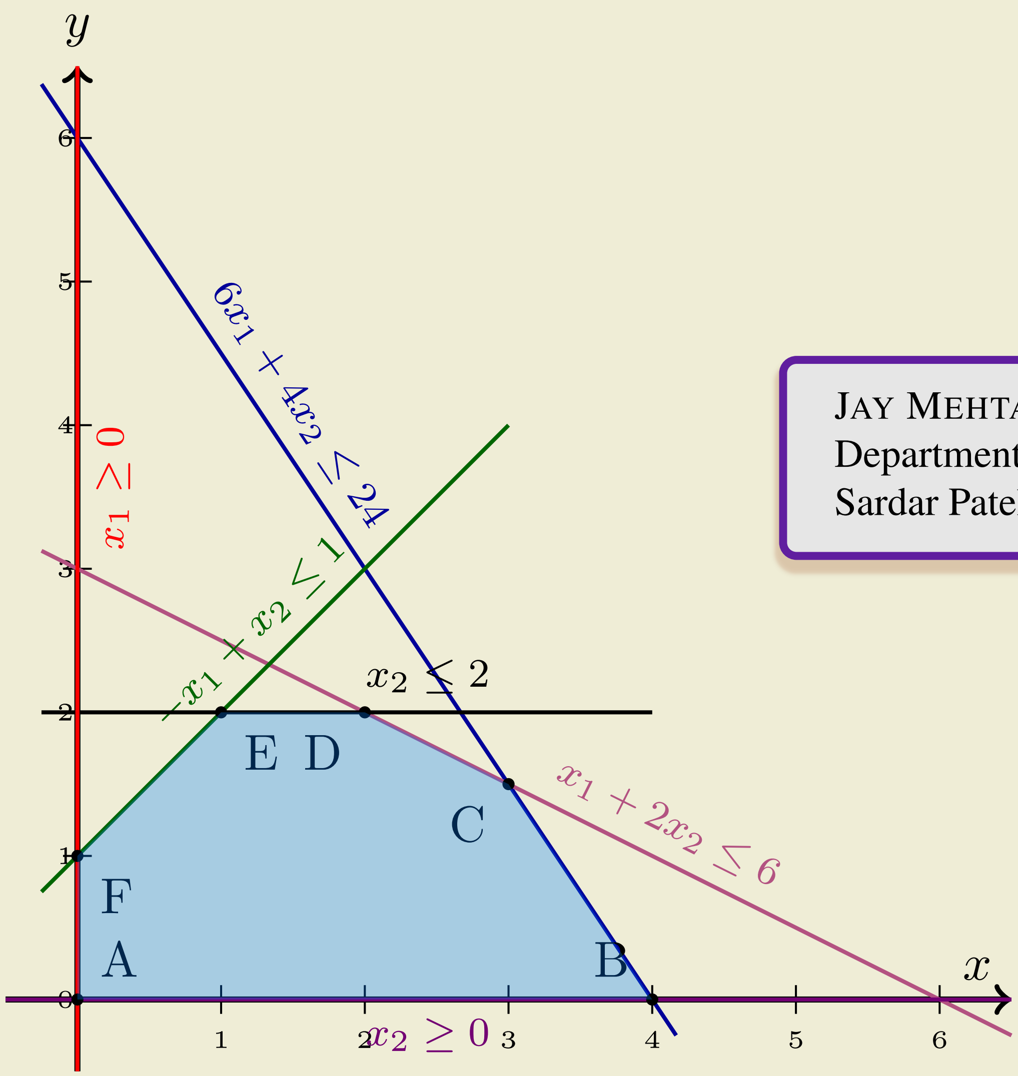
Lecture notes on

OPERATIONS RESEARCH

PS04EMTH30



JAY MEHTA
Department of Mathematics,
Sardar Patel University.

SEMESTER - IV
2017-18

LPP

Duality

TP & AP

NLP

Preface and Acknowledgments

The lecture notes outcome of the course "Operations Research" offered to the M.Sc. Semester-IV students at Department of Mathematics, Sardar Patel University, 2017-18. These handouts are tailored for the syllabus of Operations Research (PS04EMTH30) M.Sc. Semester-III and Semester-IV of the University and neither it covers all the topics of OR nor any topic is covered in detail. Solutions to the exercises are not provided as they were given as student seminar exercises and most of them discussed in the seminar sessions during the semester. Due to the limited time period of the semester, only a couple of examples for each method is discussed possibly during the classes. Students are therefore strongly advised and encouraged to solve addition exercises from the notes and from the book for their practice.

These notes are (strictly following and) prepared from the recommended text "Operations Research: An Introduction" (Ninth Edition) by Hamdy A. Taha in the university syllabus. However, to discuss more examples, few examples and theory may have been taken from other references. Besides, things like the names of places, company, people, the currency from $ to ₹, etc. have been changed for convenience of student readers. The data is however from the text book and not my original work. All the credits for the material of this course to H. A. Taha and respective authors of other literature sources from which the notes are prepared. My role was just of a course instructor, a mediator, to explain the topics listed in the syllabus to the students.

As it always happens with my lecture notes, this being the first time course taken by me, there might be a few errors and typos. We welcome the students and interested readers to give their valuable suggestions, comments, criticism or point out errors, if they find any.

Acknowledgment

All praise and thanks to the Almighty for "not even a dry leaf rustles in wind without His divine wish". Just a few months back when this course was assigned to me, this course and the (tex-wise) detailed notes were nowhere nearer, even in my thoughts.

During this short span of less than three months, I not only did learn Operations Research but also a bit of tex and related techniques like to plot graphical region, to construct simplex and transportation tables. For this, I thank the internet tex-community and contributers. Final thanks are due to the enthusiastic students for their active participation in the classes and course encouraged me more, some of them already learned a part of OR in their bachelors. Special thanks to the students Rutul and Sajeed for volunteering to explain "assignment model" in their respective classes.

Dr. Jay Mehta

Contents

Syllabus 9

1 Linear Programming 11

1.1 Introduction to Linear Programming 11

1.1.1 Terminology 11

1.1.2 Mathematical formulation of an optimization problem (or LPP) 12

1.2 A two-variable LP Model 12

1.3 Graphical Method 15

1.3.1 Graphical method for solving LPP 15

1.3.2 Alternative method for optimization - ISO method 21

1.3.3 Disadvantages of graphical method 22

1.4 Graphical Sensitivity Analysis 22

1.4.1 Sensitivity analysis of graphical solution 22

1.4.2 Sensitivity of optimum to changes in availability (right hand side) 23

1.4.3 Sensitivity of optimum to changes in coefficients of objective function 25

1.5 Linear Programming: Equation Form and Basic Solutions 26

1.5.1 General Linear Programming Problem 26

1.5.2 Canonical form of an LPP 27

1.5.3 Equation (Standard) form of an LPP 28

1.5.4 Solving a system of linear equations 30

1.6 The Simplex Method 34

1.6.1 The Simplex Algorithm 38

1.6.2 Solved examples using simplex method 39

1.7 Artificial Starting Solution 42

1.7.1 Big M-Method 43

1.7.2 Two-Phase Method 48

Exercises 53

2 Duality 59

2.1 Simplex Method: Special Cases 59

2.1.1 Degeneracy 59

2.1.2 Alternative Optima 61

2.1.3 Unbounded Solution 62

2.1.4 Infeasible Solution 64

2.2 Dual Problem 65

2.2.1 Method for constructing the dual 66

2.2.2 Examples of dual obtained from primal 66

2.3 Primal-Dual Relationships 68

2.3.1 Simplex Tableau Layout 68

2.3.2 Optimal Dual Solution 69

2.3.3 Simplex Tableau Computations 73

2.4 Economic Interpretation of Duality 76

2.4.1 Economic Interpretation of Dual Variables 77

2.4.2 Economic Interpretation of Dual Constraints 78

Exercises 79

3 Transportation and Assignment models 85

3.1 Dual Simplex Method 85

3.1.1 Dual Simplex Algorithm 85

3.1.2 Examples of dual simplex method 86

3.2 Transportation Model 89

3.2.1 The Transportation Algorithm 93

3.2.2 Determining the starting solution 94

3.2.3 Northwest-corner method 94

3.2.4 Least-cost method 95

3.2.5 Vogel Approximation Method (VAM) 96

3.2.6 Iterative computations of the Transportation algorithm 99

3.3 The Assignment Model 103

3.3.1 The Hungarian Method 104

Exercises 109

4 Nonlinear Programming 111

4.1 Unconstrained Algorithms 111

4.1.1 Direct Search Method 111

4.1.2 Gradient Method 114

4.2 Constrained Algorithms 116
4.2.1 Separable Programming 117
4.2.2 Quadratic Programming 120
4.2.3 Linear Combinations Method 122
Exercises 124

Index 127

Syllabus

PS04EMTH30: Operation Research

Unit I: Introduction to linear programming, LP model, Graphical solution and sensitivity analysis, simplex method, basic solutions, artificial starting solution.

Unit II: Degeneracy, alternative optima, duality, dual problem, economic interpretation.

Unit III: Dual simplex method, transportation models and assignment models and applications.

Unit IV: Non-linear programming; and applications.

Reference Books

1. Hamdy A. Taha, Operations Research: An introduction, Prentice-Hall, 1997.

CHAPTER 1

Linear Programming

1.1 Introduction to Linear Programming

In this section, we shall study optimization problems, in particular linear programming problems (LPP) and their mathematical formulation.

1.1.1 Terminology

Decision variables: The variables responsible for optimization are called *decision variables*. Decision variables are also called *alternatives*. In a optimization problem, these variables are to be determined for achieving the goal.

Constraints: Constraints are restriction put on the decision variables. The restrictions are due to availability of resources.

Objective function: The *objective function* is mathematical form of the goal to be optimized. The objective function is a function of decision variables. If $x_1, x_2, \ldots, x_n$ are decision variables, then the objective function has the form

$$z = f(x_1, x_2, \ldots, x_n).$$

Optimization problem: A problem in which given objective function is to be optimized subject to the given constraints is called an *optimization problem*.

Non-negativity constraints: In most of the optimization problems the decision variables will be assumed to be non-negative. These constraints are called *non-negative constraints*. The non-negativity constraints that arise are not due to availability but from the practical point of view.

Feasible solution: *Feasible solution* is a solution to the optimization problem which is consistent to all constraints. There may be many feasible solutions to the given problem.

Optimal solution: The solution among all the feasible solutions for which objective function is optimized is called *optimal solution*.

Definition 1.1.1. An optimization problem is said to be a *linear programming problem* (LPP) if all constraints and objective function are in linear form.

1.1.2 Mathematical formulation of an optimization problem (or LPP)

Any OR model including LPP comprises of three basic components. They are **decision variables, objective function** and **constraints**. The following are the steps for mathematical formulation of any optimization problem.

1. Identify and declare the decision variables.
2. Identify the objective function and express it mathematically.
3. Identify the constraints and express them mathematically.

Let us see an example of optimization problem and how to formulate it mathematically.

Example 1.1.2. Formulate the following optimization problem in mathematical form. My firm Ltd. manufactures pendrives (8 GB) and memory cards (8 GB). Each pendrive takes 3 minutes of manufacturing time and costs ₹ 25. Each memory card takes 2 minutes of manufacturing time and costs ₹ 20. The company has 500 minutes of manufacturing time and a provision of ₹ 5000 per day. Ther profit per pendrive and memory card are ₹ 5 and ₹ 7 respectively. Maximize the profit.

Solution. Let the number of pendrives and memory cards to be manufactured be x and y respectively. Then the above problem can be formulated in mathematical form as follows:

$$\max \ z = 5x + 7y.$$

$$\begin{aligned} 3x + 2y &\leq 500 \\ 25x + 20y &\leq 5000 \\ x \geq 0,\ y &\geq 0 \end{aligned}$$

□

1.2 A two-variable LP Model

Example 1.2.1 (The Reddy Mikks Company)**.** Reddy Mikks produces both interior and exterior paints from two raw materials M_1 and M_2. The following table provides the basic data of the problem:

A market survery indicates the the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons.

Reddy Mikks wants to determine the optimum product mix of interior and exterior paints that maximizes the total daily profit.

	Tons of raw material per ton of		**Maximum daily availability (tons)**
	Exterior paint	**Interior paint**	
Raw material, M_1	6	4	24
Raw material, M_2	1	2	6
Profit per ton (in ₹1000)	5	4	

Solution. Decision variables: Here we have to determine the daily amounts of the exterior and interior paints to be produced so as to maximize the total daily profit. So the decision variables are

$$x_1 = \text{tons of exterior paint produced daily.}$$
$$x_2 = \text{tons of interior paint produced daily.}$$

Objective function: The objective of the Reddy Mikks is to determine the total daily profit from both the paints.

$$\text{Profit from exterior paint} = 5x_1 \text{ (thousand) ₹}$$
$$\text{Profit from interior paint} = 4x_2 \text{ (thousand) ₹.}$$

If z denotes the total daily profit, then the goal of Reddy Mikks company is

$$\text{Maximize } z = 5x_1 + 4x_2.$$

Constraints: We have the constraints due to daily usage (availability) of raw material and the constraints due to demand. We have

$$\left(\begin{array}{c}\text{Usage of a raw material}\\ \text{by both the paints}\end{array}\right) \leq \left(\begin{array}{c}\text{Maximum availability}\\ \text{of the raw material}\end{array}\right)$$

$$\text{Daily usage of raw material } M_1 \text{ by both paints} = 6x_1 + 4x_2 \text{ tons per day.}$$
$$\text{Daily usage of raw material } M_2 \text{ by both paints} = 1x_1 + 2x_2 \text{ tons per day.}$$

Maximum daily availability of the raw materials M_1 and M_2 are respectively 24 tons and 6 tons. Therefore the constraints are

$$\begin{aligned} 6x_1 + 4x_2 &\leq 24 \\ x_1 + 2x_2 &\leq 6. \end{aligned}$$

The constraints due to restrictions on the product demand are as follows

$$\begin{aligned} x_2 - x_1 &\leq 1 \\ x_2 &\leq 2. \end{aligned}$$

Mathematical formulation: The Reddy Mikks model can be formulated mathematically as

$$\text{Maximize } z \quad = \quad 5x_1 + 4x_2 \,\} \quad \text{objective function}$$

subject to

$$\left.\begin{array}{rcl} 6x_1+4x_2 & \leq & 24 \\ x_1+2x_2 & \leq & 6 \\ -x_1+x_2 & \leq & 1 \\ x_2 & \leq & 2 \\ x_1,x_2 & \geq & 0. \end{array}\right\} \text{constraints}$$

Feasible solution: Any values of x_1 and x_2 satisfying the above constraints is a *feasible solution*, otherwise it is *infeasible*. For example, $x_1 = 3$ tons per day and $x_2 = 1$ ton per day is a feasible solution since it satisfies all the five constraints. On the other hand, observe that the $x_1 = 4$ and $x_2 = 1$ is infeasible as it does not satisfy at least one constraint.

The goal of the problem is to find the optimum (i.e. the best) feasible solution that maximizes the total daily profit z. □

Example 1.2.2. An electronic company produces two components C_1 and C_2 used in manufacturing of a TV set. Each unit C_1 costs ₹ 25 in labor and ₹ 25 for the required material. Each unit C_2 costs ₹ 125 in labor and ₹ 75 in material. The company's labor and material expenses are to be paid in cash. The selling price of C_1 is ₹ 150 and that of C_2 is ₹ 350 per unit. Due to strong monopoly of the company for these components, it is assumed that the company can sell at the prevailing prices as many units as it produces. The product capacity is however limited to two considerations. First is, at the beginning of a period the company has an initial balance of ₹ 20000. Second, the company has available in each period 4000 hours of machine time and 2800 hours of assembly time. The production of each unit of C_1 requires 6 hours of machine time and 4 hours of assembly time. Machine time and assembly time for each of C_2 are 4 hours and 6 hours respectively.

Formulate the problem as an LPP to maximize the profit of the company. Also express the problem in tabular form.

Solution. Tabular form of the given LPP:

Resources/Constraints	**Components**		**Availability**
	C_1	C_2	
Machine time	6	4	4000 hours
Assembly time	4	6	2800 hours
Budget	50	200	₹ 20000
Selling price	150	350	
Profit	100	150	

Decision variables: Here the problem is to maximize the profit. The profit is earned by producing components C_1 and C_2. Thus, the decision variables are

$$x_1 = \text{number of units of component } C_1$$
$$x_2 = \text{number of units of component } C_2.$$

Objective function: The quantity to be maximized is the profit. The profit on one unit of component C_1 is given by selling price - production cost $= 150 - 50 = 100$ ₹. Similarly, the profit on one unit of component C_2 is ₹ 150. Hence the objective function is given by

$$\text{Maximize } z = 100x_1 + 150x_2.$$

Constraints: One unit of component C_1 consumes 6 hours of machine time while one unit of C_2 consumes 4 hours of machine time. The total availability of machine time is 4000 hours. Hence the constraint is

$$6x_1 + 4x_2 \leq 4000.$$

Similarly, the constraint due to assembly time is

$$4x_1 + 6x_2 \leq 2800.$$

One unit of C_1 costs ₹ 50 and one unit of component C_2 costs ₹ 200. The availability of funds is ₹ 20000. Hence, the constraint is given by

$$50x_1 + 200x_2 \leq 20000.$$

Finally, the non-negativity constraints are:

$$x_1,\ x_2 \geq 0.$$

Hence, the given problem can be formulated mathematically as

$$\begin{array}{rcl} \text{Maximize } z & = & 100x_1 + 150x_2 \\ \text{subject to the constraints} & & \\ 6x_1 + 4x_2 & \leq & 4000 \\ 4x_1 + 6x_2 & \leq & 2800 \\ 50x_1 + 200x_2 & \leq & 20000 \\ x_1, x_2 & \geq & 0. \end{array}$$

□

1.3 Graphical Method

1.3.1 Graphical method for solving LPP

Graphical method is useful only when then number of decision variables is two. For a Linear Programming Problem (LPP) in two variables x_1 and x_2 the constraints are linear inequalities two variables. We first convert these inequalities into equalities by replacing the inequality sign by equality sign. Now each equality can be ploted as a straight line in x_1x_2-plane. This divides the plane into three parts.

1. The region in which constraint is satisfied with the given inequality.
2. The region in which constraint is not satisfied.
3. The straight line itself where the equality is satisfied. Here constraint is satisfied if it is in the form which includes equality also.

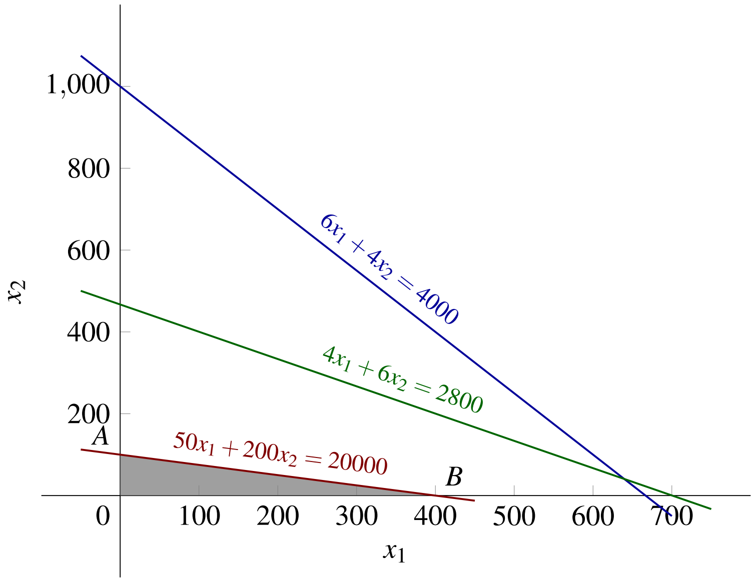
Let us consider couple of examples to demonstrate the graphical representation of an LPP.

Definition 1.3.1. The region in which all the constraints are satisfied is called the *feasible region*.

Example 1.3.2. Express the problem in Example 1.2.2 graphically i.e.

$$\begin{aligned} \text{Maximize } z &= 100x_1+150x_2 \\ \text{subject to} & \\ 6x_1+4x_2 &\leq 4000 \\ 4x_1+6x_2 &\leq 2800 \\ 50x_1+200x_2 &\leq 20000 \\ x_1,x_2 &\geq 0. \end{aligned}$$

Solution.



The constraints can be represented with the help of straight lines. For the given problem, we plot the lines

$$\begin{aligned} 6x_1+4x_2 &= 4000 \\ 4x_1+6x_2 &= 2800 \\ 50x_1+200x_2 &= 20000 \end{aligned}$$

From the graphical representation it is clear that all the constraints are satisfied in the triangular region covered by $\triangle\, OAB$ which is the feasible region.

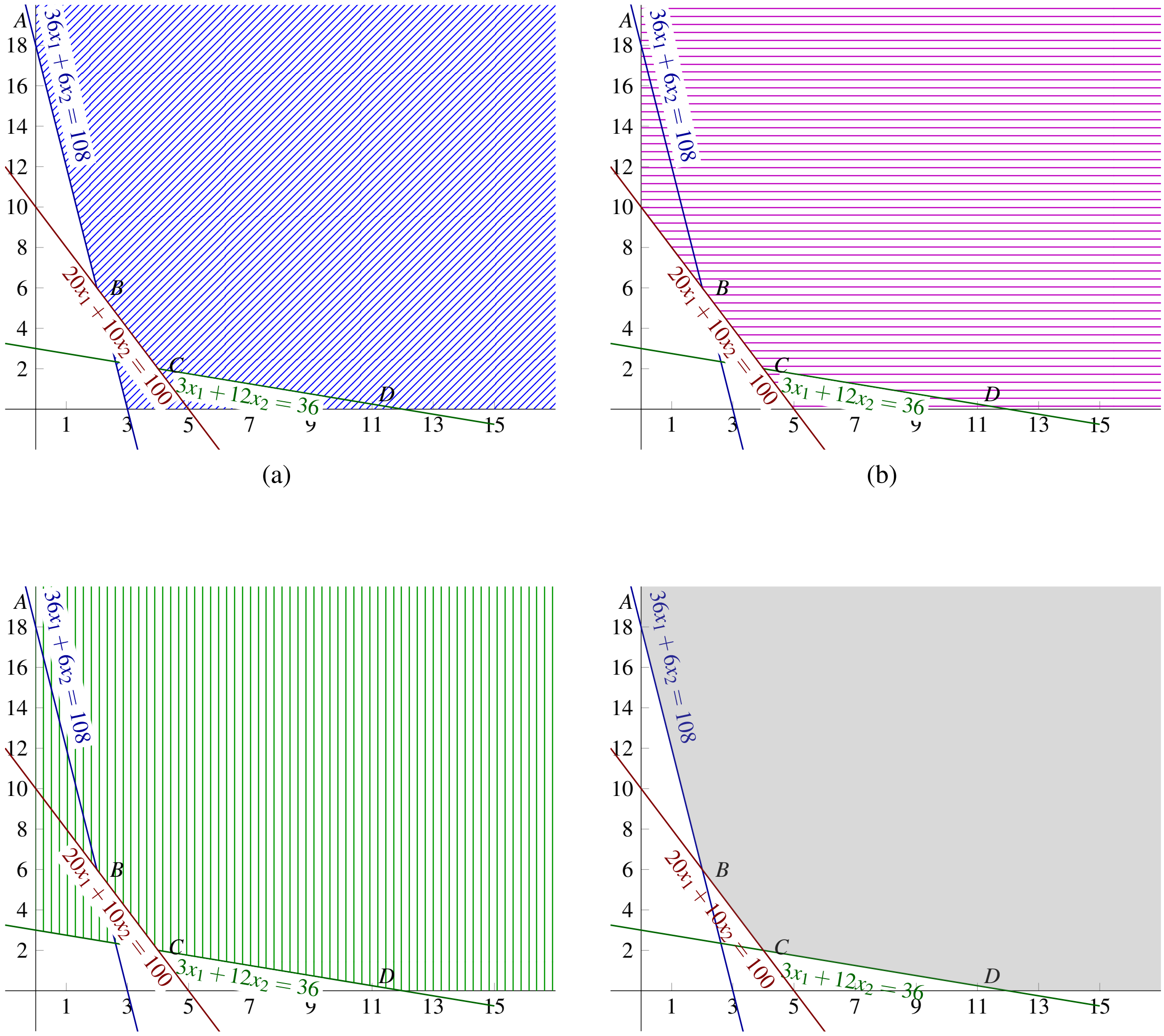
□

Example 1.3.3. Determine the feasible region for the following optimization problem

$$\begin{aligned} \text{Minimize } z = &\ 20x_1+40x_2. \\ \text{subject to} & \\ 36x_1+6x_2 \geq &\ 108 \\ 3x_1+12x_2 \geq &\ 36 \\ 20x_1+10x_2 \geq &\ 100 \\ x_1,\ x_2 \geq &\ 0. \end{aligned}$$

Solution. We plot the lines $36x_1+6x_2=108, 3x_1+12x_2=36$ and $20x_1+10x_2=100$. Since $36x_1+6x_2 \geq 108$, the region in which this constraint is satisfied is the unbounded region as shown in Figure (a). Similarly, the regions in which constraints $3x_1+12x_2 \geq 36$ and $20x_1+10x_2 \geq 100$ are satisfied are shown in Figure (b) and Figure (c) respectively.

The feasible region, i.e. the solution space is the unformulated, unbounded region which is the region above the union of line-segments $\overline{AB},\overline{BC},\overline{CD}$ in the first quadrant (due to non-negativity constraint).



(a) (b)

(c) (d)

□

Definition 1.3.4. The corner points of the feasible region in the graphical representation of an optimization problem are called *extremum points*.

If the feasible region is bounded, i.e. if it is a polygon then the corner points are the vertices of the polygon.

