frac{4z^2}{(2n-1)^2}\right]$.

2 . Find a factorization of $\sinh z$ and $\cosh z$.

4 . Prove Walli's formula: $\frac{\pi}{2}=\prod\limits_{n=1}^{\infty}\frac{(2n)^2}{(2n-1)(2n+1)}$.

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