The graphical method to solve an LPP includes the following steps.

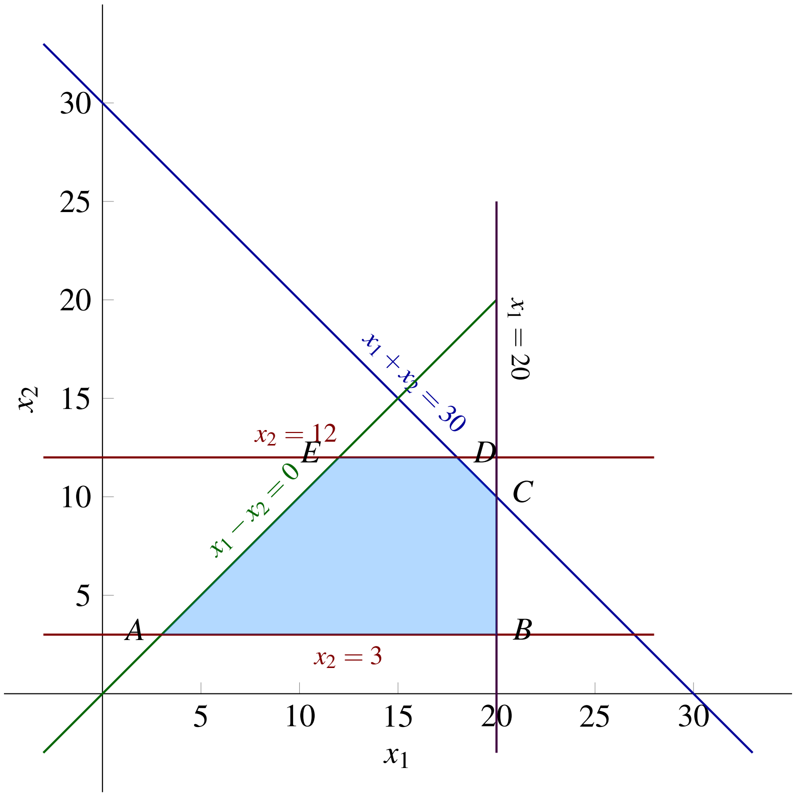
1. Identify the decision variable, objective function and constraints. If the problem has only two decision variables, then the graphical method can be used.
2. Express the objective function and constraints mathematically.
3. Represent the constraints graphically and identify the feasible region.
4. If the feasible region is a bounded polygon then the extremum of the objective function lies on any of the vertices.
5. Evaluate the objective function on the vertices and choose the vertex which has the maximum (i.e. most) or the minimum (i.e. least) value as the desired solution.
6. If the feasible region is an unbounded region, then there are two cases.

(a) **Maximization Problem**: The solution does not exists because in the feasible region the objective function goes on increasing.
(b) **Minimization Problem**: The solution will be on some corner point (vertex).

Example 1.3.5. Using graphical method solve

$$\begin{aligned} \text{Max } z &= 2x_1 + 3x_2 \\ \text{subject to} & \\ x_1 + x_2 &\le 30 \\ x_1 - x_2 &\ge 0 \\ x_2 &\ge 3 \\ 0 \le x_1 &\le 20 \\ 0 \le x_2 &\le 12. \end{aligned}$$

Solution. As shown in figure the feasible region is the pentagon with vertices A, B, C, D, E. Now, we evaluate the function z at these points as follows:



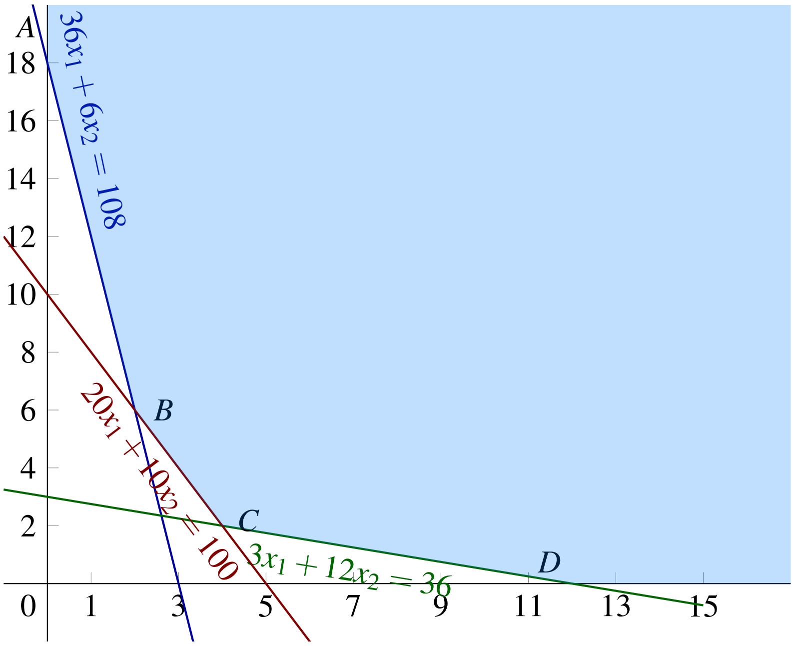
Point	Coordinates	Max $z = 2x_1 + 3x_2$
A	$(3,3)$	15
B	$(20,3)$	49
C	$(20,10)$	70
D	$(18,12)$	72
E	$(12,12)$	60

Hence, the maximum is attained at point $D(18,12)$. ☐

Example 1.3.6. Determine the solution of the following optimization problem by graphical method

$$\begin{aligned} \text{Minimize } z = & \; 20x_1 + 40x_2. \\ \text{subject to} & \\ 36x_1 + 6x_2 \geq & \; 108 \\ 3x_1 + 12x_2 \geq & \; 36 \\ 20x_1 + 10x_2 \geq & \; 100 \\ x_1,\, x_2 \geq & \; 0. \end{aligned}$$

Solution. The graph of this problem is given in Example 1.3.3. The feasible region is as shown in the graph below.



Here the feasible region is an unbounded region but the objective function is to be minimized. Therefore, the optimal solution exists on the corner points. In this case the corner points are A, B, C, D.

Point	**Coordinates**	**Min** $z = 20x_1 + 40x_2$
A	$(0,18)$	720
B	$(2,6)$	280
C	$(4,2)$	160
D	$(12,0)$	240

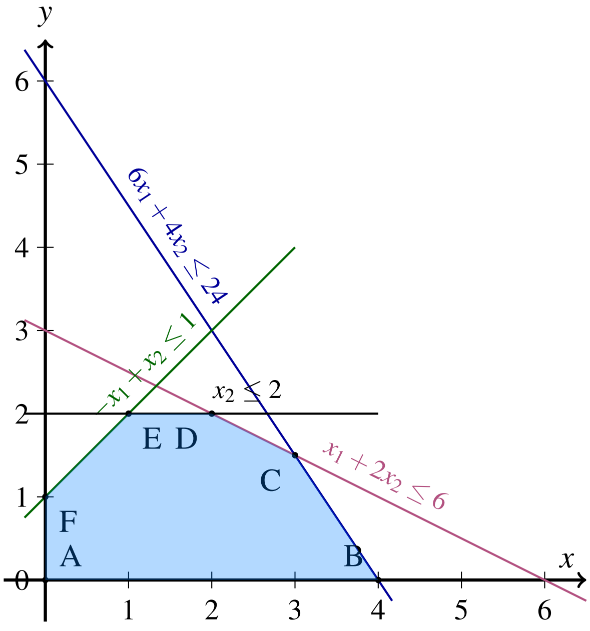
Hence, the minimum is obtained at point $C(4,2)$. □

Example 1.3.7 (Reddy Mikks model)**.** Obtain the optimal solution of the Reddy Mikks problem using graphical method. Recall that the mathematical formulation of Reddy Mikks model is given (in Example 1.2.1) by

$$\begin{array}{rrcl} & \text{Maximize } z & = & 5x_1+4x_2 \\ \text{subject to} & & & \\ & 6x_1+4x_2 & \leq & 24 \\ & x_1+2x_2 & \leq & 6 \\ & -x_1+x_2 & \leq & 1 \\ & x_2 & \leq & 2 \\ & x_1,\, x_2 & \geq & 0. \end{array}$$

Solution. We first plot the lines $6x_1+4x_2=24$, $x_1+2x_2=6$, $-x_1+x_2=1$ and $x_2=2$ and determine the feasible region.

It is clear from the figure that the feasible region is the bounded and shaded region which bounded by the polygon $ABCDEF$. Now, we compute the objective function z at the vertices of this polygon and determine the optimum (here maximum) solution.



Point	**Coordinates**	**Max** $z=5x_1+4x_2$
A	$(0,0)$	0
B	$(4,0)$	10
C	$(3,\frac{3}{2})$	21
D	$(2,2)$	18
E	$(1,2)$	13
F	$(0,1)$	4

Hence, by graphical method we obtain the optimal solution at the point $C(3,1.5)$ and the optimal (maximum) value of the objective function is $z=21$, i.e. to maximize the profit the Reddy Mikks company must produce 3 tons of exterior paint and 1.5 tons of interior paint daily and its daily profit will be ₹ 21000. □

1.3.2 Alternative method for maximization (or minimization) - ISO-profit (or cost) method

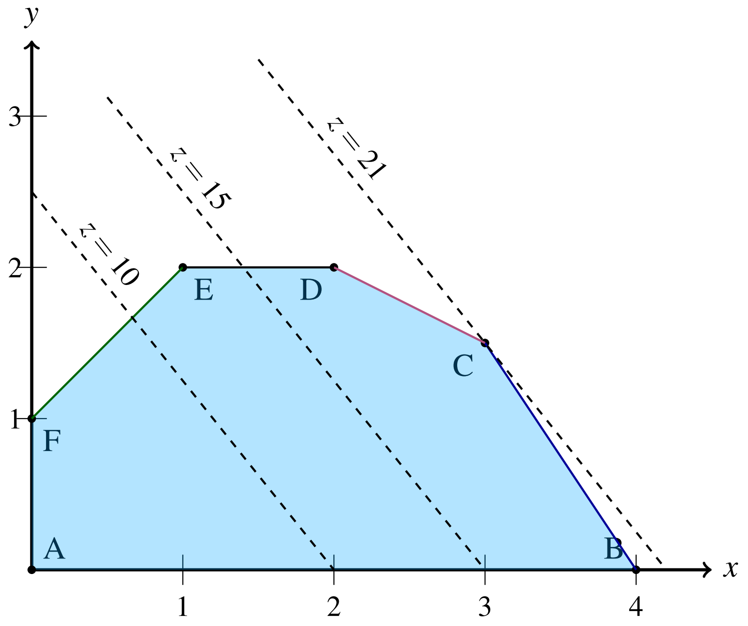
We describe an alternative method for graphically obtaining the maximum of the objective function called the ISO-profit method and the minimum of the function called the ISO-cost method.

1. First plot the graph of the problem and determine the feasible region.
2. Draw a straight line for some reasonable profit (or cost) by assigning the profit (or cost) to the objective function. This line is called *ISO-profit (or ISO-cost) line* and it must fall inside the feasible region.
3. Determine the direction in which the objective function increases (or correspondingly decreases). This direction will be usually away (or closer) from the origin.
4. Draw lines parallel to the ISO-profit (or cost) line in the direction away (or close) from the origin depending on the case.
5. The line farthest (or closest) from the origin which passes through a vertex determines the optimal value of the objective function and the corresponding vertex is the required optimal solution.

In the example below, we obtain the solution of the Reddy Mikks model by ISO-profit method.

Example 1.3.8. Obtain the solution of the Reddy Mikks model (Example 1.2.1) ISO-profit method.

Solution. We first have to plot the graph of the Reddy Mikks model and determine the feasible region which is already shown in Example 1.3.7. Next we determine in which direction the profit function $z = 5x_1 + 4x_2$ increases. This is done by arbitrarily assigning values to z. For example, assigning $z = 10$ and $z = 15$, we obtain two lines falling in the feasible region given by $5x_1 + 4x_2 = 10$ and $5x_1 + 4x_2 = 15$ respectively. Thus, we can identify the direction in which z increases. Drawing parallel lines, we finally find that optimum solution occurs at the point $C(3, \frac{3}{2})$ and the optimum value is $z = 21$.



□

Remark 1.3.9. In case the the objective function gives the ISO-profit lines parallel to one of the edges (i.e. lines represented by the constraints), then when the ISO lines are shifted away from the origin one of them will coincide with the constraint line. In this case, before leaving the feasible region, an edge of the region is a subset of the ISO-profit line and we obtain alternative optima. This means all the points on the edge give the same optimal value of the objective function.

Hence, in such case, we get infinitely many optimal solutions.

1.3.3 Disadvantages of graphical method

The graphical method has certain disadvantages and hence it is not efficient or useful in many cases. The disadvantages are:

1. It works in two decision variables only.
2. It works efficiently if the number of constraints is small.
3. If the coefficient in constraints have abnormal ranges, then it becomes difficult to plot the corresponding lines. Hence the method is not suitable.

1.4 Graphical Sensitivity Analysis

In a linear programming model, there is a scope for change in the values of the parameters like constraints, profit/cost, etc. without altering the optimum. This is called *sensitivity analysis*. There are two types of sensitivity analysis of a linear programming problem which are discussed in the reference book of this course. They are *graphical sensitivity analysis* and *algebraic sensitivity analysis*.

In our course, we limit our discussion only to sensitivity analysis of graphical solution of a linear programming problem. Thus, our case will be restricted to two decision variables only.

1.4.1 Sensitivity analysis of graphical solution

In graphical sensitivity analysis, the following two cases are considered.

1. Sensitivity of the optimum solution to changes in the availability of the resources (i.e. the right hand side of the constraints).
2. Sensitivity of the optimum solution to changes in unit profit or unit cost (i.e. the coefficients of the objective function).

We shall discuss the above two cases by an example. Consider the following example.

Example 1.4.1 (JOBCO problem)**.** JOBCO manufactures two product on two machines. A unit of product 1 requires 2 hours on machine 1 and 1 hour on machine 2. For product 2, one unit requires 1 hour on machine 1 and 3 hours on machine 2. The revenues per unit of products 1 and 2 are \$30 and \$20, respectively. The total daily processing time available for each machine is 8 hours. The goal is to maximize the revenue.

Solve the problem by graphical method.

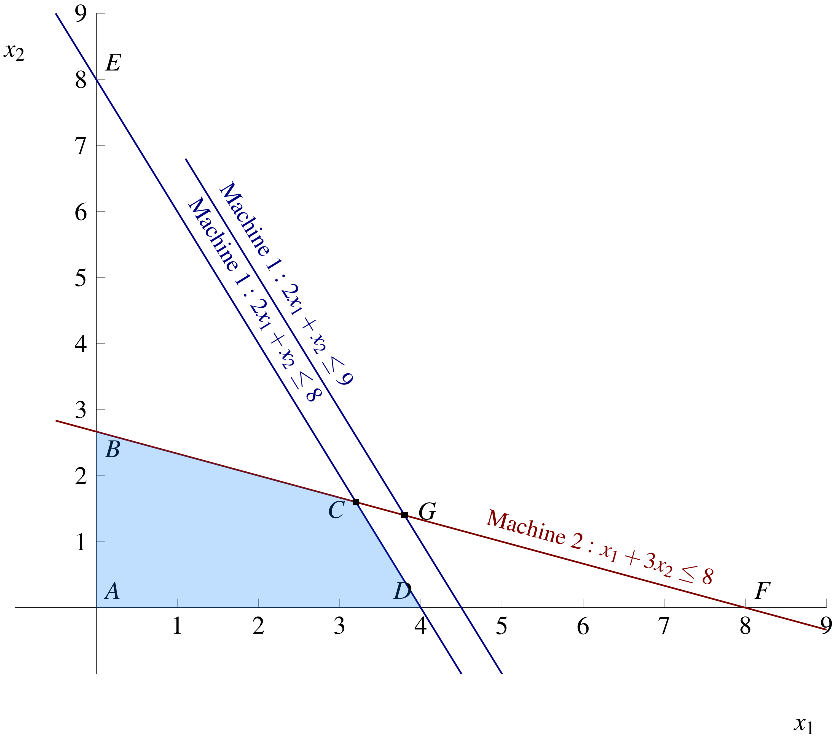
Solution. Let x_1 and x_2 be the daily number of units of products 1 and 2 respectively. Then the mathematical formulation of the given LP model is given by

$$\text{Maximize } z = 30x_1 + 20x_2$$

subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 8 && \text{(Machine 1)} \\ x_1 + 3x_2 &\leq 8 && \text{(Machine 2)} \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

We plot the graph of the given problem. As shown in the figure, the feasible region is the bounded (shaded) region bounded by the polygon $ABCD$. We now compute the objective function at the vertices A, B, C and D of this polygon and determine the corner point which gives the optimum.



Point	Coordinates	Max $z = 30x_1 + 20x_2$
A	$(0,0)$	0
B	$(0,2.67)$	53.34
C	$(3.2,1.6)$	128
D	$(4,0)$	120
G	$(3.8,1.4)$	142

Hence, we find that the optimum is $z = 128$ at point $C(3.2, 1.6)$. □

1.4.2 Sensitivity of optimum to changes in availability (right hand side)

Above figure demonstrates that optimum changes with the change in the capacity of machine 1. If the daily capacity of machine 1 is increased from 8 hours to 9 hours, then the new optimum moves to the point G (as shown in figure). The rate of change in the objective function z due to change in machine 1 capacity can be computes as follows:

$$\left(\begin{array}{c}\text{Rate of change in } z \text{ due to}\\ \text{increase in machine 1 capacity}\\ \text{by 1 hour (point } C \text{ to } G)\end{array}\right) = \frac{z_G - z_C}{\text{(Capacity change)}} = \frac{142-128}{9-8} = \$14/\text{hr}.$$

Thus, a unit increase (or decrease) in machine 1 capacity leads to increase (or decrease) in revenue by \$14. This price is called the *dual price*.

From the above figure, we can say that the dual price of \$14/hour remains valid for any change (increase or decrease) in machine 1 capacity that moves its constraint parallel to itself and passing through any point on line-segment $\overline{BF}$. Machine 1 capacity at points B and F are computes as:

$$\text{Machine 1 capacity at } B(0,2.67) = 2\times 0 + 1\times 2.67 = 2.67 \text{ hours.}$$
$$\text{Machine 2 capacity at } F(8,0) = 2\times 8 + 1\times 0 = 16 \text{ hours.}$$

Thus, the valid range for dual price of \$14/hr is

$$2.67 \text{ hr } \leq \text{Machine 1 capacity} \leq 16 \text{ hr.}$$

Changes outside this range may produce a different dual price (worth per unit).

Similarly, the dual price for machine 2 is \$2/hr and the range is (**Exercise**)

$$4 \text{ hr } \leq \text{Machine 2 capacity} \leq 24 \text{ hr.}$$

These valid ranges for machine 1 and 2 computed above are called **feasible ranges**. The dual price allows us to make economic decisions about the problem. For example, consider the following questions:

Question 1: If JOBCO can increase the capacity of both machines, which machine should receive the priority?

The dual prices of machine 1 and 2 are \$14 and \$2 respectively. Thus, machine 1 should receive the first priority.

Question 2: It is suggested that the capacities of machine 1 and 2 should be increased at the cost of \$10/hr for each of them. Is it advisable?

For machine 1, the additional revenue per hour is $\$14 - \$10 = \$4$. However, for machine 2 it is $\$2 - \$10 = -\$8$. Hence, it is advisable that only machine 1 should be considered for increase in capacity at \$10/hr.

Question 3: If the capacity of machine 1 is increased from 8 hours to 13 hours, how will it impact the optimum revenue?

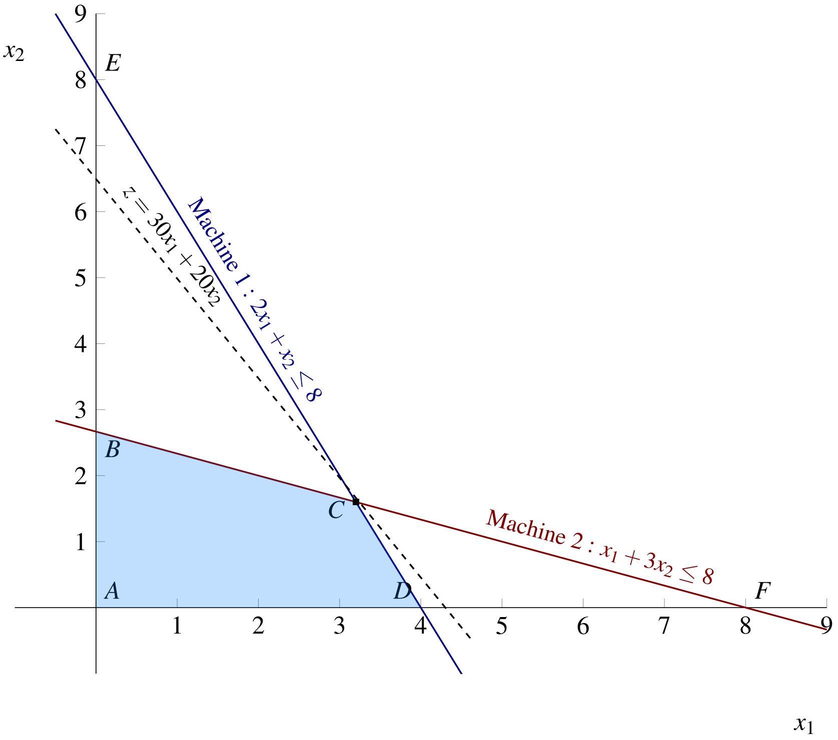
The valid range of dual price of machine 1 is $[2.67, 16]$ hour. The proposed increase to 13 hours falls in this feasible range. Hence the increase in revenue is $\$14(13-8) = \$70$, i.e. the revenue will raise from \$128 to \$198.

Question 4: Suppose the capacity of machine 1 is increased to 20 hours, how will it affect the optimum revenue?

The proposed change is outside the feasible range $[2.67, 16]$ hour. Hence nothing can be said. Note that the proposed value is outside the feasible range does not mean that the problem has no solution. It just means that the information is insufficient to make a complete decision.

1.4.3 Sensitivity of optimum to changes in coefficients of objective function

From the graph, it is clear that optimum occurs at point $C(3.2, 1.6)$ and the value is $z = 128$. Changes in revenues (i.e. objective-function coefficients) changes the slope of z. However, from the graph, it can be seen that the optimal solution remains at the point C unchanged as long as the objective function lies between lines BF and DE.



Now, we determine the ranges for the coefficients of the objective function such that the optimum solution remains unchanged at C. We write

$$\text{Maximize } z = c_1x_1 + c_2x_2.$$

Now, keeping the line z fixed at C it can rotate clockwise or anticlockwise such that it lies between the lines $x_1 + 3x_2 = 8$ and $2x_1 + x_2 = 8$. This means the ratio $\frac{c_1}{c_2}$ lies between $\frac{1}{3}$ and $\frac{2}{1}$. Therefore

$$\frac{1}{3} \leq \frac{c_1}{c_2} \leq \frac{2}{1} \quad \text{or} \quad 0.333 \leq \frac{c_1}{c_2} \leq 2.$$

This gives answer to the following questions.

Question 1: Suppose that the unit revenues for products 1 and 2 are changed to \$35 and \$25 respectively. Will the current optimum remain the same?

The new objective function becomes

$$\text{Maximize } z = 35x_1 + 25x_2.$$

The ratio $\frac{c_1}{c_2} = \frac{35}{25} = 1.4$ lies in the valid range $(0.33, 2)$. Hence the optimum point remains at C. However, the optimum value z changes to $35 \times 3.2 + 25 \times 1.6 = \$152$.

Question 2: Suppose that the unit revenue of product 2 is fixed at its current value $c_2 = \$20$. What is the associated optimality range for the unit revenue for product 1, c_1, that will keep the optimum unchanged?

Substituting $c_2 = 20$ in the condition $\frac{1}{3} \leq \frac{c_1}{c_2} \leq 2$, we get the range for c_1 as

$$6.67 \leq c_1 \leq 40.$$

Similarly, we can obtain feasible range for c_2 when c_1 is fixed at \$30 (**Exercise**).

1.5 Linear Programming: Equation Form and Basic Solutions

1.5.1 General Linear Programming Problem

Consider an optimization problem in n-decision variables $x_1, x_2, \ldots, x_n$ and m-constraints. Let z be the objective function which is a linear function of decision variables given by

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

where c_j's are constants. The constraints are also in the form of linear inequalities (or equalities) given by

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq \text{ or } \geq \text{ or } = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq \text{ or } \geq \text{ or } = b_2 \\
\vdots \qquad &\qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq \text{ or } \geq \text{ or } = b_m,
\end{aligned}$$

where the coefficients a_{ij}'s are real constants and can be represented by a $m \times n$ matrix. In addition, there may be restriction on the sign of decision variables (non-negativity restriction). The problem problem of determining an n-tuple $(x_1, x_2, \ldots, x_n)$ which optimizes the objective function z satisfying constraints described above is called the *general linear programming problem.*

Example 1.5.1. Airforce is experimenting with three types of bombs P, Q and R. They are made using explosives say A, B, C. It is decided to use at most 600 kg of explosive A, at least 480 kg of explosive B and exactly 540 kg of explosive C. The composition of three types of bombs is given below.

Bomb	**Explosive required**		
	A	**B**	**C**
P	3	2	2
Q	1	4	3
R	6	2	3

The bombs P, Q and R are equivalent to explosion $2,3$ and 4 respectively. Under what production schedule can airforce achieve biggest target. Formulate this problem in mathematical term.

Solution. Decision variables: Let

x_1 be the number of bombs of type P.
x_2 be the number of bombs of type Q.
x_3 be the number of bombs of type R.

Objective function: Maximize $z = 2x_1 + 3x_2 + 4x_3$ subject to
Constraints:

$$\begin{aligned} 3x_1 + x_2 + 6x_3 &\leq 600 \\ 2x_1 + 4x_2 + 2x_3 &\geq 480 \\ 2x_1 + 3x_2 + 3x_3 &= 540 \\ x_1,\ x_2,\ x_3 &\geq 0. \end{aligned}$$

□

After an LPP is mathematically formulated, the next step is to solve it. For solution purpose, the problem must be reduced to some specific form. The two types of such specific forms that we will see are *canonical form* and *standard form*. They are described in the succeeding subsections.

1.5.2 Canonical form of an LPP

It is always possible to convert a general LPP in the following form:

$$\text{Maximize } \ z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to the constraints

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i, \quad i = 1,2,\ldots,m$$

and

$$x_1, x_2, \ldots, x_n \geq 0.$$

This form of a linear programming is called the *canonical form* of LPP.

Note that here the objective function is of maximum type and all the constraints are of $\leq$ type. The general LPP can be converted into canonical form by following steps:

1. If the objective function is of minimum type, say

$$\text{Minimize } \ z = c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

then it is converted to maximum type by multiplying it with -1. Equivalently, we have

$$\text{Maximize } \ h = -z = -c_1x_1 - c_2x_2 - \cdots - c_nx_n.$$

2. If a constraint is of $\geq$ type then it can be converted to $\leq$ type by multiplying both sides by -1, i.e. the constraint given by

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i,$$

then it can be equivalently written as

$$-a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n \leq -b_i.$$

3. If a constraint is of equality type then it is replace by two weak constraints of $\leq$ type and $\geq$ type. For example, the constraint

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

is replaced by

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &\leq b_i \text{ and} \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &\geq b_i. \end{aligned}$$

4. If a variable is of $\leq$ type (i.e. non-positive) then it is converted to non-negative constraint by multiplying it with -1. That is

$$x_k \leq 0 \Leftrightarrow y_k (= -x_k) \geq 0.$$

5. A variable which is unrestricted in sign (i.e. neither non-positive nor non-negative) is written as difference of two non-negative variables. For example, if x_j is unrestricted then, we write

$$x_j = x'_j - x''_j, \text{ where } x'_j,\ x''_j \geq 0.$$

Example 1.5.2. Convert the general LPP in the (above) Example 1.5.1 into canonical form.

Solution. **Canonical form:** Maximize $z = 2x_1 + 3x_2 + 4x_3$
subject to constraints

$$\begin{aligned} 3x_1 + x_1 + 6x_3 &\leq 600 \\ -2x_1 - 4x_2 - 2x_3 &\leq -480 \\ 2x_1 + 3x_2 + 3x_3 &\leq 540 \\ -2x_1 - 3x_2 - 3x_3 &\leq -540 \\ x_1,\ x_2,\ x_3 &\geq 0. \end{aligned}$$

□

1.5.3 Equation (Standard) form of an LPP

To solve a given LPP by the simplex method, it has to be first converted into some specific form. This form is called the *standard form* of LPP and it has the following requirements:

1. All the constraints are equations with non-negative right hand side.
2. All the variables are non-negative.

Converting inequalities into equations

Definition 1.5.3 (Slack variable)**.** The constraint of $\leq$ type is converted into an equation by adding a non-negative variable on the left hand side of the constraint. For example, the constraint

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$

is converted into an equality as

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + s_1 = b_i, \quad s_1 \geq 0.$$

The added non-negative variable s_1 is called a *slack variable*.

Definition 1.5.4 (Surplus variable)**.** The constraint of $\geq$ type is converted into an equation by subtracting a non-negative variable from the left hand side of the constraint. For example, the constraint

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i$$

is converted into an equality as

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - S_1 = b_i, \quad S_1 \geq 0.$$

The non-negative variable S_1 which is subtracted is called a *surplus variable*.

If the right hand side of the constraint is negative, then it is converted into non-negative form by multiplying the constraint by -1.

Remark 1.5.5. Note that, multiplying both sides of an inequality constraint by -1 and then converting into an equation is same as converting it first into an equation and then multiplying both sides by -1.

Variables with unrestricted sign

An unrestricted variable is replaced by the difference of two non-negative integers. For example if S_i has no restriction on its sign, then we write

$$S_i = S_i^- - S_i^+, \quad S_i^- \geq 0, \, S_i^+ \geq 0.$$

Or if x_j is an unrestricted variable, then we write $x_j = x_j' - x_j''$, where $x_j', \, x_j'' \geq 0$.

Standard LPP in matrix form

Note that the equation form (i.e. standard form) of an LPP with n-decision variables and m-constraints can be written in terms of matrix notation as

$$A\mathbf{x} = \mathbf{b},$$

where $A = (a_{ij})_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$.

Example 1.5.6. Convert the general LPP in the (above) Example 1.5.1 into standard form.

Solution. **Standard form:** Maximize $z = 2x_1 + 3x_2 + 4x_3$
subject to constraints

$$\begin{aligned} 3x_1 + x_1 + 6x_3 + s_1 &= 600 \\ 2x_1 + 4x_2 + 2x_3 - S_2 &= 480 \\ 2x_1 + 3x_2 + 3x_3 &= 540 \\ x_1,\ x_2,\ x_3,\ s_1,\ S_2 &\geq 0, \end{aligned}$$

where s_1 is a slack variable and S_2 is a surplus variable. □

1.5.4 Solving a system of linear equations

When an LPP with n-decision variables and m-constraints is expressed in the standard form, the constraints are linear equations. Consider a system of m-linear equations in n-unknowns. We have the following three cases:

1. $m > n$.
   In this case, the number of equations exceeds than the number of unknowns. So not all constraints will be linearly independent and they are called *redundant constraints*. It is possible to discard such constraints and the case can be reduced to one of the following two cases.
2. $m = n$.
   If the system is consistent then it has unique solution which is optimum. In this case, the solution of the problem is insignificant from OR perspective.
3. $m < n$.
   In this case, the system has infinitely many feasible solutions and the problem is to determine the optimal solution by OR techniques. In all non-trivial LPPs, the number of equations m is always less than the number of variables n.

Now, we describe how to obtain some of the feasible solutions (corresponding to the corner points in graphical method) and then determine the optimum solution among them.

Definition 1.5.7 (Basic solution)**.** Consider an LPP with n-decision variables and m-constraints in standard form, where $m < n$. To determined a solution, $n - m$ variables are set equal to zero and thus it suffices to solve m equations for the remaining m variables. This provides a unique solution which is called a *basic solution*.

Definition 1.5.8 (Basic and non-basic variables)**.** To solve an LPP with m equations and n variables with $m < n$, a set of $n - m$ variables are set equal to zero. These variables are called *non-basic variables*. The remaining m variables which give a unique basic solution are called *basic variables*.

Definition 1.5.9 (Basic feasible solution)**.** A basic solution is called a *basic feasible solution* if it is feasible i.e., if the basic variables are non-negative.

Remark 1.5.10. Consider an LPP with m equations and n variables, $m < n$. We choose m basic variables among the total n variables to solve the m equations by setting the remaining $n-m$ variables to zero. Thus, the number of basic solutions will be

$$^nC_m = \frac{n!}{(n-m)!\, m!}.$$

In the following example, we demonstrate the above described method to obtain all the possible basic solutions.

Example 1.5.11. Express the following LPP in two variables into equation form and obtain all the basic solutions.

$$\begin{aligned} \text{Maximize } z &= 2x_1 + 3x_2 \\ \text{subject to} \\ 2x_1 + x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 5 \\ x_1,\, x_2 &\geq 0. \end{aligned}$$

Solution. First we convert the given LPP in equation (standard) form. Since the two constraints are of $\leq$ type inequalities, we add two slack variables s_1, s_2 and express the given LPP as

$$\begin{aligned} \text{Maximize } z &= 2x_1 + 3x_2 \\ \text{subject to} \\ 2x_1 + x_2 + s_1 &= 4 \\ x_1 + 2x_2 + s_2 &= 5 \\ x_1,\, x_2,\, s_1,\, s_2 &\geq 0. \end{aligned}$$

We have to solve $m = 2$ equations for $n = 4$ variables. Hence, as describe in the above remark, the number of basic solutions will be $^4C_2 = 6$. These 6 solutions will be obtained by setting $n-m = 2$ variables equal to zero and solving for the remaining $m = 2$ variables. We have the following possibilities for basic solutions.

1. Setting x_1, x_2 as non-basic variables, i.e. $x_1 = x_2 = 0$, from the above listed constraints, we have

   $$s_1 = 4 \text{ and } s_2 = 5.$$

   Thus, the solution is $(s_1, s_2) = (4, 5)$ at which the value of the objective function $z = 0$.
2. Non-basic variables $x_1 = s_1 = 0$. Then we have

   $$\begin{aligned} x_2 &= 4 \\ 2x_2 + s_2 &= 5 \end{aligned}$$

   solving which we get $(x_2, s_2) = (4, -3)$. Note that this (basic) solution is not feasible as the basic variable s_2 is not non-negative and hence we do not compute the value of objective function z in this case.
3. Non-basic variables $x_1 = s_2 = 0$. Then we have

   $$\begin{aligned} x_2 + s_1 &= 4 \\ 2x_2 &= 5 \end{aligned}$$

   solving which we get $(x_2, s_1) = \left(\frac{5}{2}, \frac{3}{1}\right)$ and $z = \frac{15}{2} = 7.5$.

4. Non-basic variables $x_2 = s_1 = 0$. Then we have

$$\begin{aligned} 2x_1 &= 4 \\ x_1 + s_2 &= 5. \end{aligned}$$

This gives the solution $(x_1, s_2) = (2, 3)$ and the objective function $z = 4$.

5. Non-basic variables $x_2 = s_2 = 0$. Then we have

$$\begin{aligned} 2x_1 + s_1 &= 4 \\ x_1 &= 5. \end{aligned}$$

In this case the solution is $(x_1, s_1) = (5, -6)$. This is infeasible basic solution as the basic variable s_1 is negative.

6. Non-basic variables $s_1 = s_2 = 0$. Then we have

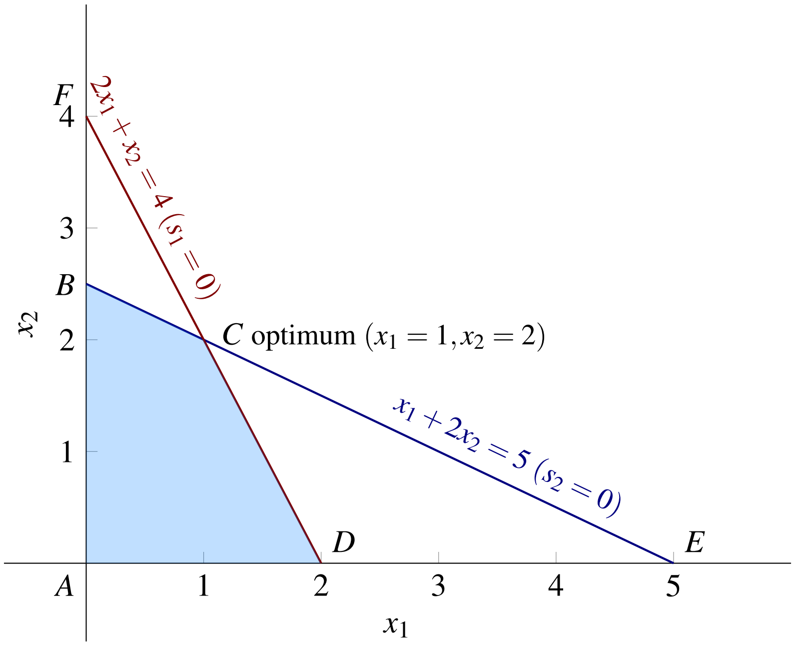
$$\begin{aligned} 2x_1 + x_2 &= 4 \\ x_1 + 2x_2 &= 5. \end{aligned}$$

Solving these equations, we get $(x_1, x_2) = (1, 2)$. The value of z at this point is 8.

□

Remark 1.5.12. From the six basic solutions obtained in the above example it appears that the optimum solution is 8 and the optimum occurs at the point $(x_1, x_2) = (1, 2)$. This is in fact, true. We verify this solving the example by graphical method below. This will also reflect how these basic solutions are analogously related to the corner points we obtain graphically.

The graph of the above LP is shown below.



The table shown below gives an idea of the analogy between the corner points and the basic solutions. It reflects how the corner points of graph are obtained as basic solutions. In fact the vertices of the feasible region are obtained as the basic feasible solutions of the given LP.

Non-basic (zero) variables	**Basic variables**	**Basic solution**	**Associated corner point** (x_1,x_2)	**Feasible?**	**Objective function value** z
(x_1,x_2)	(s_1,s_2)	$(4,5)$	$A\ (0,0)$	✔	0
(x_1,s_1)	(x_2,s_2)	$(4,-3)$	$F\ (0,4)$	✘	–
(x_1,s_2)	(x_2,s_1)	$(2.5,1.5)$	$B\ (0,2.5)$	✔	7.5
(x_2,s_1)	(x_1,s_2)	$(2,3)$	$D\ (2,0)$	✔	4
(x_2,s_2)	(x_1,s_1)	$(5,-6)$	$E\ (5,0)$	✘	–
(s_1,s_2)	(x_1,x_2)	$(1,2)$	$C\ (1,2)$	✔	8

Definition 1.5.13 (Degenerate basic solution)**.** For a system of linear equations, a basic solution is called a *degenerate basic solution* if any of the basic variables is zero.

If all the (basic) variables in a basic solution are non-zero then that basic solution is called *non-degenerate basic solution*.

In the above example, all the 6 basic solutions are non-degenerate basic solutions because in all of the basic solutions, both basic variables are non-zero. Let us consider one more example below in which we compute the basic solutions and determine which among them are feasible and non-degenerate basic solutions.

Example 1.5.14. Obtain all basic solutions to the system

$$\begin{aligned} 2x_1+x_2-x_3&=2\\ 3x_1+2x_2+x_3&=3. \end{aligned}$$

Solution. Here the given constraints of the problem are already in equation form. There are $m=2$ equations, both are linearly independent, and $n=3$ variables. So we set $n-m=1$ variable to zero to obtain ${}^3C_2=3$ basic solutions. We have the following solutions:

1. $x_1=0$ (non-basic variable). Then we have

$$\begin{aligned} x_2-x_3&=2\\ 2x_2+x_3&=3. \end{aligned}$$

   Thus, the basic solution in this case is $(x_2,x_3)=\left(\frac{5}{3},-\frac{1}{3}\right)$. This is not a feasible solution as the basic variable, x_3 are negative. However, the solution is non-degenerate basic solution as both the basic variables x_2,x_3 are non-zero.

2. $x_2=0$ (non-basic variable). Then we have

$$\begin{aligned} 2x_1-x_3&=2\\ 3x_1+x_3&=3. \end{aligned}$$

   Thus, the basic solution in this case is $(x_1,x_3)=(1,0)$. This is feasible solution as both the basic variables, x_1 and x_3 are non-negative. This is, however, a degenerate basic solution as the basic variable $x_3=0$.

3. $x_3=0$ (non-basic variable). Then we have

$$2x_1+x_2=2$$

$$3x_1 + 2x_2 = 3.$$

Thus, the basic solution in this case is $(x_1, x_2) = (1, 0)$. This is feasible solution as both the basic variables, x_1 and x_2 are non-negative. This is, however, a degenerate basic solution as the basic variable $x_2 = 0$.

We summarize the above basic solutions by writing them in tabular form as follows:

Non-basic variables	**Basic variables**	**Basic solution**	**Feasible?**	**Non-degenerate?**
x_1	(x_2, x_3)	$(\frac{5}{3}, -\frac{1}{3})$	✘	✔
x_2	(x_1, x_3)	$(1, 0)$	✔	✘
x_3	(x_1, x_2)	$(1, 0)$	✔	✘

☐

1.6 The Simplex Method

Before we describe the computational algorithm of the procedure to solve a given LP by simplex method, we explain the simplex method by considering an example demonstration.

We begin with an already seen example, the Reddy Mikks model, which is converted into equation form below. We now obtain the solution to the Reddy Mikks model by simplex method. Since this is the first example of simplex method being discussed, we include the description of the method and full computational details and not just tableau steps only.

Example 1.6.1. Consider the Reddy Mikks model (Example 1.2.1) expressed in standard form as follows:

$$\text{Maximize } z = 5x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

subject to

$$\begin{aligned} 6x_1 + 4x_2 + s_1 \qquad\qquad &= 24 \\ x_1 + 2x_2 \qquad + s_2 \qquad &= 6 \\ -x_1 + x_2 \qquad\qquad + s_3 &= 1 \\ x_2 \qquad\qquad\qquad + s_4 &= 2 \end{aligned}$$

$$x_1,\ x_2,\ s_1,\ s_2,\ s_3,\ s_4 \geq 0,$$

where the variables s_1, s_2, s_3, s_4 are slack variables.

Solution. We write the objective equation as

$$z - 5x_1 - 4x_2 = 0.$$

The initial simplex table can be represented as follows:

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	**Solution**	
z	1	-5	-4	0	0	0	0	0	z-row
s_1	0	6	4	1	0	0	0	24	s_1-row
s_2	0	1	2	0	1	0	0	6	s_2-row
s_3	0	-1	1	0	0	1	0	1	s_3-row
s_4	0	0	1	0	0	0	1	2	s_4-row

The process of obtaining solution by simplex method begins at origin, in this case $(x_1,x_2) = (0,0)$. Thus, x_1, x_2 are set to be non-basic variables and the remaining variables (s_1,s_2,s_3,s_4) are taken to be basic variables. The basic variables are listed in the first and leftmost column ("Basic column") and their solutions are given in the last and rightmost column ("Solution column"). Thus, the starting simplex table formulation itself gives us the initial basic solution. In our case, the starting basic solution is $(s_1,s_2,s_3,s_4) = (24,6,1,2)$ which is obtained from the constraint equations by taking $x_1 = x_2 = 0$. Note that (at every stage) the submatrix formed by the basic variables in the simplex table forms an identity matrix. This is shown in the above table by shaded columns.

The objective function z is written in form of an equation with right hand side equal to 0. The coefficients of z and all the variables x_1,x_2,s_1,s_2,s_3,s_4 gives us the first row (i.e. "z-row") of starting simplex table. Accordingly here we write $z - 5x_1 - 4x_2 = 0$. Since the right hand side of objective function is 0, the entry in Solution column of z-row is also 0.

The initial solution of z can be improved by increasing the value of a non-basic variable x_1 or x_2 from 0 to some positive value. The variable with the most negative coefficient (in the z-row) is chosen for this purpose. This rule is called **simplex optimality condition**. The variable chosen is called the **entering variable** and since the number of basic variables m and non-basic variables $n - m$ is fixed, one of the basic variables is to be replaced by the entering non-basic variable (x_1 in this case). This variable is called the **leaving variable**. We compute the **ratios** of the solutions to the coefficients of entering variable. The row with the minimum ratio decides the leaving variable. This rule is called the **simplex feasibility condition**.

In our example, x_1 has the most negative coefficient -5 and hence x_1 is the entering variable. Now, we compute the ratios of $\frac{\text{Solution}}{\text{coefficient of } x_1}$ for "s_1-row, s_2-row, s_3-row and s_4-row". The table of ratios is given below.

Basic	**Entering** x_1	**Solution**	**Ratio (or intercept)**
s_1	6	24	$x_1 = \frac{24}{6} = 4 \leftarrow$ minimum
s_2	1	6	$x_1 = \frac{6}{1} = 6$
s_3	-1	1	$x_1 = \frac{1}{-1} = -1$ (negative denominator, ignore)
s_4	0	2	$x_1 = \frac{2}{0}$ (zero denominator, ignore)

Note that from the above table, it is clear that this ratio is minimum for the basic variable s_1 and hence s_1 is the leaving variable. The the row of leaving variable is called the **pivot row** while the column of entering variable is called the **pivot column** and their intersection cell is called the **pivot element**. The same starting simplex table with pivot row ("s_1-row"), pivot column ("x_1-column") and pivot element (6 in this case) is re-sketched in the following table.

			Enter ↓							
	Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	**Solution**	
	z	1	-5	-4	0	0	0	0	0	
Leave ←	s_1	0	6	4	1	0	0	0	24	Pivot row
	s_2	0	1	2	0	1	0	0	6	
	s_3	0	-1	1	0	0	1	0	1	
	s_4	0	0	1	0	0	0	1	2	
			Pivot column							

Next, we compute the **Gauss-Jordan row operations** to obtain the new basic solutions. These computations are carried out in the following two steps:

1. For pivot row:
  - **(a)** Replace the leaving variable in the "Basic" column by the entering variable.
  - **(b)** New pivot row = Current pivot row ÷ Pivot element.

2. For all other rows:

$$\text{New row} = \big(\text{Current row}\big) - \big((\text{Pivot column coefficient}) \times (\text{New pivot row})\big).$$

These computations are shown below:

1. Replace s_1 by x_1 in the Basic column.

$$\begin{aligned} \text{New } x_1-\text{row} &= \text{Current } s_1\text{-row} \div 6 \\ &= \frac{1}{6}(0 \quad 6 \quad 4 \quad 1 \quad 0 \quad 0 \quad 0 \quad 24) \\ &= \left(0 \quad 1 \quad \frac{2}{3} \quad \frac{1}{6} \quad 0 \quad 0 \quad 0 \quad 4\right). \end{aligned}$$

2.

$$\begin{aligned} \text{New } z-\text{row} &= \text{Current } z\text{-row} - (-5)\times \text{ New } x_1 \text{ row} \\ &= (1 \quad -5 \quad -4 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0) - (-5)\left(0 \quad 1 \quad \frac{2}{3} \quad \frac{1}{6} \quad 0 \quad 0 \quad 0 \quad 4\right) \\ &= \left(1 \quad 0 \quad -\frac{2}{3} \quad \frac{5}{6} \quad 0 \quad 0 \quad 0 \quad 20\right). \end{aligned}$$

3.

$$\begin{aligned} \text{New } s_2-\text{row} &= \text{Current } s_2\text{-row} - (1)\times \text{ New } x_1 \text{ row} \\ &= (0 \quad 1 \quad 2 \quad 0 \quad 1 \quad 0 \quad 0 \quad 6) - (1)\left(0 \quad 1 \quad \frac{2}{3} \quad \frac{1}{6} \quad 0 \quad 0 \quad 0 \quad 4\right) \\ &= \left(0 \quad 0 \quad \frac{4}{3} \quad -\frac{1}{6} \quad 1 \quad 0 \quad 0 \quad 2\right). \end{aligned}$$

4. New s_3 − row $= \text{Current } s_3\text{-row} - (-1) \times \text{New } x_1 \text{ row}$

$$= (0 \quad -1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1) - (-1)\left(0 \quad 1 \quad \frac{2}{3} \quad \frac{1}{6} \quad 0 \quad 0 \quad 0 \quad 4\right)$$

$$= \left(0 \quad 0 \quad \frac{5}{3} \quad \frac{1}{6} \quad 0 \quad 1 \quad 0 \quad 5\right).$$

5. New s_4 − row $= \text{Current } s_4\text{-row} - (0) \times \text{New } x_1 \text{ row}$

$$= (0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 2) - (0)\left(0 \quad 1 \quad \frac{2}{3} \quad \frac{1}{6} \quad 0 \quad 0 \quad 0 \quad 4\right)$$

$$= (0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 2).$$

The new simplex table with the above rows is shown below.

				↓					
	Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	**Solution**
	z	1	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	20
	x_1	0	1	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	4
←	s_2	0	0	$\frac{4}{3}$	$-\frac{1}{6}$	1	0	0	2
	s_3	0	0	$\frac{5}{3}$	$\frac{1}{6}$	0	1	0	5
	s_4	0	0	1	0	0	0	1	2

The above table gives the new basic solution $(x_1, s_2, s_3, s_4) = (4, 2, 5, 2)$. Equivalently the new value of the objective $z = 5 \times x_1 + 4 \times x_2 + 0 \times s_1 + 0 \times s_2 + 0 \times s_3 + 0 \times s_4 = 20$.

Note that the optimality condition (most negative coefficient) shows that x_2 is the entering variables in the next step. We compute the ratios in the following table and find out that s_2 is the leaving variable as it has the minimum ratio.

Basic	**Entering** x_2	**Solution**	**Ratio (or intercept)**
x_1	$\frac{2}{3}$	4	$x_2 = 4 \div \frac{2}{3} = 6$
s_2	$\frac{4}{3}$	2	$x_2 = 2 \div \frac{4}{3} = 1.5$ (minimum)
s_3	$\frac{5}{3}$	5	$x_2 = 5 \div \frac{5}{3} = 3$
s_4	1	2	$x_2 = \frac{2}{1} = 2$

Again computing the Gauss-Jordan row operations as before, we obtain the new simplex table as follows:

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	**Solution**
z	1	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
x_1	0	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
x_2	0	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
s_3	0	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
s_4	0	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

Observe that none of the coefficients of the z-row in the above table have negative coefficients. Hence the above table is optimal. The optimum is $z = 21$ and the optimal point is $(x_1,x_2) = \left(3,\frac{3}{2}\right)$. The complete basic solution is $(x_1,x_2,s_3,s_4) = \left(3,\frac{3}{2},\frac{5}{2},\frac{1}{2}\right)$. □

1.6.1 The Simplex Algorithm

The above example is of maximization type in which by *optimality condition* the variable with the most negative coefficient in the z-row of simplex table is the entering variable. In minimization type problems, it is the opposite, i.e. the variable with the most positive coefficient in the z-row becomes the entering variable. However, the *feasibility condition* for the leaving variable remains unchanged.

Now, we describe below the terms we used in simplex process in the above example.

Definition 1.6.2 (Optimality condition)**.** The entering variable in a maximization type (or minimization type) problem is the non-basic variable with the most negative (or most positive) coefficient in the z-row. If two variables have the same most negative coefficient then any one is chosen arbitrarily. The optimum is reached at the iterative step where all the coefficients of the z-row are non-negative (or non-positive).

Definition 1.6.3 (Feasiblility condition)**.** For both the maximization and minimization type problems, the leaving variable is the basic variable associated with the smallest non-negative ratio and strictly positive denominator. In case of any tie, any one of the variable is chosen as leaving variable arbitrarily.

Definition 1.6.4 (Gauss-Jordan row operations)**.**

1. For pivot row:
   - **(a)** Replace the leaving variable in the “Basic” column by the entering variable.
   - **(b)** New pivot row = Current pivot row ÷ Pivot element.
2. For all other rows (including the z-row):

$$\text{New row} = \big(\text{Current row}\big) - \big((\text{Pivot column coefficient}) \times (\text{New pivot row})\big).$$

The following are the steps of the simplex method:

Simplex Method Algorithm

1. Express the given LPP into equation form (i.e. standard form) by introducing slack or surplus variables if necessary.
2. Determine a initial basic feasible solution and frame the starting simplex table.
3. Choose the entering variable by optimality condition. If there is no entering variable, then the latest solution is optimal. If optimal solution is obtained, terminate the process. Else go to next step.
4. Select the leaving variable using the feasibility condition.
5. Compute Gauss-Jordan row operations to determine the new basic solution.
6. Go to Step 1 and repeat the process till optimum is obtained.

Let us consider few more examples of the simplex method.

1.6.2 Solved examples using simplex method

Example 1.6.5. Solve the following problem by simplex method.

$$\begin{aligned} \text{Max. } z &= 4x_1 + 10x_2 \\ \text{subject to} \\ 2x_1 + x_2 &\leq 50 \\ 2x_1 + 3x_2 &\leq 90 \\ 2x_1 + 5x_2 &\leq 100 \\ x_1,\ x_2, &\geq 0. \end{aligned}$$

Solution. The equation form of the given LPP is written below.

$$\text{Max. } z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$$

subject to

$$\begin{array}{ccccccccccc} 2x_1 & + & x_2 & + & s_1 & & & & & = & 50 \\ 2x_1 & + & 3x_2 & & & + & s_2 & & & = & 90 \\ 2x_1 & + & 5x_2 & & & & & + & s_3 & = & 100 \end{array}$$
$$x_1,\ x_2,\ s_1,\ s_2,\ s_3 \geq 0,$$

where the variables s_1, s_2, s_3 are slack variables.

We write the objective function equation as $z - 4x_1 - 10x_2 = 0$. The starting simplex table giving initial basic solution $(s_1, s_2, s_3) = (50, 90, 100)$ is represented below.

	Basic	z	x_1	↓ x_2	s_1	s_2	s_3	Solution	Ratio	
	z	1	-4	-10	0	0	0	0		
	s_1	0	2	1	1	0	0	50	$\frac{50}{1} = 50$	
	s_2	0	2	3	0	1	0	90	$\frac{90}{3} = 30$	
←	s_3	0	2	5	0	0	1	100	$\frac{100}{5} = 20$	Pivot row
				Pivot column						

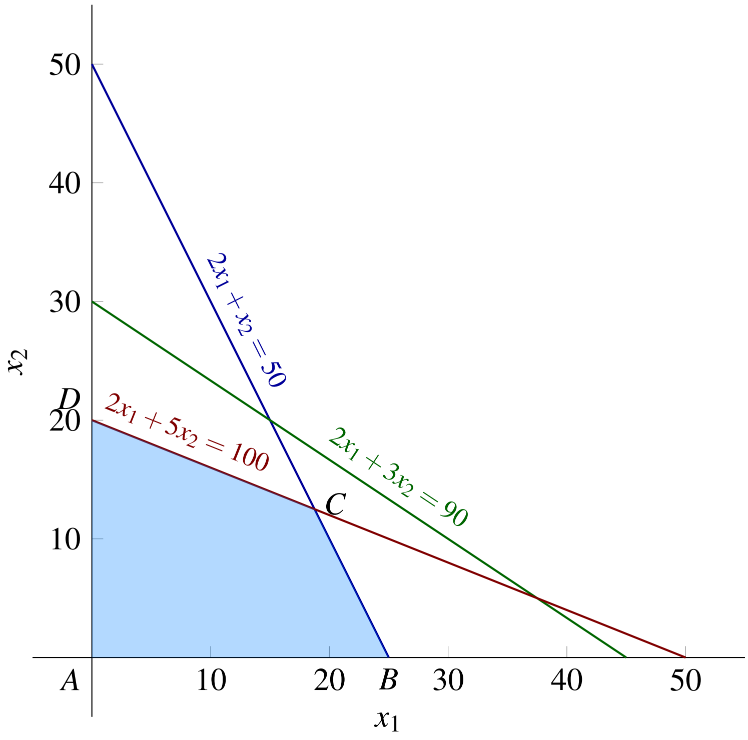
Here x_2 has the most negative coefficient and hence x_2 is the entering variable. The ratios computed in the rightmost column indicates that s_3 is the leaving variable. Hence, “s_3-row” is the pivot row, “x_2-column” is the pivot column and their intersection entry 5 is the pivot element.

We carry out the Gauss-Jordan computations to obtain the following simplex table.

Basic	z	x_1	x_2	s_1	s_2	s_3	**Solution**
z	1	0	0	0	0	2	200
s_1	0	$\frac{8}{5}$	0	1	0	$-\frac{1}{5}$	30
s_2	0	$\frac{4}{5}$	0	0	1	$-\frac{3}{5}$	30
x_2	0	$\frac{2}{5}$	1	0	0	$\frac{1}{5}$	20

Since all the entries in the “z-row” are non-negative, we obtain the the solution of the given problem which is $z = 200$ with $x_1 = 0$ and $x_2 = 20$. ☐

We now verify the optimal solution of the above problem by graphical method. The graph of the above problem is sketched below.



The feasible region is the shaded region bounded by the polygon $ABCD$. We compute the objective value at these corner points and determine the optimum point.

Point	**Coordinates**	**Max** $z = 4x_1 + 10x_2$
A	$(0,0)$	0
B	$(25,0)$	100
C	$(\frac{75}{4}, \frac{25}{2})$	200
D	$(0,20)$	200

Thus, by graphical method we have verified that $D(0,20)$ is one of the point where optimum occurs and the optimum value is $z = 200$.

Example 1.6.6. Solve the following problem by simplex method.

$$\begin{aligned} \text{Max. } z &= 4x_1 + 3x_2 + 6x_3 \\ \text{subject to} \\ 2x_1 + 3x_2 + 2x_3 &\leq 440 \\ 4x_1 + 3x_3 &\leq 470 \\ 2x_1 + 5x_2 &\leq 430 \\ x_1,\ x_2,\ x_3 &\geq 0. \end{aligned}$$

Solution. The given LPP can be expressed in equation form as

$$\text{Max. } z = 4x_1 + 3x_2 + 6x_3 + 0s_1 + 0s_2 + 0s_3$$

subject to

$$\begin{array}{lllllllll} 2x_1 & + & 3x_2 & + & 2x_3 & + & s_1 & & & & = & 440 \\ 4x_1 & & & + & 3x_3 & & & + & s_2 & & = & 470 \\ 2x_1 & + & 5x_2 & & & & & & & + \ s_3 & = & 430 \end{array}$$

$$x_1,\ x_2,\ x_3,\ s_1,\ s_2,\ s_3 \geq 0,$$

where the variables s_1, s_2, s_3 are slack variables.

We write the objective equation as $z - 4x_1 - 3x_2 - 6x_3 = 0$. The starting simplex table giving the initial basic solution $(s_1, s_2, s_3) = (440, 470, 430)$ is framed below.

Basic	z	x_1	x_2	x_3	s_1	s_2	s_3	**Solution**	**Ratio**
z	1	-4	-3	-6	0	0	0	0	
s_1	0	2	3	2	1	0	0	440	$\frac{440}{2} = 220$
s_2	0	4	0	3	0	1	0	470	$\frac{470}{3}$
s_3	0	2	5	0	0	0	1	430	$\frac{430}{0} = \infty$

The optimality condition (most negative coefficient) shows that x_3 is the entering non-basic variable and the feasibility condition (ratios) shows that s_2 is the leaving basic variable. After computing the Gauss-Jordan row operations, we get the simplex table as follows:

Basic	z	x_1	x_2	x_3	s_1	s_2	s_3	**Solution**	**Ratio**
z	1	4	-3	0	0	2	0	940	
s_1	0	$-\frac{2}{3}$	3	0	1	$-\frac{2}{3}$	0	$\frac{380}{3}$	$\frac{380}{9}$
x_3	0	$\frac{4}{3}$	0	1	0	$\frac{1}{3}$	0	$\frac{470}{3}$	$\frac{470}{0} = \infty$
s_3	0	2	5	0	0	0	1	430	$\frac{430}{5} = 83$

From the above table it follows that, in the next iteration, the non-basic variable x_2 is the entering variable and the basic variable s_1 leaves. We have the following simplex table in the next iterative step.

Basic	z	x_1	x_2	x_3	s_1	s_2	s_3	**Solution**
z	1	2	0	0	1	0	0	$\frac{3200}{3}$
x_2	0	$-\frac{4}{3}$	1	0	$\frac{1}{3}$	$-\frac{2}{9}$	0	$\frac{380}{9}$
x_3	0	$\frac{4}{3}$	0	1	0	$\frac{1}{3}$	0	$\frac{470}{3}$
s_3	0	$\frac{26}{3}$	0	0	$-\frac{5}{3}$	$\frac{10}{9}$	1	$\frac{1970}{9}$

Thus, the optimal solution is $z = \frac{3200}{3} = 1066.67$ and the point at which it occurs is $(x_1, x_2, x_3) = \left(0, \frac{470}{3}, \frac{380}{9}\right)$. The final basic solution is $(x_2, x_3, s_3) = \left(\frac{470}{3}, \frac{380}{9}, \frac{1970}{9}\right)$. ☐

1.7 Artificial Starting Solution

Example 1.7.1. Express the following LP into equation form and check if it has a starting solution consisting of all slack variables.

$$\text{Max } z = 5x_1 - 4x_2 + 3x_3$$

subject to

$$\begin{aligned} 2x_1 + x_2 - 6x_3 &= 20 \\ 6x_1 + 5x_2 + 10x_3 &\leq 76 \\ 8x_1 - 3x_2 + 6x_3 &\leq 50 \\ x_1,\ x_2,\ x_3 &\geq 0. \end{aligned}$$

Solution. The equation form of given LP is written below.

$$\text{Max } z = 5x_1 - 4x_2 + 3x_3 + 0s_1 + 0s_2$$

subject to

$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & x_2 & - & 6x_3 & & & & & = & 20 \\ 6x_1 & + & 5x_2 & + & 10x_3 & + & s_1 & & & = & 76 \\ 8x_1 & - & 3x_2 & + & 6x_3 & & & + & s_2 & = & 50 \end{array}$$

$$x_1,\ x_2,\ x_3,\ s_1,\ s_2, \geq 0,$$

where the variables s_1, s_2 are slack variables. Here there are three equations and only two slack variables. Hence, we cannot get an initial all-slack variable basic solution if we take $x_1 = x_2 = x_3 = 0$. The reason is the fact that the first constraint is an equation and not "$\leq$" type. ☐

As seen in the above example if the a given LP has constraints of type "$=$" or "$\geq$", then it does not give a starting basic solution consisting of all slack variables only. In other words, in such cases, we cannot initiate the simplex process at origin i.e. by taking given variables $x_1, x_2, \ldots, x_n$ as non-basic variables.

To deal with such problems, we introduce *artificial variables* which play the same role as that of the slack variables in the first iteration. The artificial variables are then disposed of at a later iteration. There are two methods involving the use of artificial variables for solving a given LP for which initial basic feasible solution cannot be given by all the slack variables. These methods are: the **Big M-method** and the **Two-Phase method**.

1.7.1 Big M-Method

In a given LPP represented in equation form, if an equation i does not have a slack variable then an *artificial variable* R_i is added. This helps in forming a starting basic solution similar to all-slack basic solution. However, these artificial variables thus added, are not part of the original problem and they are reduced to zero by the time we reach the optimum solution (assuming that feasible solution exists). This is done by penalizing the artificial variables in the objective function by the following rule.

Penalty rule for artificial variables:

Given a sufficiently large positive value M ($M \to \infty$), the objective function coefficient of an artificial variable represents an appropriate penalty if:

$$\text{Artificial variable objective coefficient} = \begin{cases} -M, & \text{in maximization problems} \\ M, & \text{in minimization problems} \end{cases}.$$

The Big M-method is sometimes also called the *M-method* or the *penalty method*. Let us consider couple of examples below to understand this method.

Example 1.7.2. Solve: Max. $z = 6x_1 + 4x_2$
subject to

$$\begin{aligned} 2x_1 + 3x_2 &\leq 30 \\ 3x_1 + 2x_2 &\leq 24 \\ x_1 + x_2 &\geq 3 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

Solution. First we express the given problem in equation form as below:

$$\text{Max. } z = 6x_1 + 4x_2 + 0s_1 + 0s_2 + 0S_3$$

subject to

$$\begin{array}{rcrcrcrcrcl} 2x_1 & + & 3x_2 & + & s_1 & & & & & = & 30 \\ 3x_1 & + & 2x_2 & & & + & s_2 & & & = & 24 \\ x_1 & + & x_2 & & & & & - & S_3 & = & 3 \end{array}$$

$$x_1,\, x_2,\, x_3,\, s_1,\, s_2,\, S_3 \geq 0,$$

where s_1, s_2 are slack variables and S_3 is a surplus variable.

If we start the simplex method at origin, i.e. setting x_1, x_2 as non-basic variables, then the initial basic solution is given by $(s_1, s_2, S_3) = (30, 24, -3)$ which is infeasible. The reason is that the third constraint does not have a slack variable (here S_3 is a surplus variable). So we add an artificial variable R_1 in the third constraint and penalize it in the objective function with $-MR_1$. The resultant LP is given below.

$$\text{Max. } z = 6x_1 + 4x_2 - MR_1$$

subject to

$$\begin{array}{rcrcrcrcrcrcl} 2x_1 & + & 3x_2 & + & s_1 & & & & & & & = & 30 \\ 3x_1 & + & 2x_2 & & & + & s_2 & & & & & = & 24 \\ x_1 & + & x_2 & & & & & - & S_3 & + & R_1 & = & 3 \end{array}$$

$$x_1,\, x_2,\, x_3,\, s_1,\, s_2,\, S_3,\, R_1 \geq 0.$$

The starting basic solution is given by $(s_1, s_2, R_1) = (30, 24, 3)$. The computations during the iterations can be carried out algebraically considering M as a large value. However, for computational convenience, we assign a large value to M, say $M = 100$ (looking at the coefficients in the objective function, it would suffice). Writing the objective equation as $z - 6x_1 - 4x_2 + 100R_1 = 0$, the starting simplex table takes the following form:

Basic	x_1	x_2	s_1	s_2	S_3	R_1	**Solution**
z	-6	-4	0	0	0	100	0
s_1	2	3	1	0	0	0	30
s_2	3	2	0	1	0	0	24
R_1	1	1	0	0	-1	1	3

Note that the initial value of z is

$$z = 6x_1 + 4x_2 - 100R_1 = 6 \times 0 + 4 \times 0 - 100 \times 3 = -300.$$

So to make the z-row in the above simplex table consistent with the rest of the table, we substitute out R_1 in the z-row by the following operation:

$$\text{New } z\text{-row} = \text{Old } z\text{-row} - (100 \times R_1\text{-row}).$$

Then the starting simplex table takes the following form:

Basic	x_1	x_2	s_1	s_2	S_3	R_1	**Solution**	**Ratio**
z	-106	-104	0	0	100	0	-300	
s_1	2	3	1	0	0	0	30	$\frac{30}{2}=15$
s_2	3	2	0	1	0	0	24	$\frac{24}{3}=8$
R_1	1	1	0	0	-1	1	3	$\frac{3}{1}=3$ (min.)

The above table is now ready for execution of simplex algorithm. It can be seen, by optimality and feasibility conditions respectively, that x_1 is the entering variable and R_1 is the leaving variable. We compute the Gauss-Jordan operations to obtain the following table:

Basic	x_1	x_2	s_1	s_2	S_3	R_1	**Solution**	**Ratio**
z	0	2	0	0	-6	106	18	
s_1	0	1	1	0	2	-2	24	$\frac{24}{2}=12$
s_2	0	-1	0	1	3	-3	15	$\frac{15}{3}=5$ (min.)
x_1	1	1	0	0	-1	1	3	$\frac{3}{-1}=-3$

Basic	x_1	x_2	s_1	s_2	S_3	R_1	**Solution**
z	0	0	0	2	0	100	48
s_1	0	$\frac{5}{3}$	1	$-\frac{2}{3}$	0	0	14
S_3	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	1	-1	5
x_1	1	$\frac{2}{3}$	0	$\frac{1}{3}$	0	0	8

Hence, the optimum value of the objective function is $z = 48$ and the optimum point is $(x_1, x_2) = (8, 0)$. ☐

The above LP involves two variables and so the solution can be verified graphically. It is left as an **exercise** to check the solution by graphical method.

Example 1.7.3. Solve: Min. $z = 4x_1 + x_2$
subject to

$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 &\geq 6 \\ x_1 + 2x_2 &\leq 4 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

Solution. Introducing x_3 as a surplus variable in the second constraint and x_4 as a slack variable in the third constraint, the given LP can be expressed in equation form as

$$\text{Min. } z = 4x_1 + x_2 + 0x_3 + 0x_4$$

subject to

$$\begin{array}{rcrcrcrcl} 3x_1 & + & x_2 & & & & & = & 3 \\ 4x_1 & + & 3x_2 & - & x_3 & & & = & 6 \\ x_1 & + & 2x_2 & & & + & x_4 & = & 4 \end{array}$$

$$x_1,\ x_2,\ x_3,\ x_4 \geq 0.$$

The first and second constraint equations written above do not have any slack variable. Hence it is not possible in this case to have a starting basic feasible solution of all slack variables only. As a result, we add artificial variables R_1, R_2 to the first and second constraint respectively and penalize them in the objective function with $MR_1 + MR_2$ (since it is a minimization problem) for a sufficiently large number M. The resultant LP can be written as:

$$\text{Min. } z = 4x_1 + x_2 + MR_1 + MR_2$$

subject to

$$\begin{array}{rcrcrcrcrcrcl} 3x_1 & + & x_2 & & & & & + & R_1 & & & = & 3 \\ 4x_1 & + & 3x_2 & - & x_3 & & & & & + & R_2 & = & 6 \\ x_1 & + & 2x_2 & & & + & x_4 & & & & & = & 4 \end{array}$$

$$x_1,\ x_2,\ x_3,\ x_4,\ R_1,\ R_2 \geq 0.$$

The starting basic solution is given by $(R_1, R_2, x_4) = (3, 6, 4)$. Looking at the coefficients of the variables in the objective function, it appears that the value of $M = 100$ is reasonable. Writing the objective equation as

$$z - 4x_1 - x_2 - 100R_1 - 100R_2 = 0,$$

the starting simplex table can be written as follows:

Basic	x_1	x_2	x_3	R_1	R_2	x_4	**Solution**
z	-4	-1	0	-100	-100	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

The initial value of z is $z = 4x_1 + x_2 + 100R_1 + 100R_2 = 4 \times 0 + 0 + 100 \times 3 + 100 \times 6 = 900$. The above table shows the starting value of $z = 0$. So to make the z-row consistent with the rest of the table, we substitute out R_1 and R_2 in the z-row by the following row operation:

New z-row = Old z-row + ($100 \times R_1$-row + $100 \times R_2$-row).

Then the simplex table then takes the form as shown below.

Basic	x_1	x_2	x_3	R_1	R_2	x_4	**Solution**	**Ratio**
z	696	399	-100	0	0	0	900	
R_1	3	1	0	1	0	0	3	$\frac{3}{3} = 1$ (min.)
R_2	4	3	-1	0	1	0	6	$\frac{6}{4} = \frac{3}{2}$
x_4	1	2	0	0	0	1	4	$\frac{4}{1} = 4$

The above table is now ready for simplex method computations. Since the problem is of minimization type, we choose the variable with the most positive coefficient in the z-row as the entering variable. Consequently, it can be seen that, x_1 is the entering variable and R_1 is the leaving variable. We carry out the simplex computations till we reach the optimum.

Basic	x_1	x_2	x_3	R_1	R_2	x_4	**Solution**	**Ratio**
z	0	167	-100	-232	0	0	204	
x_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	1	$\frac{1}{1/3} = 3$
R_2	0	$\frac{5}{3}$	-1	$-\frac{4}{3}$	1	0	2	$\frac{2}{5/3} = \frac{6}{5}$ (min.)
x_4	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	0	1	3	$\frac{3}{5/3} = \frac{9}{5}$

Basic	x_1	x_2	x_3	R_1	R_2	x_4	**Solution**	**Ratio**
z	0	0	$\frac{1}{5}$	$-\frac{492}{5}$	$-\frac{501}{5}$	0	$\frac{18}{5}$	
x_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$	$\frac{3/5}{1/5} = 3$
x_2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$	$\frac{6/5}{-3/5} = -2$
x_4	0	0	1	1	-1	1	1	1 (min.)

Basic	x_1	x_2	x_3	R_1	R_2	x_4	**Solution**
z	0	0	0	$-\frac{493}{5}$	-100	$-\frac{1}{5}$	$\frac{17}{5}$
x_1	1	0	0	$\frac{2}{5}$	0	$-\frac{1}{5}$	$\frac{2}{5}$
x_2	0	1	0	$-\frac{1}{5}$	0	$\frac{3}{5}$	$\frac{9}{5}$
x_3	0	0	1	1	-1	1	1

Since the given problem is of minimization type and all the variables in the z-row of the above table have non-positive coefficient, we have reached the optimum.

Hence, the optimum value of the objective function is $z = \frac{17}{5}$, the optimum occurs at the point $(x_1, x_2) = \left(\frac{2}{5}, \frac{9}{5}\right)$ and the complete basic solution is $(x_1, x_2, x_3) = \left(\frac{2}{5}, \frac{9}{5}, 1\right)$. □

Again, the above LP is in two variables and so one can verify the solution of the above problem by graphical method. The verification is left as an **exercise**.

Remark 1.7.4. Note that, in any given LP, an artificial variable need not always necessarily vanish at the end of simplex method by imposing a penalty M to it in the objective function. If, at the final iteration, an artificial variable has a positive value, then the given LP does not have feasible solution (see Exercise 1.20).

1.7.2 Two-Phase Method

The Two-Phase method also involves adding artificial variables to the constraint just as in the Big M-method except that the penalty M is not used in the objective function in this case. As the name suggest, the solution of the given LP is obtained in two phases. Phase I obtains the starting basic feasible solution and if one such solution is found then Phase II solves the problem by obtaining the optimum.

The Two-Phase method can be summarized as follows:

Phase I:

1. Express the given problem in equation form and add necessary, slack, surplus, and artificial variables (same as in the Big M-method) to obtain the starting basic feasible solution.
2. Thereafter, find a basic solution of the resulting equations that minimizes the sum of artificial variables, irrespective of whether the given LP is maximization type or minimization type.
3. If the minimum value of the sum of the artificial variables is positive, then the given LP has no feasible solution (see Exercise 1.22). Else proceed to Phase II.

Phase II:

Using the feasible solution obtained from Phase I as the starting basic feasible solution, solve the original LP, i.e. the columns of artificial variables can now be deleted.

We solve the above discussed two examples by Two-Phase method (Examples 1.7.2 and 1.7.3) that we already solved by Big M-method in the preceding subsection.

Example 1.7.5. Solve by Two-Phase method

$$\text{Max. } z = 6x_1 + 4x_2$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 &\leq 30 \\ 3x_1 + 2x_2 &\leq 24 \\ x_1 + x_2 &\geq 3 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

Solution. **Phase I:**

$$\text{Min } r = R_1$$

subject to

$$\begin{array}{ccccccccccccc} 2x_1 & + & 3x_2 & + & s_1 & & & & & & & = & 30 \\ 3x_1 & + & 2x_2 & & & + & s_2 & & & & & = & 24 \\ x_1 & + & x_2 & & & & & - & S_3 & + & R_1 & = & 3 \end{array}$$

$$x_1,\ x_2,\ x_3,\ s_1,\ s_2,\ S_3,\ R_1 \geq 0,$$

where s_1, s_2 are slack variables, S_3 is a surplus variable, and R_1 is an artificial variable.

The associated table is

Basic	x_1	x_2	s_1	s_2	S_3	R_1	**Solution**
r	0	0	0	0	0	-1	0
s_1	2	3	1	0	0	0	30
s_2	3	2	0	1	0	0	24
R_1	1	1	0	0	-1	1	3

The starting basic solution is $(s_1, s_2, R_1) = (30, 24, 3)$ and the initial value of r is $r = R_1 = 3$. However, the above table shows $r = 0$. So, to make the r-row consistent with the rest of the table, we substitute out R_1 in r-row by the following operation:

New r-row = Old r-row + $(1 \times R_1$-row).

Consequently, we have the following table:

Basic	x_1	x_2	s_1	s_2	S_3	R_1	**Solution**	**Ratio**
r	1	1	0	0	-1	0	3	
s_1	2	3	1	0	0	0	30	$\frac{30}{2} = 15$
s_2	3	2	0	1	0	0	24	$\frac{24}{3} = 8$
R_1	1	1	0	0	-1	1	3	$\frac{3}{1} = 3$ (min.)

Since the problem is to "minimize" r, we choose the variable with the most "positive" coefficient in the r-row as the entering variable. Here, there is a tie between x_1 and x_2. We choose x_1 arbitrarily. As seen in the above table, R_1 is the leaving variable and we have the following table as the next iteration.

Basic	x_1	x_2	s_1	s_2	S_3	R_1	**Solution**
r	0	0	0	0	0	-1	0
s_1	0	1	1	0	2	-2	24
s_2	0	-1	0	1	3	-3	15
x_1	1	1	0	0	-1	1	3

We have reached the optimum as the above table indicates and Phase I ends here. Since minimum $r = 0$, by Phase I, the basic feasible solution is $(s_1, s_2, x_1) = (24, 15, 3)$. Now, we eliminate the column of the artificial variable R_1 and go to Phase II.

Phase II: The original problem can now be written as

$$\text{Max } z = 6x_1 + 4x_2$$

subject to

$$\begin{array}{rcrcrcrcrcl} & & x_2 & + & s_1 & & & + & 2S_3 & = & 24 \\ & - & x_2 & & & + & s_2 & + & 3S_3 & = & 15 \\ x_1 & + & x_2 & & & & & - & S_3 & = & 3 \end{array}$$

$$x_1,\ x_2,\ x_3,\ s_1,\ s_2,\ S_3 \geq 0.$$

The associated table (obtained by deleting R_1-column and replacing r-row by z-row) is as follows:

Basic	x_1	x_2	s_1	s_2	S_3	**Solution**
z	-6	-4	0	0	0	0
s_1	0	1	1	0	2	24
s_2	0	-1	0	1	3	15
x_1	1	1	0	0	-1	3

Again, initial value of z is $z = 6 \times x_1 + 4 \times x_2 = 6 \times 3 + 4 \times 0 = 18$. To make the z-row consistent with the rest of the table, we substitute out x_1 in the above table by the following operation:

New z-row = Old z-row + ($6 \times x_1$-row)

Then the new table obtained is as follows:

Basic	x_1	x_2	s_1	s_2	S_3	**Solution**	**Ratio**
z	0	2	0	0	-6	18	
s_1	0	1	1	0	2	24	$\frac{24}{2} = 8$
s_2	0	-1	0	1	3	15	$\frac{15}{3} = 5$ (min.)
x_1	1	1	0	0	-1	3	$\frac{3}{-1} = -3$

Here S_3 is the entering variable and s_2 is the leaving variable. Computing Gauss-Jordan operations, we have the following tableau:

Basic	x_1	x_2	s_1	s_2	S_3	**Solution**
z	0	0	0	2	0	48
s_1	0	$\frac{5}{3}$	1	$-\frac{2}{3}$	0	14
S_3	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	1	5
x_1	1	$\frac{2}{3}$	0	$\frac{1}{3}$	0	8

Hence, the maximum value of the objective function is $z = 48$ and the optimum point is $(x_1, x_2) = (8, 0)$. The complete basic solution is given by $(s_1, S_3, x_1) = (14, 5, 8)$. ☐

Example 1.7.6. Solve Example 1.7.3 by Two-Phase method, i.e.
Solve by two-phase method: Minimize $z = 4x_1 + x_2$
subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$

$$\begin{aligned} x_1 + 2x_2 &\leq 4 \\ x_1,\, x_2 &\geq 0. \end{aligned}$$

Solution. **Phase I:**

$$\text{Min } r = R_1 + R_2$$

subject to

$$\begin{array}{rcrcrcrcrcrcr} 3x_1 & + & x_2 & & & & & + & R_1 & & & = & 3 \\ 4x_1 & + & 3x_2 & - & x_3 & & & & & + & R_2 & = & 6 \\ x_1 & + & 2x_2 & & & + & x_4 & & & & & = & 4 \end{array}$$

$$x_1,\, x_2,\, x_3,\, x_4,\, R_1,\, R_2 \geq 0,$$

where x_3 is a surplus variable, x_4 is a slack variable and R_1, R_2 are artificial variables. The associated table is

Basic	x_1	x_2	x_3	R_1	R_2	x_4	**Solution**
r	0	0	0	-1	-1	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

We substitute out R_1 and R_2 in the r-row just as in Big M-method, by the following operation:

New r-row = Old r-row + $(1 \times R_1$-row $+1 \times R_2$-row$)$.

The resultant table and further iterations are as given below:

Basic	x_1	x_2	x_3	R_1	R_2	x_4	**Solution**	**Ratio**
r	7	4	-1	0	0	0	9	
R_1	3	1	0	1	0	0	3	$\frac{3}{3} = 1$ (min.)
R_2	4	3	-1	0	1	0	6	$\frac{6}{4} = \frac{3}{2}$
x_4	1	2	0	0	0	1	4	$\frac{4}{1} = 4$

Basic	x_1	x_2	x_3	R_1	R_2	x_4	**Solution**	**Ratio**
r	0	$\frac{5}{3}$	-1	$-\frac{7}{3}$	0	0	2	
x_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	1	$\frac{1}{1/3} = 3$
R_2	0	$\frac{5}{3}$	-1	$-\frac{4}{3}$	1	0	2	$\frac{2}{5/3} = \frac{6}{5}$ (min.)
x_4	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	0	1	3	$\frac{3}{5/3} = \frac{9}{5}$

Basic	x_1	x_2	x_3	R_1	R_2	x_4	**Solution**
r	0	0	0	-1	-1	0	0
x_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	-1	1	1

Since minimum $r = 0$, Phase I provides the basic feasible solution $(x_1, x_2, x_4) = \left(\frac{3}{5}, \frac{6}{5}, 1\right)$. The role of artificial variables is over, their columns can be deleted now and we move to Phase II.

<u>Phase II</u>: The original problem can be written as

$$\text{Min } z = 4x_1 + x_2$$

subject to

$$\begin{array}{rcrcrcrcl} x_1 & & & + & \frac{1}{5}x_3 & & & = & \frac{3}{5} \\ & + & x_2 & - & \frac{3}{5}x_3 & & & = & \frac{6}{5} \\ & & & & x_3 & + & x_4 & = & 1 \end{array}$$

$$x_1,\ x_2,\ x_3,\ x_4 \geq 0.$$

The table associated with Phase II is gives as

Basic	x_1	x_2	x_3	x_4	**Solution**
z	-4	-1	0	0	0
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	1

Again, to make the z-row consistent, we substitute out basic variables x_1 and x_2 in the z-row by the following row operation:

$$\text{New } z\text{-row} = \text{Old } z\text{-row} + (4 \times x_1\text{-row} + 1 \times x_2\text{-row}).$$

Then the initial table of Phase II becomes

Basic	x_1	x_2	x_3	x_4	**Solution**	**Ratio**
z	0	0	$\frac{1}{5}$	0	$\frac{18}{5}$	
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$	$\frac{3/5}{1/5} = 3$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$	$\frac{6/5}{-3/5} = -2$
x_4	0	0	1	1	1	1 (min.)

Here, x_3 is the entering variable and x_4 is the leaving variable. By Gauss-Jordan row operations, we obtain the optimum in the next iteration.

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	0	$-\frac{1}{5}$	$\frac{17}{5}$
x_1	1	0	0	$-\frac{1}{5}$	$\frac{2}{5}$
x_2	0	1	0	$\frac{3}{5}$	$\frac{9}{5}$
x_3	0	0	1	1	1

The maximum value of the objective function is $z = \frac{17}{5}$, the optimum occurs at the point $(x_1, x_2) = \left(\frac{2}{5}, \frac{9}{5}\right)$ and the complete basic solution is $(x_1, x_2, x_3) = \left(\frac{2}{5}, \frac{9}{5}, 1\right)$.

□

Remark 1.7.7. The columns of artificial variables at the end of Phase I can only be deleted if all of them are non-basic variables. If an artificial variable is basic at zero level (i.e. having value 0) at the end of Phase I, then it can be removed by the following two additional steps:

1. Select a zero artificial variable to leave the basic solution by considering its row as pivot row. The entering variable can be chosen as any non-basic variable and non-artificial variable with a non-zero coefficient in the pivot row. Go to the next iteration by simplex computations (see Exercise 1.23).
2. Delete the column of that (left) artificial variable from the table. If there are no more zero artificial variables then move to Phase II, else repeat above step.

In continuation of this topic, in the next chapter, we shall see some special cases in the simplex method and their interpretations.

Exercises

Exercise 1.1

The owner of metro sports company wishes to determine how many advertisements to place in three selected magazines A, B and C. His objective is to advertise in such a way that total exposure to potential buyers of sports goods is maximized. Percentage of readers for each magazine are known. Exposure in any particular magazine is the number of advertisements placed multiplied by the number of potential buyers. The following data is given:

	A	B	C
Readers	100000	60000	40000
Potential buyers	20%	15%	8%
Cost per advertisement (in ₹)	8000	6000	5000

The budget is ₹ 100000. He has also decided that no more than 15 advertisements should be placed in magazine A, and in B and C he wants to place at least 8 advertisements.

Formulate this optimization problem in mathematical form.

Exercise 1.2

Check whether the following solutions are feasible or not and hence determine the best feasible solution among the following solutions for the Reddy Mikks model in Example 1.2.1.

(a) $x_1 = 1,\ x_2 = 2$

(b) $x_1 = 3,\ x_2 = 1$

(c) $x_1 = 3,\ x_2 = 1.5$

(d) $x_1 = 2,\ x_2 = 1$

(e) $x_1 = 2,\ x_2 = -1$

Exercise 1.3

A company buying scrap metal has two types of scrap available to them. The first type of scrap metal has 20% metal A, 10% of impurity and 20% of metal B by weight. The second type of scrap has 30% metal A, 10% impurity and 15% of metal B by weight. The company requires at least 120 kg of metal A, at most 40 kg of impurity and at least 90 kg of metal B. The prices of two scraps are ₹ 200 and ₹ 300 per kg respectively.

Determine the optimum quantities of two scraps to be purchased at minimum cost.

Exercise 1.4

A farm uses at least 800 kg of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions:

	kg per kg of feedstuff		
Feedstuff	**Protein**	**Fibre**	**Cost (₹/kg)**
Corn	0.09	0.02	0.30
Soybean meal	0.60	0.06	0.90

The dietary requirements of the special feed are at least 30% protein and at most 5% fibre. The goal is to determine the daily minimum-cost feed mix.

Exercise 1.5

A farmer has a supply of chemical fertilizer of Type-I which contains 10% Nitrogen, and 6% of Phosphoric acid and Type-II fertilizer which contains 5% Nitrogen and 10% Phosphoric acid. After testing the soil conditions of a field it is found that at least 14 kg of Nitrogen and 14 kg of Phosphoric acid is required for a good crop. The fertilizer of Type-I costs ₹ 2 per kg and that of Type-II costs ₹ 3. How many kilograms of each fertilizer should be used to meet the requirement and the cost be minimum? (Use graphical method).

Exercise 1.6

The ABC company has a producer of picture tubes for television sets and certain printed circuits for radios. The company has just expanded into full scale production and making of AM and AM-FM radios. It has built a new plant that can operate 48 hours per week. Production of an AM radio in the new plant will require 2 hours and production of an AM-FM radio will require

3 hours. Each AM radio will contribute ₹ 40 to profit while an AM-FM radio will contributes ₹ 80 to the profit. The marketing department, after extensive research, has determined that a maximum of 15 AM radios and 10 AM-FM radios can be sold each week.

(a) Formulate a linear programming model to determine the optimum production mix of AM-FM radios that will maximize the profit.

(b) Solve the problem using graphical method.

Exercise 1.7

A firm makes two products X and Y and has a total production capacity of 9 tons per day, X and Y requiring the same production capacity. The firm has a permanent contract to supply at least 2 tons of X and at least 3 tons of Y per day to another company. Each ton of X requires 20 machine hours of production time and each ton of Y requires 50 machine hours of production time. The daily maximum possible number of machine hours is 360. All the firm's output can be sold, and the profit made is ₹ 80 per ton of X and ₹ 120 per ton of Y. It is required to determine the production schedule for maximum profit and to calculate this profit.

Exercise 1.8

An aeroplane can carry maximum of 200 passengers. A profit of Rs. 400 is made on each first class ticket and a profit of Rs. 300 is made on each economy class ticket. The airline reserves at least 20 tickets for first class seats. However, at least 4 times as many passengers prefer to travel by economy class as to the first class. Determine how many tickets of each type must be sold in order to maximize the total profit for the airline.

Formulate a linear programming model to determine the optimal mix that will maximum the profit. Solve it by using graphical method.

Exercise 1.9

Obtain the optimal solution in Examples 1.3.3 and 1.3.5 by ISO-profit or ISO-cost method, whichever applicable.

Exercise 1.10

A company manufactures two products P and Q. The unit profit on P and Q are ₹ 2 and ₹ 3 respectively. Two raw materials A and B are used in products P and Q. The daily availability of of A and B are 8 and 18 units respectively. One unit of P uses 2 units of A and 3 units of B. One unit of Q uses 2 units of A and 6 units of B.

(a) Determine the dual prices of A and B and their feasibility ranges.

(b) Suppose 2 additional units of A can be obtained at the cost of 25 paise per unit, would it be advisable to go for additional purchase?

(c) What is the maximum price for per unit of B that the company should pay?

(d) Determine the optimum profit if the availability of material B is increased by 3 units.

Exercise 1.11

Consider Exercise 1.10. Answer the following questions:

(a) Determine the optimality condition for the ratio $\frac{c_P}{c_Q}$ of coefficients of the objective function that will keep the optimum unchanged.

(b) If the unit profits c_P and c_Q are changed simultaneouly to ₹ 5 and ₹ 4 respectively, then determine the new optimum.

Exercise 1.12

Answer the following questions for the Reddy Mikks model (Example 1.2.1).

(a) If the profit per ton of exterior paint remains constant at ₹ 6000 per ton, then determine the maximum unit profit on interior paint that will keep the present optimum solution unchanged.

(b) If the unit profit on interior paint is reduced to ₹ 2500, will it affect the current optimum?

Exercise 1.13

Determine the optimum solution for each of the following LPs by enumerating all the basic solutions. Also state which are feasible and non-degenerate solutions.

(a)

$$\begin{aligned} \text{Maximize } z &= 2x_1 - 4x_2 + 5x_3 - 6x_4 \\ \text{subject to} & \\ x_1 + 4x_2 - 2x_3 + 8x_4 &\leq 2 \\ -x_1 + 2x_2 + 3x_3 + 4x_4 &\leq 1 \\ x_1,\ x_2,\ x_3,\ x_4 &\geq 0. \end{aligned}$$

(b)

$$\begin{aligned} \text{Maximize } z &= x_1 + 2x_2 - 3x_3 - 2x_4 \\ \text{subject to} & \\ x_1 + 2x_2 - 3x_3 + x_4 &= 4 \\ x_1 + 2x_2 + x_3 + 2x_4 &= 4 \\ x_1,\ x_2,\ x_3,\ x_4 &\geq 0. \end{aligned}$$

Exercise 1.14

Consider the LP as follows:

$$\begin{aligned} \text{Maximize } z &= x_1 + 3x_2 \\ \text{subject to} & \\ x_1 + x_2 &\leq 2 \\ -x_1 + x_2 &\leq 4 \\ x_1 & \quad \text{unrestricted} \\ x_2 &\geq 0. \end{aligned}$$

(a) Determine all the basic feasible solutions of the problem.

(b) Use direct substitution in the objective function to determine the best basic solution.

(c) Solve the problem graphically and verify that the solution obtained is the optimum.

Exercise 1.15

Solve the problem for each of the following objective functions by simplex method.

(a) Maximize $z = 2x_1 + x_2 - 3x_3 + 5x_4$
(b) Maximize $z = 8x_1 + 6x_2 + 3x_3 - 2x_4$
(c) Maximize $z = 3x_1 - x_2 + 3x_3 + 4x_4$
(d) Minimize $z = 5x_1 - 4x_2 + 6x_3 - 8x_4$

subject to the constraints

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & 2x_3 & + & 4x_4 & \leq & 40 \\ 2x_1 & - & x_2 & + & x_3 & + & 2x_4 & \leq & 8 \\ 4x_1 & - & 2x_2 & + & x_3 & - & x_4 & \leq & 10 \end{array}$$

$$x_1,\, x_2,\, x_3,\, x_4 \geq 0.$$

Exercise 1.16

Solve the LP for each of the following objective functions by Big M-method

(a) Maximize $z = 2x_1 + 3x_2 - 5x_3$
(b) Minimize $z = 2x_1 + 3x_2 - 5x_3$
(c) Maximize $z = x_1 + 2x_2 + x_3$
(d) Minimize $z = 4x_1 - 8x_2 + 3x_3$

subject to the constraints

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 7 \\ 2x_1 & - & 5x_2 & + & x_3 & \geq & 10 \end{array}$$

$$x_1,\, x_2,\, x_3 \geq 0.$$

Exercise 1.17

Using x_3 and x_4 as starting basic variables (i.e. do not use artificial variables), solve the following LP.

$$\text{Maximize } \; z = 2x_1 + 4x_2 + 4x_3 - 3x_4$$

subject to

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & + & x_3 & & & = & 4 \\ x_1 & + & 4x_2 & & & + & x_4 & = & 8 \end{array}$$

$$x_1,\, x_2,\, x_3,\, x_4 \geq 0.$$

Exercise 1.18

Using x_3 and x_4 as starting basic variables (i.e. do not use artificial variables), solve the following problem.

$$\text{Minimize } \; z = 3x_1 + 2x_2 + 3x_3$$

subject to

$$\begin{array}{rcrcrcrcl} x_1 & + & 4x_2 & + & x_3 & & & \geq & 14 \\ 2x_1 & + & x_2 & & & + & x_4 & \geq & 20 \end{array}$$

$$x_1,\, x_2,\, x_3,\, x_4 \geq 0.$$

Exercise 1.19

Solve (by Big M-method): Minimize $z = x_1 + 5x_2 + 3x_3$

subject to

$$\begin{array}{rcrcrl} x_1 & + & 2x_2 & + & x_3 & = 6 \\ 2x_1 & - & x_2 & & & = 8 \end{array}$$

$$x_1,\ x_2,\ x_3 \geq 0$$

by using x_3 as a slack variable and one artificial variable R in the second constraint.

Exercise 1.20

Show, by Big M-method, that the LP given below has no feasible solution. Also exhibit this by graphical method.

$$\text{Maximization } \ z = 2x_1 + 5x_2$$

subject to

$$\begin{array}{rcrcl} 3x_1 & + & 2x_2 & \geq & 6 \\ 2x_1 & + & x_2 & \leq & 2 \end{array}$$

$$x_1,\ x_2 \geq 0.$$

Exercise 1.21

Solve all the problems in Exercise 1.16 by the two-phase method.

Exercise 1.22

Show that (by two-phase method) that the following problem has no feasible solution.

$$\text{Maximize } \ z = 2x_1 + 5x_2$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 &\geq 12 \\ 2x_1 + x_2 &\leq 4 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

Exercise 1.23

Solve the following problem by Two-Phase method:

$$\text{Maximize } \ z = 2x_1 + 2x_2 + 4x_3$$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 &\leq 2 \\ 3x_1 + 4x_2 + 2x_3 &\geq 8 \\ x_1,\ x_2,\ x_3 &\geq 0. \end{aligned}$$

CHAPTER 2

Duality

2.1 Simplex Method: Special Cases

In this section, we shall study four special cases that arise in using the simplex method and also see their theoretical and practical properties. They are

1. Degeneracy
2. Alternative optima
3. Unbounded solutions
4. Infeasible solutions

Our present syllabus include **only first two cases**, namely, **degeneracy** and **alternative optima**. However, for the completion of this section and the notes on this topic, we will discuss all four of them. The students, though encouraged study all the four special cases, may skip the last two cases (i.e. unbounded solutions and infeasible solutions) from exam point of view.

2.1.1 Degeneracy

Recall that, a basic solution is called degenerate if any of the basic variables is zero.

During simplex method, while determining the leaving variable if two or more basic variables have the same least non-negative ratio, then by feasibility condition we can arbitrarily choose any one of them as the leaving variable. This will lead to atleast one basic variable with zero value in the succeeding iteration. The new basic solution is thus said to be *degenerate*.

Consider the following example.

Example 2.1.1 (Degenerate Optimal Solution)**.**

$$\text{Maximize } \; z = 3x_1 + 9x_2$$

subject to

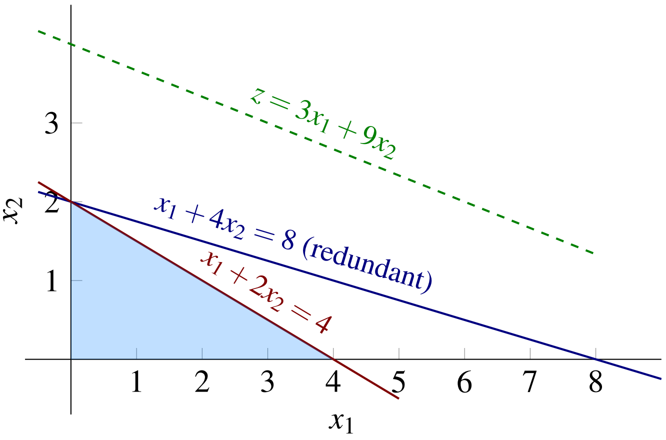
$$\begin{aligned} x_1 + 4x_2 &\leq 8 \\ x_1 + 2x_2 &\leq 4 \\ x_1,\, x_2 &\geq 0. \end{aligned}$$

Solution. Writing the given LP in equation form by adding slack variables x_3 and x_4 in first two constraints respectively, we get the simplex table, its iterations and solution as given below:

Iteration	**Basic**	x_1	x_2	x_3	x_4	**Solution**	**Ratio**
0$^{\text{th}}$	z	-3	-9	0	0	0	
x_2 enters	x_3	1	4	1	0	8	$\frac{8}{4}=2$
x_3 leaves	x_4	1	2	0	1	4	$\frac{4}{2}=2$
1$^{\text{st}}$	z	$-\frac{3}{4}$	0	$\frac{9}{4}$	0	18	
x_1 enters	x_2	$\frac{1}{4}$	1	$\frac{1}{4}$	0	2	8
x_4 leaves	x_4	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	0
2$^{\text{nd}}$	z	0	0	$\frac{3}{2}$	$\frac{3}{2}$	18	
(optimum)	x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	2	
	x_1	1	0	-1	2	0	

As seen in the above table, in the initial iteration, i.e. 0th iteration, there is a tie between x_3 and x_4 for the leaving variable. Here, we choose x_3 as the leaving variable arbitrarily. This tie results in degeneracy in the next basic solution (x_2, x_4) appearing in the next iteration (i.e. 1st iteration) as it can be seen that the basic variable $x_4 = 0$.

The graph of the given problem is sketched below.



Remark 2.1.2.

1. From practical point of view, degeneracy means one or more constraints are *redundant* (unnecessary or additional), i.e. the optimum solution can be determined even if such constraints are removed. For example, in the graph above, the optimum point is $(0, 2)$ and it can be determined by only one constraint $x_1 + 2x_2 \leq 4$ along with non-negativity constraint. Thus, here the constraint $x_1 + 4x_2 \leq 8$ is redundant.
2. From theoretical point of view, degeneracy can lead to cycling of simplex iteration, thus never terminating the simplex algorithm. In the above example, the iterations 1 and 2 have the same value of objective function $z = 18$ which is optimum. However, due to degeneracy, it may be possible that the simplex method enters a repetitive sequence of iterations without improving the objective value and thus never reaching optimum.

☐

2.1.2 Alternative Optima

In a given LP, if the objective function is parallel to a (non-redundant) binding constraint, then it may have infinite number of *alternative optima*. Consider the following example in which the objective function has infinitely many solution points.

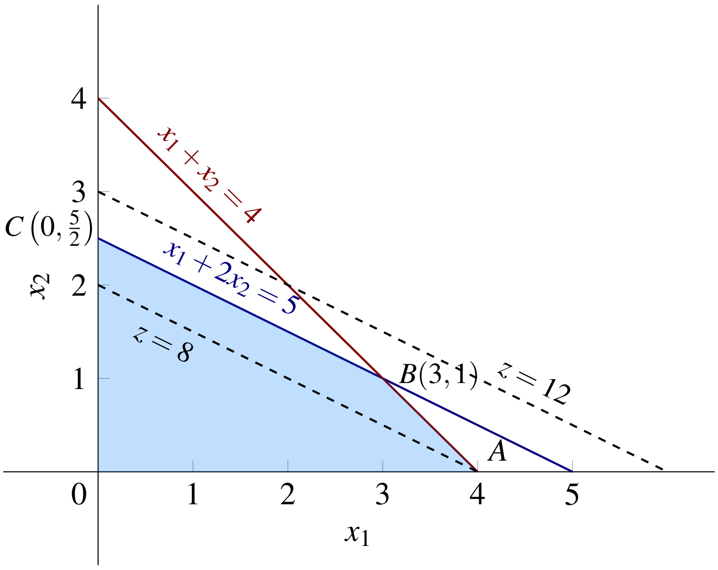
Example 2.1.3. Solve the following optimization problem using graphical method.

$$\text{Maximize } z = 2x_1 + 4x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 5 \\ x_1 + x_2 &\leq 4 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

Solution. The graph of the given LP is given below and the optimum is $z = 10$.



Here the feasible region is the shaded region as shown in the graph. It is bounded by the polygon with vertices 0, A, B and C. We have

Point	Coordinates	Max $z = 2x_1 + 4x_2$
0	$(0,0)$	0
A	$(4,0)$	8
B	$(3,1)$	10
C	$(0,2.5)$	10

Note: Here the solution is not unique. In fact, for any point on the line-segment $\overline{BC}$, the value of the function z is 10. For example at the midpoint $\left(\frac{3}{2},\frac{7}{4}\right)$ of line-segment $\overline{BC}$, the value of z is 10. Hence there are infinitely many solutions.

The simplex iterations of the given problem are as follows:

Iteration	**Basic**	x_1	x_2	x_3	x_4	**Solution**	**Ratio**
0$^{\text{th}}$	z	-2	-4	0	0	0	
x_2 enters	x_3	1	2	1	0	5	$\frac{5}{2}$
x_3 leaves	x_4	1	1	0	1	4	$\frac{4}{1}=4$
1$^{\text{st}}$ (optimum)	z	0	0	2	0	10	
x_1 enters	x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{5}{2}$	5
x_4 leaves	x_4	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	$\frac{3}{2}$	3
2$^{\text{nd}}$	z	0	0	2	0	10	
(alternative optimum)	x_2	0	1	1	-1	1	
	x_1	1	0	-1	2	3	

Iteration 1 gives the solution $(x_1,x_2)=\left(0,\frac{5}{2}\right)$ and $z=10$ (point C in the graph). Whether the given problem has an alternative optima or not can be determined by observing the coefficients of non-basic variables in the z-row. If a non-basic variable has 0 coefficient in the z-row then it can be made basic. This gives a different basic solution without changing the value of objective function z.

In iteration 2, choosing x_1 as entering variable and x_4 as leaving variable, we get a new optimum point $(x_1,x_2)=(3,1)$ (point B in the graph) without changing the value of $z=10$.

Practically, the alternative optima situations are useful as one can choose from many solutions without changing the optimum objective value. For example, at point C in the graph only activity 2 has a positive level while at point B both activities are at a positive level. Hence in practical situations, if the activities represents selling 2 different products, then it is beneficial to sell two different products in market instead of selling just one. □

2.1.3 Unbounded Solution

If a constraint permits the indefinite increase or decrease of the decision variables then the objective function will also accordingly increase or decrease. Now if the the objective function is increases indefinitely then the solution space is *unbounded* at least in one variable and we cannot determine the optimum.

An unbounded solution means the given problem is poorly formulated. In such cases, some constraints may not have been considered or the existing constraints may not be accurate.

In a problem with an unbounded solution space, at a simplex iteration there is no leaving variable by feasibility condition as there may not be any non-negative ratio. Consider the following example.

Example 2.1.4 (Unbounded Objective Value)**.**

$$\text{Maximize } z=2x_1+x_2$$

subject to

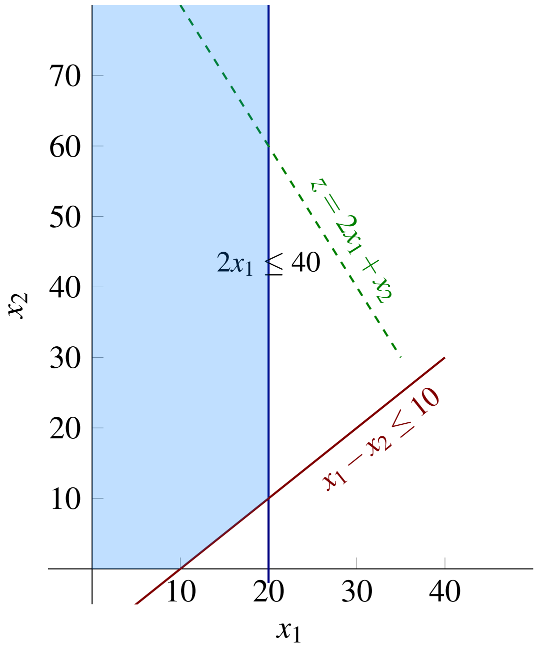
$$x_1-x_2\leq 10$$

$$2x_1 \leq 40$$
$$x_1,\, x_2 \geq 0.$$

Solution. The simplex iterations are shown below, where x_3 and x_4 are slack variables.

Iteration	**Basic**	x_1	x_2	x_3	x_4	**Solution**	**Ratio**
0th	z	-2	-1	0	0	0	
x_1 enters	x_3	1	-1	1	0	10	$\frac{10}{1} = 10$
x_3 leaves	x_4	2	0	0	1	40	$\frac{40}{2} = 20$
1st	z	0	-3	2	0	20	
x_2 enters	x_1	1	-1	1	0	10	$\frac{10}{-1}$
x_4 leaves	x_4	0	2	-2	1	20	$\frac{20}{2} = 10$
2nd	z	0	0	-1	$\frac{3}{2}$	50	
x_3 enters	x_2	1	0	0	$\frac{1}{2}$	20	$\frac{20}{0}$
(no leaving variable)	x_1	0	1	-1	$\frac{1}{2}$	10	$\frac{10}{-1}$

Note that in the initial iteration, both x_1 and x_2 have negative coefficients in the z-row. This means that an increase in their values will increase the value of z. Here x_1 is entering variable but note that all the coefficients of x_2 are non-positive which means that x_2 can increase indefinitely without violating any constraints. Hence, z can be increased indefinitely resulting into unbounded solution. This is shown in the graph given below.



□

2.1.4 Infeasible Solution

If in a given LP the constraints are inconsistent, then it has no feasible solution. This situation does not arise if all the constraints are of "$\leqslant$" type as a trivial feasible solution is provided by all the slack variables. For other types of constraints, we use artificial variables. If at least one of the artificial variable remains basic variable with a positive value in the optimum iteration, then the given LP has no feasible solution. In other words, the solution space is infeasible. Consider the following example.

Example 2.1.5 (Infeasible Solution Space)**.**

$$\text{Maximize } z = 3x_1 + 2x_2$$

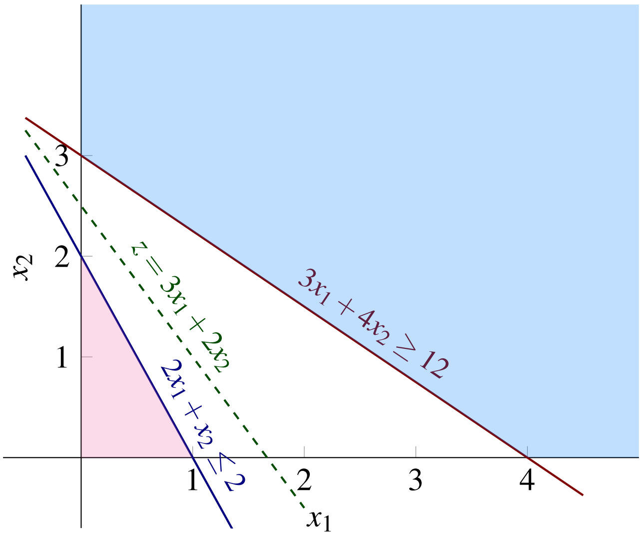
subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 2 \\ 3x_1 + 4x_2 &\geq 12 \\ x_1,\, x_2 &\geq 0. \end{aligned}$$

Solution. Adding a slack variable x_4 to the first constraint, subtracting a surplus variable x_3 and adding an artificial variable R to the second constraint with a penalty $M = 100$ in the objective function, we get the following simplex tableau:

Iteration	**Basic**	x_1	x_2	x_4	x_3	R	**Solution**	**Ratio**
$\mathbf{0^{th}}$	z	-303	-402	100	0	0	-1200	
x_2 enters	x_3	2	1	0	1	0	2	$\frac{2}{1} = 2$
x_3 leaves	R	3	4	-1	0	1	12	$\frac{12}{4} = 3$
$\mathbf{1^{st}}$	z	501	0	100	402	0	-396	
(pseudo-optimum)	x_2	2	1	0	1	0	2	
	R	-5	0	-1	-4	1	4	

The optimum is reached in the 1st iteration, where the artificial variable R is a basic variable and has positive value ($R = 4$). This shows that the given LP has no feasible solution. This is also verified in the following graph. If can be seen that there is no region in which all the constraints are consistent, i.e. there is no feasible region at all.



□

2.2 Dual Problem

The original LP model is called *primal* or *primal problem* and the *dual* problem is systematically defined from the primal. The optimum solution of any one of the problem gives the optimum solution of the other. In this aspect, dual is important and is closely related to the primal.

The dual can be defined for different types of primal problems, depending on the objective type (maximization or minimization), constraints type ($\leq$, $\geq$, or $=$), the sign of the variables (unrestricted or non-negative).

To frame the dual problem, the primal (given) problem first needs to be expressed into equation form as in case of simplex method. Before we describe the rules and method to frame a dual, we see an example of how to construct the dual to a given primal to better understanding of the method.

Example 2.2.1. Frame the dual of the following problem:

Primal	**Primal in equation form**	**Dual variables**
Maximize $z = 5x_1 + 12x_2 + 4x_3$	Maximize $z = 5x_1 + 12x_2 + 4x_3 + 0x_4$	
subject to	subject to	
$x_1 + 2x_2 + x_3 \leq 10$	$x_1 + 2x_2 + x_3 + x_4 = 10$	y_1
$2x_1 - x_2 + 3x_3 = 8$	$2x_1 - x_2 + 3x_3 = 8$	y_2
$x_1,\ x_2,\ x_3 \geq 0.$	$x_1,\ x_2, x_3,\ \ x_4 \geq 0$	

Solution. The (given) primal problem is first expressed into equation form as shown above. The dual of the above primal is written as follows:

Dual Problem:

$$\text{Minimize } \ w = 10y_1 + 8y_2$$

subject to

$$\begin{array}{rcrcr} y_1 & + & 2y_2 & \geq & 5 \\ 2y_1 & - & y_2 & \geq & 12 \\ y_1 & + & 3y_2 & \geq & 4 \end{array}$$

$$\left.\begin{array}{rcrcr} y_1 & + & 0y_2 & \geq & 0 \\ \multicolumn{5}{c}{y_1,\ y_2 \text{ unrestricted}} \end{array}\right\} \Rightarrow (y_1 \geq 0,\ y_2 \text{ unrestricted}).$$

$\square$

2.2.1 Method for constructing the dual

Now we describe the method of constructing the dual from the given primal. The steps are as follows:

1. Express the given primal in equation form.
2. Construct a dual constraint for each primal variable.
3. The (column) constraint coefficients and the objective coefficient of the jth primal variable define the left hand sides and right hand sides of the the jth dual constraint respectively.
4. The right hand sides of the primal constraint equations becomes the dual objective coefficients.
5. The sense of optimization, direction of inequalities, and the signs of the variables in the dual are governed by the rules given in the following table.

Primal objective	**Dual Problem**		
	Objective	**Constraints type**	**Variables sign**
Maximization	Minimization	$\geq$	Unrestricted
Minimization	Maximization	$\leq$	Unrestricted

2.2.2 Examples of dual obtained from primal

Example 2.2.2. Write the dual of the following problem:

Primal	**Primal in equation form**	**Dual variables**
Minimize $z = 15x_1 + 12x_2$ subject to	Minimize $z = 15x_1 + 12x_2 + 0x_3 + 0x_4$ subject to	
$x_1 + 2x_2 \geq 3$	$x_1 + 2x_2 - x_3 + 0x_4 = 3$	y_1
$2x_1 - 4x_2 \leq 5$	$2x_1 - 4x_2 + 0x_3 + x_4 = 5$	y_2
$x_1,\ x_2 \geq 0.$	$x_1,\ x_2,\ x_3,\ x_4 \geq 0.$	

Solution. **Dual Problem**:

$$\text{Maximize} \;\; w = 3y_1 + 5y_2$$

subject to

$$\left.\begin{array}{rcrcl} y_1 & + & 2y_2 & \leq & 15 \\ 2y_1 & - & 4y_2 & \leq & 12 \end{array}\right.$$

$$\left.\begin{array}{rcl} -y_1 & \leq & 0 \\ y_2 & \leq & 0 \\ \multicolumn{3}{l}{y_1,\ y_2 \text{ unrestricted}} \end{array}\right\} \Rightarrow (y_1 \geq 0,\ y_2 \leq 0).$$

$\square$

Example 2.2.3. Frame the dual of the following problem:

Primal	**Primal in equation form**	**Dual variables**
	Substitute $x_1 = x_1^- - x_1^+$	
Maximize $z = 5x_1 + 6x_2$	Maximize $z = 5x_1^- - 5x_1^+ + 6x_2$	
subject to	subject to	
$x_1 + 2x_2 = 5$	$x_1^- - x_1^+ + 2x_2 = 5$	y_1
$-x_1 + 5x_2 \geq 3$	$-x_1^- + x_1^+ + 5x_2 - x_3 = 3$	y_2
$4x_1 + 7x_2 \leq 8$	$4x_1^- - 4x_1^+ + 7x_2 + x_4 = 8$	y_3
x_1 unrestricted, $x_2 \geq 0$	$x_1^-,\ x_1^+, x_2,\ x_3, x_4 \geq 0$	

Solution. **Dual Problem**:

$$\text{Minimize}\ \ w = 5y_1 + 3y_2 + 8y_3$$

subject to

$$\left.\begin{array}{r} y_1 - y_2 + 4y_3 \geq \ \ 5 \\ -y_1 + y_2 - 4y_3 \geq -5 \end{array}\right\} \Rightarrow \left.\begin{array}{r} y_1 - y_2 + 4y_3 \geq 5 \\ y_1 - y_2 + 4y_3 \leq 5 \end{array}\right\} \Rightarrow y_1 - y_2 + 4y_3 = 5$$

$$2y_1 + 5y_2 + 7y_3 \geq 6$$

$$\left.\begin{array}{r} -y_2 \quad \geq 0 \\ y_3 \geq 0 \\ y_1,\ y_2,\ y_3 \text{ unrestricted} \end{array}\right\} \Rightarrow (y_1 \text{ unrestricted},\ y_2 \leq 0,\ y_3 \geq 0)$$

$\square$

Example 2.2.4 (NBHM 2009, 4.8)**.** Use duality to find the optimal value of the cost function in the following linear programming problem:

$$\text{Max}\ \ x + y + z$$

subject to

$$\begin{aligned} 3x + 2y + 2z &= 1 \\ x,\ y,\ z &\geq 0. \end{aligned}$$

Solution. Here, the given primal is already in equation form. Its dual is the following problem:

$$\text{Minimize } w = y_1$$

subject to

$$\left.\begin{aligned} 3y_1 &\geq 1 \\ 2y_1 &\geq 1 \\ 2y_1 &\geq 1 \\ y_1 &\text{ unrestricted} \end{aligned}\right\} \Rightarrow 2y_1 \geq 1 \text{ or } y_1 \geq \frac{1}{2}.$$

Thus, $w = \frac{1}{2}$ is the optimum value of the given cost function. □

Observing the above examples of dual, the rules of constructing the dual can be summarized as follows:

Maximization problem		**Minimization problem**
Constraints		Variables
$\geq$	$\Leftrightarrow$	≤ 0
$\leq$	$\Leftrightarrow$	≥ 0
$=$	$\Leftrightarrow$	Unrestricted
Variables		Constraints
≥ 0	$\Leftrightarrow$	$\geq$
≤ 0	$\Leftrightarrow$	$\leq$
Unrestricted	$\Leftrightarrow$	$=$

Note that the headings of the column in the above table do not specify primal or dual problem. Only sense of optimization matters. If the primal is maximization then dual will be minimization, and vice-versa. Also observe that there is no provision for including artificial variables in the primal as they do not change the definition of the dual.

Remark 2.2.5. Dual of the dual problem yields the original primal problem. Verify this for the above considered examples.

2.3 Primal-Dual Relationships

In this section, we shall see some aspects of primal-dual relationships and how they can be used to recompute the quantities in the optimal simplex table. These primal-dual relationships form the basis for the economic interpretation of the LP model that we will be considering at the end of this unit.

2.3.1 Simplex Tableau Layout

The *starting* simplex tableau and the *general* simplex tableau are expressed in the format as represented in the following figures.

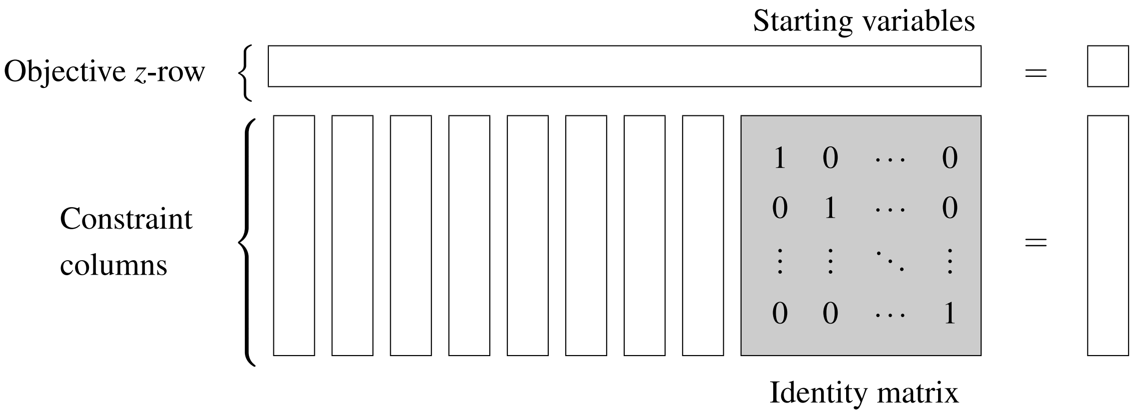


Figure 2.1: (Starting simplex tableau)

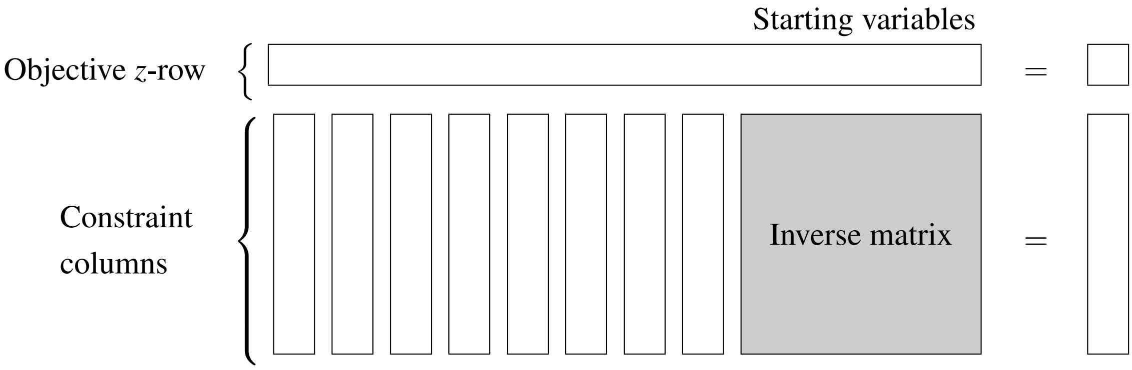


Figure 2.2: (General simplex iteration)

In the starting simplex table, the constraint coefficients under the starting variables form an identity matrix. With this tabular arrangement, the subsequent iterations of the simplex table produced by the Gauss-Jordan row operations modify the elements of the identity matrix to produce the *inverse matrix*. This inverse matrix serves as the key to compute all the elements of the simple tableau.

2.3.2 Optimal Dual Solution

The primal and the dual solutions are closely related in the sense that the optimal solution of either of the problems provides the solutions of the other. In a given LP model, if the number of variables is comparatively much smaller than the number of constraints, then it is better to solve the dual of the problem for computation convenience. This is because the amount of simplex computations depend largely (thought not completely) on the number of constraints.

In this subsection, we shall see a couple of methods of obtaining the optimum solution (point) for the dual problem from the given optimum primal solution. The two methods are described below.

Method-I.

$$\begin{pmatrix}\text{Optimal value of the}\\ \textit{dual}\text{ variable } y_i\end{pmatrix} = \begin{pmatrix}\text{Optimal primal } z\text{-coefficient}\\ \text{of the starting basic variable } x_i\\ +\\ \text{Original objective coefficient of } x_i\end{pmatrix}$$

Consider the following example.

Example 2.3.1. Find the optimum values of the dual variables from the optimal primal solution for the following problem.

$$\text{Maximize } z = 5x_1 + 12x_2 + 4x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\le 10\\ 2x_1 - x_2 + 3x_3 &= 8\\ x_1,\ x_2,\ x_3 &\ge 0. \end{aligned}$$

Solution. We express the given primal in equation form by adding a slack variable x_4 to the first constraint and an artificial variable R to the second constraint to solve the problem by simplex method. We penalize the artificial variable R in the objective function by $-M$. The resulting primal and the dual associated to it are as follows:

Primal	**Dual**
Maximize $z = 5x_1 + 12x_2 + 4x_3 - MR$ subject to $x_1 + 2x_2 + x_3 + x_4 = 10$ $2x_1 - x_2 + 3x_3 + R = 8$ $x_1,\ x_2,\ x_3,\ x_4,\ R \ge 0$	Minimize $w = 10y_1 + 8y_2$ subject to $y_1 + 2y_2 \ge 5$ $2y_1 - y_2 \ge 12$ $y_1 + 3y_2 \ge 4$ $y_1 \ge 0$ $y_2 \ge -M \Rightarrow y_2$ unrestricted

We now obtain the optimal primal solution using simplex (Big M) method. The starting simplex table is given below:

Basic	x_1	x_2	x_3	x_4	R	**Solution**
z	-5	-12	-4	0	M	0
x_4	1	2	1	1	0	10
R	2	-1	3	0	1	8

We substitute out R in the z-row by the following row operation:

$$\text{New } z\text{-row} = \text{Old } z\text{-row} - (M \times R\text{-row}).$$

Then the resulting simplex table takes the following form:

Basic	x_1	x_2	x_3	x_4	R	**Solution**	**Ratio**
z	$-2M-5$	$M-12$	$-3M-4$	0	0	$-8M$	
x_4	1	2	1	1	0	10	10
R	2	-1	3	0	1	8	$\frac{8}{3}$

Basic	x_1	x_2	x_3	x_4	R	**Solution**	**Ratio**
z	$-\frac{7}{3}$	$-\frac{40}{3}$	0	0	$M+\frac{4}{3}$	$\frac{32}{3}$	
x_4	$\frac{1}{3}$	$\frac{7}{3}$	0	1	$-\frac{1}{3}$	$\frac{22}{7}$	$\frac{22}{7}$
x_3	$\frac{2}{3}$	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	$\frac{8}{3}$	-8

Basic	x_1	x_2	x_3	x_4	R	**Solution**	**Ratio**
z	$-\frac{3}{7}$	0	0	$\frac{40}{7}$	$M-\frac{4}{7}$	$\frac{368}{7}$	
x_2	$\frac{1}{7}$	1	0	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{22}{7}$	22
x_3	$\frac{5}{7}$	0	1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{26}{7}$	$\frac{26}{5}$

Basic	x_1	x_2	x_3	x_4	R	**Solution**
z	0	0	$\frac{3}{5}$	$\frac{29}{5}$	$M-\frac{2}{5}$	$\frac{274}{5}$
x_2	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{12}{5}$
x_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{26}{5}$

Dual values by Method-I: The starting primal variables are x_4 and R which uniquely corresponds to dual variables y_1 and y_2 respectively. The optimal z-row coefficients of x_4 and R are $\frac{29}{5}$ and $M-\frac{2}{5}$ respectively and their original objective coefficients are 0 and $-M$ respectively. Thus, optimum dual solution is as presented in the table below:

Starting primal basic variables	x_4	R
Optimal z-row coefficients	$\frac{29}{5}$	$M-\frac{2}{5}$
Original objective coefficient	0	$-M$
Corresponding dual variables	y_1	y_2
Optimal dual values	$\frac{29}{5}+0=\frac{29}{5}$	$\left(M-\frac{2}{5}\right)+(-M)=-\frac{2}{5}$

The the optimal values of dual variables are $y_1 = \frac{29}{5}$ and $y_2 = -\frac{2}{5}$. Notice that, here y_2 is negative and in the dual too y_2 is an unrestricted variable. ☐

Method-II.

$$\begin{pmatrix} \text{Optimal values of} \\ \textit{dual} \text{ variables} \end{pmatrix} = \begin{pmatrix} \text{Row vector of the} \\ \text{original objective coefficients} \\ \text{of the optimal primal variables} \end{pmatrix} \times \begin{pmatrix} \text{Optimal primal} \\ \text{inverse} \end{pmatrix}$$

Note: The elements of the row vector must appear in the same order as the basic variables listed in the Basic column of the optimal simplex iteration.

Example 2.3.2. Determine the optimum dual values by Method-II for the above problem, i.e.

$$\text{Maximize } z = 5x_1 + 12x_2 + 4x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\le 10 \\ 2x_1 - x_2 + 3x_3 &= 8 \\ x_1,\ x_2,\ x_3 &\ge 0. \end{aligned}$$

Solution. Recall (from above example), the optimal simplex table obtained is as follows:

Basic	x_1	x_2	x_3	x_4	R	**Solution**
z	0	0	$\frac{3}{5}$	$\frac{29}{5}$	$M-\frac{2}{5}$	$\frac{274}{5}$
x_2	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{12}{5}$
x_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{26}{5}$

Dual values by Method-II: The optimal primal inverse matrix is the shaded matrix in the above table under the variables x_4 and R, i.e.

$$\text{Optimal inverse } = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

The order of the optimal primal basic variables in the Basic column is first x_2 and then x_1. Recall that, the original objective function of primal is Maximize $z = 5x_1 + 12x_2 + 4x_3$. Thus, the original objective coefficients of x_2 and x_1 are 12 and 5 respectively, i.e. the row vector of original objective coefficients (in the same order) is given by $(12, 5)$. Then the optimal dual values are obtained as

$$\begin{aligned} (y_1, y_2) &= \begin{pmatrix} \text{Original objective} \\ \text{coefficients of } x_2,\ x_1 \end{pmatrix} \times \begin{pmatrix} \text{Optimal primal} \\ \text{inverse} \end{pmatrix} \\ &= (12, 5) \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \end{aligned}$$

$$= \left(\frac{29}{5}, -\frac{2}{5}\right).$$

☐

Primal-dual objective values.

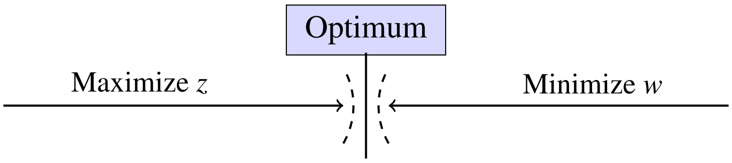


Figure 2.3: Relationship between maximum z and minimum w

For any pair of feasible solution of primal and its dual, we have the following relation:

$$\left(\begin{array}{c}\text{Objective value in the}\\ \textit{maximization}\text{ problem}\end{array}\right) \leq \left(\begin{array}{c}\text{Objective value in the}\\ \textit{minimization}\text{ problem}\end{array}\right)$$

The equality occurs, i.e. the two objective values are equal, only at the optimum. In other words, the optimum cannot occur with z strictly less than w (i.e., $z < w$) because now matter how close z and w are, there is always scope of improvement in their values as shown in the above figure.

Note that the above relationship does not mention which problem is primal and which is dual. Only the sense of optimization (i.e. maximization or minimization) of the problem is important in this case. For example

Example 2.3.3. In Example 2.3.1, $(x_1, x_2, x_3) = \left(0, 0, \frac{8}{3}\right)$ is an arbitrary feasible primal solution, while $(y_1, y_2) = (6, 0)$ is an arbitrary feasible dual solution. The associated objective values are

$$\begin{aligned} z &= 5x_1 + 12x_2 + 4x_3 = 5 \times 0 + 12 \times 0 + 4 \times \frac{8}{3} = \frac{32}{3} \\ w &= 10y_1 + 8y_2 = 10 \times 6 + 8 \times 0 = 60. \end{aligned}$$

Thus, $z = \frac{32}{3}$ for (primal) maximization problem is less that $w = 60$ for (dual) minimization problem. Note that the optimum value $z = \frac{274}{5}$ falls in the range $(\max, \min) = \left(\frac{32}{3}, 60\right)$.

2.3.3 Simplex Tableau Computations

Any given iteration of the simplex table can be generated from the given data of the problem, the inverse matrix associated with that iteration and the dual problem. Consider the format of the simplex table as shown in Figures 2.1 and 2.2. Then the computations can be divided into the following two categories:

1. Constraint columns (left hand side and right hand side)
2. Objective function row, i.e. z-row

Formula 1: Constraint column computations.

In any simplex iteration, the left hand side constraint column or the right hand side constraint column is computed by the following formula:

$$\begin{pmatrix}\text{Constraint column}\\ \text{in iteration } i\end{pmatrix} = \begin{pmatrix}\text{Inverse in}\\ \text{iteration } i\end{pmatrix} \times \begin{pmatrix}\text{Original}\\ \text{constraint column}\end{pmatrix}$$

Example 2.3.4. Compute the x_1-column, x_2-column, R-column and right hand side (solution)-column in the optimal iteration of Example 2.3.1 by using optimal inverse $= \begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix}$.

Solution. The original problem and optimal inverse are given. We compute

$$\begin{aligned}\begin{pmatrix}x_1\text{-column in}\\ \text{optimal iteration}\end{pmatrix} &= \begin{pmatrix}\text{Inverse in}\\ \text{optimal iteration}\end{pmatrix} \times \begin{pmatrix}\text{Original}\\ x_1\text{-column}\end{pmatrix}\\ &= \begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix} \times \begin{pmatrix}1\\ 2\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}.\end{aligned}$$

$$\begin{aligned}\begin{pmatrix}x_2\text{-column in}\\ \text{optimal iteration}\end{pmatrix} &= \begin{pmatrix}\text{Inverse in}\\ \text{optimal iteration}\end{pmatrix} \times \begin{pmatrix}\text{Original}\\ x_2\text{-column}\end{pmatrix}\\ &= \begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix} \times \begin{pmatrix}2\\ -1\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}.\end{aligned}$$

$$\begin{aligned}\begin{pmatrix}R\text{-column in}\\ \text{optimal iteration}\end{pmatrix} &= \begin{pmatrix}\text{Inverse in}\\ \text{optimal iteration}\end{pmatrix} \times \begin{pmatrix}\text{Original}\\ R\text{-column}\end{pmatrix}\\ &= \begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix} \times \begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}-\frac{1}{5}\\ \frac{2}{5}\end{pmatrix}.\end{aligned}$$

$$\begin{aligned}\begin{pmatrix}\text{RHS-column in}\\ \text{optimal iteration}\end{pmatrix} &= \begin{pmatrix}\text{Inverse in}\\ \text{optimal iteration}\end{pmatrix} \times \begin{pmatrix}\text{Original}\\ \text{RHS-column}\end{pmatrix}\\ &= \begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix} \times \begin{pmatrix}10\\ 8\end{pmatrix} = \begin{pmatrix}\frac{12}{5}\\ \frac{26}{5}\end{pmatrix}.\end{aligned}$$

☐

Similar computations can be used to determine the optimal columns for x_3, x_4 (Verify!).

Formula 2: Objective z-row computations.

In any simplex iteration, the objective equation coefficient of x_j is compute as follows:

$$\begin{pmatrix}\text{Primal } z\text{-coefficient}\\ \text{of variable } x_j\end{pmatrix} = \begin{pmatrix}\text{Left-hand side of}\\ j\text{th dual constraint}\end{pmatrix} - \begin{pmatrix}\text{Right-hand side of}\\ j\text{th dual constraint}\end{pmatrix}$$

In the example below, we show how the z-row coefficients are computed. The optimal values of the dual variables $(y_1, y_2) = \left(\frac{29}{6}, -\frac{2}{5}\right)$ were obtained in Example 2.3.2.

Example 2.3.5. Compute the z-coefficients x_1 and R in the optimal iteration of Example 2.3.1 by using optimal inverse $= \begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix}$.

Solution. By the above formula (i.e. Formula 2), we have

$$\begin{aligned}
z-\text{coefficient of } x_1 &= y_1 + 2y_2 - 5 = \tfrac{29}{5} + 2 \times -\tfrac{2}{5} - 5 = 0\\
z-\text{coefficient of } R &= y_2 - (-M) = -\tfrac{2}{5} - (-M) = M - \tfrac{2}{5}
\end{aligned}$$

☐

Similar computations can be used to determine the z-coefficients of x_2, x_3 and x_4 (Verify!).

In the example below, we show how the above two formulas help us to determine whether, at any given iteration, the basic solution is feasible or not and whether the objective value at that iteration is optimum or not.

Example 2.3.6. Consider the following LP model:

$$\text{Maximize } 4x_1 + 14x_2$$

subject to

$$\begin{aligned}
2x_1 + 7x_2 + x_3 &= 21\\
7x_1 + 2x_2 + x_4 &= 21
\end{aligned}$$

$$x_1,\ x_2,\ x_3,\ x_4 \geq 0.$$

Check the optimality and feasibility of the following basic solution:

$$\text{Basic variables} = (x_2, x_4),\ \text{Inverse} = \begin{pmatrix}\frac{1}{7} & 0\\ -\frac{2}{7} & 1\end{pmatrix}.$$

Solution. First we check the feasibility of the given basic solution (i.e. basic variables x_2, x_4). For this we compute the right-hand side of the iteration under consideration. By formula 1,

$$\begin{pmatrix}\text{RHS-column in}\\ \text{the iteration}\end{pmatrix} = \begin{pmatrix}\text{Inverse in}\\ \text{the iteration}\end{pmatrix} \times \begin{pmatrix}\text{Original}\\ \text{RHS-column}\end{pmatrix}$$

$$\begin{pmatrix}x_2\\ x_4\end{pmatrix} = \begin{pmatrix}\frac{1}{7} & 0\\ -\frac{2}{7} & 1\end{pmatrix} \times \begin{pmatrix}21\\ 21\end{pmatrix} = \begin{pmatrix}3\\ 15\end{pmatrix}.$$

Thus, the basic solution $(x_2, x_4) = (3, 15)$ is **feasible**.

Next we check the optimality of the basic solution. For this we compute the z-row and check that there are no negative coefficients (since the problem is of maximization). We construct the dual of the problem below:

$$\text{Minimize } w = 21y_1 + 21y_2$$

subject to

$$\begin{array}{rcrcr} 2y_1 & + & 7y_2 & \geq & 4 \\ 7y_1 & + & 2y_2 & \geq & 14 \\ y_1 & & & \geq & 0 \\ & & y_2 & \geq & 0 \end{array}$$

Now, we compute the corresponding values of dual variables y_1, y_2 (by Method-II seen above):

$$\left(\begin{array}{c}\text{Values of}\\ \textit{dual}\text{ variables}\end{array}\right) = \left(\begin{array}{c}\text{Row vector of the}\\ \text{original objective coefficients}\\ \text{of the primal variables}\end{array}\right) \times \left(\begin{array}{c}\text{Primal}\\ \text{inverse}\end{array}\right)$$

$$\begin{aligned} \therefore \quad (y_1, y_2) &= (x_2, x_4) \times \left(\begin{array}{c}\text{Primal}\\ \text{inverse}\end{array}\right) \\ &= (14, 0) \times \begin{pmatrix} \frac{1}{7} & 0 \\ -\frac{2}{7} & 1 \end{pmatrix} \\ &= (2, 0). \end{aligned}$$

Since x_2, x_4 are basic variables, their coefficient in the z-row are already 0. It suffices to check the coefficients of x_1 and x_3 in the z-row. By formula 2,

$$\left(\begin{array}{c}\text{Primal } z\text{-coefficient}\\ \text{of variable } x_1\end{array}\right) = \left(\begin{array}{c}\text{Left-hand side of}\\ \text{1st dual constraint}\end{array}\right) - \left(\begin{array}{c}\text{Right-hand side of}\\ \text{1st dual constraint}\end{array}\right)$$

$$\begin{aligned} \therefore \quad z\text{-coefficient of } x_1 &= 2y_1 + 7y_2 - 4 \\ &= 2 \times 2 + 7 \times 0 - 4 = 0. \end{aligned}$$

Similarly,

$$\text{Objective-coefficient of } x_3 = y_1 - 0 = 2.$$

Thus, objective coefficients of x_1 and x_3 are 0 and 2 respectively. Since both are non-negative, the basic solution $(x_2, x_4) = (3, 15)$ is **optimal**. □

2.4 Economic Interpretation of Duality

An LP model can be considered as a resource allocation model that maximizes revenue or profit under limited available resources. Then the dual problem yields interesting economic interpretations of the primal LP. Consider the primal and the dual expressed in the following general form:

Primal	**Dual**
Maximize $z = \sum_{j=1}^{n} c_j x_j$	Minimize $w = \sum_{i=1}^{m} b_i y_i$
subject to	subject to
$\sum_{j=1}^{n} a_{ij} x_i \leq b_i,\ i = 1,2,\ldots,m$	$\sum_{i=1}^{m} a_{ij} y_i \geq c_j,\ j = 1,2,\ldots,n$
$x_j \geq 0,\ j = 1,2,\ldots,n$	$y_i \geq 0,\ i = 1,2,\ldots,m$

The primal problem has n economic activities and m resources. The coefficients c_j denote the revenue per unit of activity j, the coefficients a_{ij} indicate the rate at which resource i is used per unit of activity j, and the constants b_i indicate the availability of the resource i.

2.4.1 Economic Interpretation of Dual Variables

We know that, as seen at the end of Subsection 2.3.2, any two finite primal and dual feasible solution (irrespective of which is maximization and which is minimization) satisfy the following relation:

$$z = \sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i = w,$$

where the equality holds in the above inequality when both the values z and w are optimum, i.e. $z = w$ when they are optimum.

As seen above, in terms of resource allocation model, z represents ₹ revenue and b_i represents the available units of resource i. Thus, dimensionally $z = w$ (i.e. optimum objective) implies

$$\text{₹ revenue } = \sum_{i=1}^{m} b_i y_i = \sum_{i=1}^{m} (\text{units of resource } i) \times (\text{₹ per unit of resource } i).$$

This means that the dual variable y_i represents the **worth per unit** (dual price) of the resource i.

By the same dimensional analysis, for any two finite basic feasible primal and dual solutions, the strict inequality $z < w$ indicates

$$(\text{Revenue}) < (\text{Worth of resources}).$$

This represents that as long as the total revenue is less than the worth of all the resources, the corresponding primal and dual solutions are not optimal. Optimality is attained only when all the resources are availed totally. This can happen only when the input (i.e. the worth of all the resources) equals the output (i.e. total revenue ₹). In economic terms, the system in this case is said to be *unstable* or *nonoptimal*.

Example 2.4.1. The Reddy Mikks model (Example 1.2.1) and its dual are given as follows:

Reddy Mikks primal	**Reddy Mikks dual**
Maximize $z = 5x_1 + 4x_2$ subject to	Minimize $w = 24y_1 + 6y_2 + y_3 + 2y_4$ subject to
$6x_1 + 4x_2 \leq 24$ (resource 1, M_1) $x_1 + 2x_2 \leq 6$ (resource 2, M_2) $-x_1 + x_2 \leq 1$ (resource 3, market) $x_2 \leq 2$ (resource 4, demand) $x_1, x_2 \geq 0$	$6y_1 + y_2 - y_3 \geq 5$ $4y_1 + 2y_2 + y_3 + y_4 \geq 4$ $y_1, y_2, y_3, y_4 \geq 0$
Optimum solution: $x_1 = 3,\ x_2 = 1.\ z = 21$	Optimum solution: $y_1 = 0.75,\ y_2 = 0.5,\ y_3 = y_4 = 0,\ w = 21$

Solution. The optimum dual solution shows that the dual price (i.e. worth per unit) of raw material M_1 is $y_1 = 0.75$ (i.e. ₹ 750 per ton) and that of raw material M_2 is $y_2 = 0.5$ (or ₹ 500 per ton). ☐

2.4.2 Economic Interpretation of Dual Constraints

Now we discuss what a dual constraint represents economically. By formula 2 (of the previous subsection), we have

$$\begin{aligned}\begin{pmatrix}\text{Objective-coefficient}\\ \text{of variable } x_j\end{pmatrix} &= \begin{pmatrix}\text{Left-hand side of}\\ j\text{th dual constraint}\end{pmatrix} - \begin{pmatrix}\text{Right-hand side of}\\ j\text{th dual constraint}\end{pmatrix}\\ &= \sum_{i=1}^{m} a_{ij}y_i - c_j.\end{aligned}$$

We know that c_j represents the revenue (in ₹) per unit of activity j. Since the quantity $\sum_{i=1}^{m} a_{ij}y_i$ has the opposite sign at that of c_j, it must represent cost (in ₹). Using above dimensional analysis, we have

$$₹\text{ cost } = \sum_{i=1}^{m} a_{ij}y_i = \sum_{i=1}^{m} \begin{pmatrix}\text{Usage of resource } i\\ \text{per unit of activity } j\end{pmatrix} \times \begin{pmatrix}\text{Cost per unit}\\ \text{of resource } i\end{pmatrix}$$

Thus, in this context, the dual variable y_i represents what is called **imputed cost** per unit of resource i. The quantity $\sum_{i=1}^{m} a_{ij}y_i$ can be considered as the imputed cost of all the resources needed to produce one unit of activity j. The quantity $\sum_{i=1}^{m} a_{ij}y_i - c_j$ is called **reduced cost**.

The maximization optimality condition (i.e. condition for entering variable) in simplex method says that an increase in the unused activity j (non-basic variable) can improve the revenue only if the reduced cost is negative. That is

$$\begin{pmatrix}\text{Imputed cost of}\\ \text{resources used by}\\ \text{one unit of activity } j\end{pmatrix} < \begin{pmatrix}\text{Revenue per unit}\\ \text{of activity } j\end{pmatrix}.$$

Exercises

Exercise 2.1

Construct the dual for the following primal problems:

(a) Maximize $z = 5x_1 + 3x_2$
subject to

$$\begin{aligned} 3x_1 + 5x_2 &\leq 15 \\ 3x_1 + 2x_2 &\leq 10 \\ x_1,\, x_2 &\geq 0. \end{aligned}$$

(b) Minimize $z = x_1 - 3x_2 - 2x_3$
subject to

$$\begin{aligned} 3x_1 - x_2 + 2x_3 &\leq 7 \\ 2x_1 - 4x_2 &\geq 12 \\ -4x_1 + 3x_2 + 3x_3 &= 10 \\ x_1,\, x_2.\, x_3 &\geq 0. \end{aligned}$$

(c) Maximize $z = 5x + 7y$
subject to

$$\begin{aligned} x - y &\leq 1 \\ 2x + y &\geq 2 \\ x + 2y &\leq 4 \\ x,\, y &\geq 0. \end{aligned}$$

(d) Maximize $z = 2x + 3y$
subject to

$$\begin{aligned} x + 2y &= 3 \\ 2x + y &\geq 4 \\ x + y &\leq 5 \\ x,\, y &\geq 0. \end{aligned}$$

(e) Maximize $z = 2x_1 + 3x_2 + x_3$
subject to

$$\begin{aligned} 4x_1 + 3x_2 + x_3 &= 6 \\ x_1 + 2x_2 + 5x_3 &\geq 4 \\ x_1,\, x_2,\, x_3 &\geq 0. \end{aligned}$$

Exercise 2.2

Write the dual for the following primal problems:

(a) Maximize $z = 66x_1 - 22x_2$
subject to

$$\begin{aligned} -x_1 + x_2 &\leq 2 \\ 2x_1 + 3x_2 &\leq 5 \\ x_1,\, x_2 &\geq 0 \end{aligned}$$

(b) Minimize $z = 6x_1 + 3x_2$
subject to

$$\begin{aligned} 6x_1 - 3x_2 + x_3 &\geq 25 \\ 3x_1 + 4x_2 + x_4 &\geq 55 \\ x_1,\, x_2,\, x_3 &\geq 0 \end{aligned}$$

(c) Maximize $z = x_1 + x_2$
subject to

$$\begin{aligned} 2x_1 + x_2 &= 5 \\ 3x_1 - x_2 &= 6 \\ x_1,\, x_2 &\text{ unrestricted} \end{aligned}$$

Exercise 2.3

Consider the following LP:

$$\text{Maximize} \quad z = 5x_1 + 2x_2 + 3x_3$$

subject to

$$\begin{array}{ccccccc} x_1 & + & 5x_2 & + & 2x_3 & = & 30 \\ x_1 & - & 5x_2 & - & 6x_3 & \leq & 40 \end{array}$$

$$x_1,\, x_2,\, x_3 \geq 0.$$

Given that the artificial variable x_4 and the slack variable x_5 form the starting basic variables and that M was set equal to 100 when solving the problem, the *optimal* tableau is given as:

Basic	x_1	x_2	x_2	x_4	x_5	**Solution**
z	0	23	7	105	0	150
x_1	1	5	2	1	0	30
x_5	0	-10	-8	-1	1	10

Write the associated dual problem, and determine its optimal solution in two ways.

Exercise 2.4

Consider the following LP:

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$\begin{array}{rcrcr} 3x_1 & + & x_2 & = & 30 \\ 4x_1 & + & 3x_2 & \geq & 60 \\ x_1 & + & 2x_2 & \leq & 40 \end{array}$$

$$x_1,\, x_2 \geq 0.$$

The starting solution consists of artificial x_4 and x_5 form the first and second constraints and slack x_6 for the third constraint. Using $M = 100$ for the artificial variables, the optimal tableau is given as

Basic	x_1	x_2	x_2	x_4	x_5	x_6	**Solution**
z	0	0	0	-98.6	-100	-0.2	34
x_1	1	0	0	0.4	0	-0.2	4
x_2	0	1	0	-0.2	0	0.6	18
x_3	0	0	1	1	-1	1	10

Write the associated dual problem, and determine its optimal solution in two ways.

Exercise 2.5

Consider the following LP (Exercise 1.17):

$$\text{Maximize } z = 2x_1 + 4x_2 + 4x_3 - 3x_4$$

subject to

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & x_3 & & & = & 4 \\ x_1 & + & 4x_2 & & & + & x_4 & = & 8 \end{array}$$

$$x_1,\, x_2,\, x_3,\, x_4 \geq 0.$$

Using x_3 and x_4 as starting basic variables, the optimal table is given as

Basic	x_1	x_2	x_2	x_4	**Solution**
z	2	0	0	3	16
x_3	0.75	0	1	-0.25	2
x_2	0.25	1	0	0.25	2

Write the associated dual problem, and determine its optimal solution in two ways.

Exercise 2.6

Consider the following LP:

$$\text{Maximize } \; z = x_1 + 5x_2 + 3x_3$$

subject to

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 3 \\ 2x_1 & - & x_2 & & & = & 4 \end{array}$$

$$x_1,\, x_2,\, x_3 \geq 0.$$

The starting solution consists of x_3 in the first constraint and an artificial variable x_4 in the second constraint with $M = 100$. The optimal tableau is given as

Basic	x_1	x_2	x_2	x_4	**Solution**
z	0	2	0	99	5
x_3	1	2.5	1	-0.5	1
x_1	0	-0.5	0	0.5	2

Write the associated dual problem, and determine its optimal solution in two ways.

Exercise 2.7

Consider the following LP model:

$$\text{Maximize } \; 4x_1 + 14x_2$$

subject to

$$\begin{aligned} 2x_1 + 7x_2 + x_3 \qquad &= 21 \\ 7x_1 + 2x_2 \qquad + x_4 &= 21 \end{aligned}$$

$$x_1,\, x_2,\, x_3,\, x_4 \geq 0.$$

Check the optimality and feasibility of each of the following basic solutions:

(a) Basic variables $= (x_2, x_3)$, Inverse $= \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{7}{2} \end{pmatrix}$

(b) Basic variables $= (x_2, x_1)$, Inverse $= \begin{pmatrix} \frac{7}{45} & -\frac{2}{45} \\ -\frac{2}{45} & \frac{7}{45} \end{pmatrix}$

(c) Basic variables $= (x_1, x_4)$, Inverse $= \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}$

Exercise 2.8

Consider the following LP model:

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 \quad &= 30 \\ 3x_1 \quad + 2x_3 \quad + x_5 \quad &= 60 \\ x_1 + 4x_2 \quad + x_6 &= 20 \end{aligned}$$

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6 \geq 0.$$

Check the optimality of and feasibility of the following basic solutions:

(a) Basic variables $= (x_4, x_3, x_6)$, Inverse $= \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) Basic variables $= (x_2, x_3, x_1)$, Inverse $= \begin{pmatrix} \frac{1}{4} & -\frac{1}{8} & \frac{1}{8} \\ \frac{3}{2} & -\frac{1}{4} & -\frac{3}{4} \\ -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

(c) Basic variables $= (x_2, x_3, x_6)$, Inverse $= \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix}$

Exercise 2.9

Consider the following LP model:

$$\text{Minimize } z = 2x_1 + x_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 - x_3 \quad &= 3 \\ 4x_1 + 3x_2 \quad - x_4 \quad &= 6 \\ x_1 + 2x_2 \quad + x_5 &= 3 \end{aligned}$$

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0.$$

Compute the entire simplex table associated with the following basic solution, and check it for optimality and feasibility.

Basic variables $= (x_1, x_2, x_5)$, Inverse $= \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 1 & -1 & 1 \end{pmatrix}$

Exercise 2.10

The following is the optimal tableau for a maximization LP problem with three $(\leq)$ constraints and all non-negative variables. The variables x_3, x_4 and x_5 are the slacks associated with the three constraints. Determine the associated optimal objective value in the two ways by using the primal and dual objective functions.

Basic	x_1	x_2	x_3	x_4	x_5	**Solution**
z	0	0	0	3	2	?
x_3	0	0	1	1	-1	2
x_2	0	1	0	1	0	6
x_1	1	0	0	-1	1	2

CHAPTER

3

Transportation and Assignment models

3.1 Dual Simplex Method

We have seen that simplex method starts with a feasible solution and continues to be feasible in all the iterations until optimum is attained. In this section, we will see **dual simplex** method which starts with an *infeasible* solution but better than optimal solution and continues to be infeasbile until feasibility is attained.

3.1.1 Dual Simplex Algorithm

In (primal) simplex method, we have seen that, the starting basic solution is feasible but need not be optimal. It remains feasible through all the iterations that continue till we obtain an optimal solution. In dual simplex method, the procedure is somewhat the opposite. We start with a solution which is better than optimal but infeasible and we carry out the iterations till feasibility is restored.

Unlike the simplex method, in the dual simplex method, we choose the leaving variables first and then the entering variable. The optimality condition (i.e. the condition for entering variable) and the feasibility condition (i.e. the condition for leaving variable) for the dual simplex method are stated as below:

Definition 3.1.1 (Dual feasibility condition)**.** The leaving variable x_r is the basic variable having the most negative value (ties are broken arbitrarily). If all the basic variables are non-negative, then feasibility is restored and terminate the algorithm.

Definition 3.1.2 (Dual optimality condition)**.** If x_r is the leaving variable, $\bar{c}_j$ is the reduced cost (z-row coefficient) of non-basic variable x_j and α_{rj} is the constraint coefficient in the x_r-row and x_j-column, then the entering variable is the non-basic variable with $\alpha_{rj} < 0$ that corresponds to

$$\min_{\text{Nonbasic } x_j} \left\{ \left| \frac{\bar{c}_j}{\alpha_{rj}} \right|, \ \alpha_{rj} < 0 \right\}.$$

Ties are broken arbitrarily. If $\alpha_{rj} \geq 0$ for all nonbasic variables x_j, then the problem has no feasible solution.

The following two conditions must be satisfied prior to starting the dual simplex algorithm of an LP with optimal and infeasible solution.

1. The objective function must satisfy the optimality condition of the regular simplex method.
2. All the constraints must be of "$\leq$" type.

If a constraint is an inequality of type "$\geq$", then it is converted into "$\leq$" type by multiplying both the sides of the inequality by -1. If the constraint is an equation i.e. "$=$" type, then the equation is replaced by two inequalities. The starting solution is infeasible if at least one of the right hand sides of the inequalities is negative.

3.1.2 Examples of dual simplex method

Consider the following LP solved by the dual simplex algorithm. Since it is the first example of this method, we include the full detailed solution below, as we have done in the above two chapters. However, one can directly show the necessary computations and tabular operations to derive the solution.

Example 3.1.3. Solve the following LP by dual simplex method:

$$\text{Minimize } \ z = 3x_1 + 2x_2 + x_3$$

subject to

$$\begin{aligned} 3x_1 + \ \ x_2 + x_3 &\geq 3 \\ -3x_1 + 3x_2 + x_3 &\geq 6 \\ x_1 + \ \ x_2 + x_3 &\leq 3 \end{aligned}$$

$$x_1,\ x_2,\ x_3 \geq 0.$$

Solution. The first two constraints are inequalities of $\geq$ type which are converted into $\leq$ by multiplying both the sides by -1. We then add slack variables x_4, x_5 and x_6 to all the three $(\leq)$ type constraints and obtain the resultant LP as

$$\begin{aligned} -3x_1 - \ \ x_2 - x_3 + x_4 \qquad\qquad &= -3 \\ 3x_1 - 3x_2 - x_3 \qquad + x_5 \qquad &= -6 \\ x_1 + \ \ x_2 + x_3 \qquad\qquad + x_6 &= \ \ \ 3 \end{aligned}$$

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6 \geq 0.$$

The starting tableau is thus given as follows:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	**Solution**
z	-3	-2	-1	0	0	0	0
x_4	-3	-1	-1	1	0	0	-3
x_5	3	-3	-1	0	1	0	-6
x_6	1	1	1	0	0	1	3

Note that, the above table is optimal as all the reduced costs (coefficients) in the z-row are non-positive ($\bar{c}_1 = -3$, $\bar{c}_2 = -2$, $\bar{c}_3 = -1$, $\bar{c}_4 = 0$, $\bar{c}_5 = 0$, $\bar{c}_6 = 0$) in the given minimization LP. The solution is also infeasible as at least one basic variable is negative (here $x_4 = -3$ and $x_5 = -6$).

By the feasibility condition x_5 (thus $r = 5$ for x_r) is the leaving variable as it x_5 the most negative value. To determine the entering variable, we compute the ratios $\left|\frac{\bar{c}_j}{\alpha_{5j}}\right|$ for each non-basic variables x_j, where $\alpha_{5j} < 0$. Here x_1, x_2, x_3 (thus $j = 1, 2, 3$) are the non-basic variables. We have the following table of computed ratios:

	$j = 1$	$j = 2$	$j = 3$
Nonbasic variable	x_1	x_2	x_3
z-row ($\bar{c}_j$)	-3	-2	-1
x_5-row, α_{5j}	3	-3	-1
Ratio $\left|\frac{\bar{c}_j}{\alpha_{5j}}\right|$, $\alpha_{5j} < 0$	–	$\frac{2}{3}$	1

The computed ratios indicate that x_2 is the entering variable in the next iteration. Computing the Gauss-Jordan row-operations gives the following table:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	**Solution**
z	-5	0	$-\frac{1}{3}$	0	$-\frac{2}{3}$	0	4
x_4	-4	0	$-\frac{2}{3}$	1	$-\frac{1}{3}$	0	-1
x_2	-1	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	2
x_6	2	0	$\frac{2}{3}$	0	$\frac{1}{3}$	1	1
Ratio	$\frac{5}{4}$	–	$\frac{1}{2}$	–	2	–	

The above table indicates that x_4 is the leaving variable (most negative value) and x_3 is the entering variable (minimum ratio for non-basic variables). Again by Gauss-Jordan row-operations, we have the following table:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	**Solution**
z	-3	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{9}{2}$
x_3	6	0	1	$-\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$
x_2	-3	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{3}{2}$
x_6	-2	0	0	1	0	1	0

The above table provides a basic solution which is both optimal and feasible and hence we stop the process here.

Observe the working of the dual simplex method. All the iterations are already optimal as all the reduced costs (coefficients of z-row) are ≤ 0. However, the solutions are not feasible. Once the feasibility is attained in the 3rd iteration, the process ends and the optimal feasible solution is given by $(x_1, x_2, x_3) = \left(0, \frac{3}{2}, \frac{3}{2}\right)$ with $z = \frac{9}{2}$. □

Let us consider one more example of dual simplex.

Example 3.1.4. Solve by dual simplex method.

$$\text{Minimize } \; z = 2x_1 + 3x_2$$

subject to

$$\begin{aligned} 2x_1 + 2x_2 &\leq 3 \\ x_1 + 2x_2 &\geq 1 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

Solution. Multiplying both the sides of the second constraint by -1 and then adding slack variables x_3 and x_4 to both the constraints, we can write the given LP as follows.

$$\text{Minimize } \; z = 2x_1 + 3x_2$$

subject to

$$\begin{array}{rcrcrcrcr} 2x_1 &+& 2x_2 &+& x_3 && &=& 3 \\ -x_1 &-& 2x_2 && &+& x_4 &=& -1 \end{array}$$

$$x_1,\ x_2,\ x_3,\ x_4 \geq 0.$$

The starting basic (infeasible) solution is given by $(x_3, x_4) = (3, -1)$ and the starting (optimal) table is given as below:

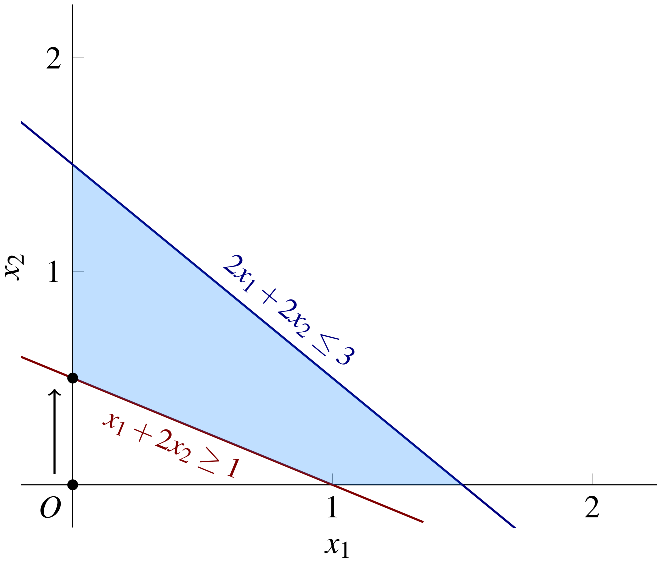
Basic	x_1	x_2	x_3	x_4	**Solution**
z	-2	-3	0	0	0
x_3	2	2	1	0	3
x_4	-1	-2	0	1	-1
Ratio	2	$\frac{3}{2}$	–	–	

In the above table, the basic variable x_4 has the most negative value and hence it is the leaving variable. The computed ratios determine that the basic variable x_2 is the entering variable. Gauss-Jordan row operations yields the following table:

Basic	x_1	x_2	x_3	x_4	**Solution**
z	$-\frac{1}{2}$	0	0	$-\frac{3}{2}$	$\frac{3}{2}$
x_3	1	0	1	1	2
x_2	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	$\frac{1}{2}$

Feasibility of the solution is restored in the above table and so we end the process. The optimal feasible solution is given by $x_1 = 0$, $x_2 = \frac{1}{2}$ with $z = \frac{3}{2}$. ☐

The solution of the above LP, being in two variables, can be verified by graphical method. We sketch the graph below to examine the course of the dual simplex algorithm in reaching a feasible optimal solution.



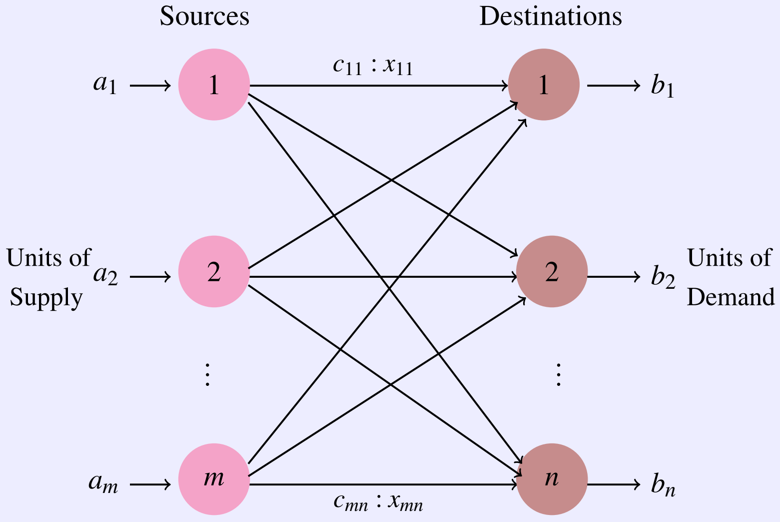
Observe that, we started with $(x_1,x_2) = (0,0)$ which is an infeasible solution but better than optimal (as minimum $z = 0$). In the next iteration itself, feasibility is restored at $(x_1,x_2) = \left(0,\frac{1}{2}\right)$ and the optimum value is $z = \frac{3}{2}$. The the path of the dual simplex method for the above example is shown in the graph above.

3.2 Transportation Model

In this section, we shall study transportation model. We will see three methods to obtain a starting solution of the transportation problem and then method to determine the optimal solution from the starting solution.

The definition of the transportation model is given below:

Definition 3.2.1 (Transportation model)**.** The transportation problem is represented by the network shown in the following figure.



There are m sources and n destinations, each represented by a **node**. The **arcs** represent the routes linking the sources and the destinations. Arc (i, j) joining source i to destination j carries two pieces of information:

- the transportation cost per unit, c_{ij}, and
- the amount shipped, x_{ij}.

The amount of supply at source i is a_i, and the amount of demand at destination j is b_j. the objective of the model is to minimize the total transportation cost while satisfying all the supply and demand constraints.

Consider an example of transportation problem given below.

Example 3.2.2 (MG Auto Model)**.** MG Auto has three plants in Rajkot, Delhi, Mumbai, and two major distribution centers in Chennai and Kolkata. The quarterly capacities of the three plants are 1000, 1500, and 1200 cars, and the demands at the two distribution centers for the same period are 2300 and 1400 cars. The mileage chart between the plants and the distribution centers is given in the following table.

Table 3.1: Mileage Chart

	Chennai	Kolkata
Rajkot	1000	2690
Delhi	1250	1350
Mumbai	1275	850

The trucking company in charge of transporting the cars charges 8 paise per km per car. The transportation costs per car on the different routes, rounded to the closest ₹, are given in the following table.

Table 3.2: Transportation Cost per Car

	Chennai (1)	Kolkata (2)
Rajkot (1)	₹ 80	₹ 215
Delhi (2)	₹ 100	₹ 108
Mumbai (3)	₹ 102	₹ 68

The LP model of the problem is

$$\text{Minimize } z = 80x_{11} + 215x_{12} + 100x_{21} + 108x_{22} + 102x_{31} + 68x_{32}$$

subject to

$$\begin{array}{cccccccl}
x_{11} & + x_{12} & & & & & = & 1000 \text{ (Rajkot)} \\
& & x_{21} & + x_{22} & & & = & 1500 \text{ (Delhi)} \\
& & & & x_{31} & + x_{32} & = & 1200 \text{ (Mumbai)} \\
x_{11} & & + x_{21} & & + x_{31} & & = & 2300 \text{ (Chennai)} \\
& x_{12} & & + x_{22} & & + x_{32} & = & 1400 \text{ (Kolkata)}
\end{array}$$

$$x_{ij} \geq 0, \; i = 1,2,3, \; j = 1,2.$$

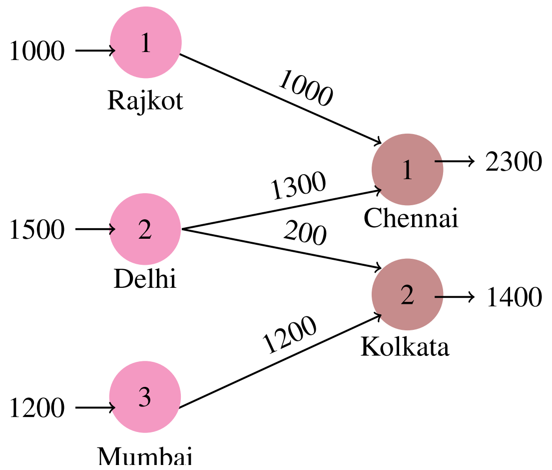
These constraints are all equations because the total supply from the three sources ($= 1000 + 1500 + 1200 = 3700$) equals the total demand at the two destinations ($= 2300 + 1400$).

The special structure of the transportation problem helps to represent the problem compactly in the tabular form called the transportation table given as follows:

Table 3.3: MG Auto Transportation Model

	Chennai	Kolkata	Supply
Rajkot	80 x_{11}	215 x_{12}	1000
Delhi	100 x_{21}	108 x_{22}	1500
Mumbai	102 x_{31}	68 x_{32}	1200
Demand	2300	1400	

Remark 3.2.3. The optimal solution of the above problem is given (already provided) to be $x_{11} = 1000$, $x_{21} = 1300$, $x_{22} = 200$, $x_{32} = 1200$. This is represented in the following diagram.



The minimum transportation cost is given by

$$z = 1000 \times ₹\ 80 + 1300 \times ₹\ 100 + 200 \times ₹\ 108 + 1200 \times ₹\ 68 = ₹\ 313200.$$

Balancing the transportation model

The transportation tableau representations already assumes that the table is balanced, i.e. the total demand is same as the total supply. If the model is unbalanced, then we can add a dummy source or a dummy destination, as required, to make it balanced.

Let us understand the balancing of the transportation model by the above example of MG Auto model.

Example 3.2.4. Suppose that the Delhi plant in the MG Auto model has the production capacity of only 1300 cars instead of 1500 cars. Then the supply (= 3500) is less than the demand (= 3700). This means that the part of demand at Chennai and Kolkata will not be met.

Table 3.4: MG model with Dummy plant

	Chennai	Kolkata	Supply
	80	215	
Rajkot	1000		1000
	100	108	
Delhi	1300		1500
	102	68	
Mumbai		1200	1200
	0	0	
Dummy plant		200	200
Demand	2300	1400	

Since the demand exceeds the supply (by 200 cars), we add a dummy source (dummy plant) with a capacity of 200 cars to restore the balance of the transportation model. The unit transportation cost from the dummy plant to the two destinations is set to zero as the plant does not exists. This is represented in the above table (Table 3.4) along with the optimal solution in this case.

Now, consider another case. Suppose that the demand at the Chennai center is only 1900 cars instead of 2300 cars. In this case the supply exceeds the demand by 400 cars. Hence to restore the balance of the transportation model, we add a dummy destination (distribution center) to receive the surplus supply. Again, the unit transportation cost to this dummy destination from the three plants is zero. This case, along with its optimal solution is shown in the following table.

Table 3.5: MG model with Dummy destination

	Chennai	Kolkata	Dummy	Supply
Rajkot	80 1000	215	0	1000
Delhi	100 900	108 200	0 400	1500
Mumbai	102	68 1200	0	1200
Demand	1900	1400	400	

3.2.1 The Transportation Algorithm

The basic steps of the transportation algorithm are similar to those of the simplex method. However, in the transportation problem, we have the advantage of the special transportation table format of the model as compared the usual simplex table. This makes the computations more convenient.

Step 1. Determine the starting basic feasible solution and go to Step 2.
Step 2. Use the optimality condition of the simplex method to determine the entering variable from among all the non-basic variables. If the optimality condition is satisfied, then stop the process. Else, go to Step 3.
Step 3. Use the feasibility condition of the simplex method to determine the leaving variable form among all the basic variables, and find the new basic solution. Go back to Step 2.

Before going into the details of the algorithm, consider the following example. We solve (obtain starting solution of) the example given below by three methods described in the preceding subsection.

Example 3.2.5 (SunRay Transport)**.** SunRay Transport Company ships truckloads of grain from three silos to four mills. The supply (in truckloads) and the demand (in truckloads) together with the unit transportation costs per truckload on the different routes are given in the following table. The unit transportation costs, c_{ij} (indicated in the top right corner of each box) are in hundreds of ₹. The objective of the model is to determine a shipping schedule between the silos and the mills that minimizes the transportation costs.

Table 3.6: Sunray Transport model

	Mill 1	Mill 2	Mill 3	Mill 4	Supply
Silo 1	10 x_{11}	2 x_{12}	20 x_{13}	11 x_{14}	15
Silo 2	12 x_{21}	7 x_{22}	9 x_{23}	20 x_{24}	25
Silo 3	4 x_{31}	14 x_{32}	16 x_{33}	18 x_{34}	10
Demand	5	15	15	15	

3.2.2 Determining the starting solution

A general form of transportation model with m sources and n destinations has $m+n$ constraint equations, one for each source and each destination. Since, the transportation model is always balanced (sum of the units of supply = sum of the units of demand), the model can be considered to be reduced to $m+n-1$ independent equations and $m+n-1$ basic variables. For instance, in the above example (SunRay model), the starting solution has $3+4-1=6$ basic variables.

The special structure of the transportation problem provides a non-artificial starting basic solution using one of the following three methods:

1. Northwest corner method
2. Least-cost method
3. Vogel approximation method

3.2.3 Northwest-corner method

The method starts at the *nothwest-corner* cell of the table.

Step 1. Allocate as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount.

Step 2. Cross out the row or column with zero supply or demand to indicate that no further assignments can be made in that row or column. If both a row and a column net to zero simultaneously, then cross out only one, and leave a zero supply (or demand) in the uncrossed row (or column).

Step 3. If exactly one row or column is left uncrossed, stop. Otherwise, move to the cell to the right if a column has just been crossed out or below if a row has just been crossed out. Go to Step 1.

The starting solution of the Sunray model (Example 3.2.5) is obtained by Northwest-corner method in the following example.

Example 3.2.6. The application of the procedure to the Sunray model (Example 3.2.5) gives the starting basic solution in table given below. The arrows show the order in which the allocated amounts are generated.

Table 3.7: North-West corner starting solution

	Mill 1	Mill 2	Mill 3	Mill 4	Supply
Silo 1	10 5	2 10	20	11	15
Silo 2	12	7 5	9 15	20 5	25
Silo 3	4	14	16	18 10	10
Demand	5	15	15	15	

The starting solution is

$$x_{11} = 5, x_{12} = 10,$$
$$x_{22} = 5, x_{23} = 15,\ x_{24} = 5,$$
$$x_{34} = 10.$$

The associated cost of the schedule is

$$z = 5 \times 10 + 10 \times 2 + 5 \times 7 + 15 \times 9 + 5 \times 20 + 10 \times 18 = ₹\ 520.$$

3.2.4 Least-cost method

The least-cost method yields a better starting solution by determining the cheapest routes. That is, it finds the cell with the smallest unit cost and assigns to it as much as possible. Ties are broken arbitrarily. The row or the column that is satisfied is crossed out and the amounts of supply and demand are adjusted accordingly. If both, a row and a column, are satisfied simultaneously then only one is crossed out just as in case of northwest-corner method. Next, select the uncrossed cell with the smallest unit cost and repeat the process until exactly one row or column is left uncrossed.

Example 3.2.7. Example 3.2.5 is solved by the least-cost method below to obtain a starting solution of the Sunray model.

1. Cell $(1,2)$ has the least unit cost in the table (i.e. ₹ 2). The most that can be shipped through $(1,2)$ is $x_{12} = 15$ truckloads. This satisfies both row 1 and column 2 simultaneously. We arbitrarily cross out column 2 and adjust the supply in row 1 to 0.
2. Cell $(3,1)$ has the smallest uncrossed unit cost (which is ₹ 4). Again $x_{31} = 5$, and cross out column 1 as it is satisfied, and adjust the demand or row 3 to $10 - 5 = 5$ truckloads.
3. Continuing in this way, we successively assign 15 truckloads to cell $(2,3)$, 0 to cell $(1,4)$, 5 truckloads to cell $(3,4)$, and 10 to cell $(2,4)$ (Check!).

The resulting starting solution is given in the following table. The arrows show the order in which the allocations are made.

Table 3.8: Least-cost starting solution

	Mill 1	Mill 2	Mill 3	Mill 4	Supply
Silo 1	10	(start) 2 15	20	11 0	15
Silo 2	12	7	9 15	(end) 20 10	25
Silo 3	4 5	14	16	18 5	10
Demand	5	15	15	15	

The starting solution consists of 6 variables and is given by

$$x_{12} = 15,\ x_{14} = 0,\ x_{23} = 15,\ x_{24} = 10,\ x_{31} = 5,\ x_{34} = 5.$$

The associated objective value is $z = 15\times2+0\times11+15\times9+10\times20+5\times4+5\times18 =$ ₹ 475.

Note that this starting solution turns out to be better than that of the northwest-corner method.

3.2.5 Vogel Approximation Method (VAM)

Vogel approximation method (VAM) is an improved version of the least-cost method that usually, not always, produces better starting solutions.

The process is given below:

Step 1. For each row (or column), determine a penalty measure by subtracting the smallest unit cost element in the row (or column) from the next smallest unit cost element in the same row (or column).

Step 2. Identify the row of column with the largest penalty (Ties are broken arbitrarily). Allocate as much as possible to the variable with the least unit cost in the selected row or column. Adjust the supply and demand, and cross out the satisfied row or column. If a row or column are satisfied simultaneously then only one of the two is crossed out, and the remaining row (or column) is assigned the zero supply (or demand).

Step 3.
- (a) If exactly one row or column with zero supply or demand remain uncrossed, then stop.
- (b) If one row (or column) with positive supply (or demand) remains uncrossed, determine the basic variables in the row (or column) by the least-cost method. Stop.
- (c) If all the uncrossed rows and columns have (remaining) zero supply and demand, determine the zero basic variables by the least-cost method. Stop.
- (d) Otherwise, go to Step 1.

Example 3.2.8. Example 3.2.5 is solved by VAM. The following table shows the computed first set of penalties as per the above described algorithm.

Table 3.9: Row and Column Penalties in VAM

	Mill 1	Mill 2	Mill 3	Mill 4	Supply	Row penalty
Silo 1	10	2	20	11	15	$10-2=8$
Silo 2	12	7	9	20	25	$9-7=2$
Silo 3	4 5	14	16	18	10	$14-4=$ 10
Demand	5	15	15	15		
Column penalty	$10-4=6$	$7-2=5$	$16-9=7$	$18-11=7$		

Here row 3 has the largest penalty ($=10$) among the row and the column penalties. Hence, row 3 is selected. In row 3, cell $(3,1)$ has the least unit cost. The maximum amount that can be allotted to that cell is 5. Thus, $x_{31}=5$ is assigned and column 1 is now satisfied and must be crossed out.

Again the row and column penalties are computed by subtracting the smallest element in the rows (and similarly in the columns) from the next smallest element. This is shown in the following table.

Table 3.10: First assignment in VAM ($x_{31}=5$)

	Mill 1	Mill 2	Mill 3	Mill 4	Supply	Row penalty
Silo 1	10	2	20	11	15	9
Silo 2	12	7	9	20	25	2
Silo 3	4 5	14	16	18	10	2
Demand	5	15	15	15		
Column penalty	–	5	7	7		

Now, row 1 has the largest penalty ($=9$) among all rows and column penalties. In row 1, cell $(1,2)$ has the smallest unit cost and hence cell $(1,2)$ is selected. The maximum amount that can be assigned to cell $(1,2)$ is 15 and hence second assignment becomes $x_{12}=15$. This satisfies both row 1 and column 2. We arbitrarily cross out column 2 and set the supply in row 1 to zero.

The second assignment along with the row and column penalties for the next iteration is represented in the following table.

Table 3.11: Second assignment in VAM ($x_{31} = 5$, $x_{12} = 15$)

	Mill 1	Mill 2	Mill 3	Mill 4	Supply	Row penalty
Silo 1	10	2 15	20	11	15	9
Silo 2	12	7	9	20	25	11
Silo 3	4 5	14	16	18	10	2
Demand	5	15	15	15		
Column penalty	–	–	7	7		

Next, row 2 has the largest penalty (i.e. 11) and least unit cost (= 9) is in cell $(2,3)$. We assign $x_{23} = 15$. This satisfies column 3 and hence it is crossed out. The number of units left for row 2 is now 10 and we have the following table.

Table 3.12: Third assignment in VAM ($x_{31} = 5$, $x_{12} = 15$, $x_{23} = 15$)

	Mill 1	Mill 2	Mill 3	Mill 4	Supply
Silo 1	10	2 15	20	11	15
Silo 2	12	7	9 15	20	25
Silo 3	4 5	14	16	18	10
Demand	5	15	15	15	

Now only column 4 is left uncrossed and hence there is no penalty to be computed. Column 4 has a positive demand of 15 units. We apply the least cost method. Cell $(1,4)$ has the least unit cost (= 11) in column 4. But row 1 has 0 units of supply left. Hence we assign $x_{14} = 0$ and cell $(1,4)$ is crossed out. Next, in column 4, cell $(3,4)$ has the least unit cost (= 18). Since row 3 has only 5 units of supply left, we assign $x_{34} = 5$. Finally we have $x_{24} = 10$ and we obtain the starting solution and the table representing the starting solution as follows.

The objective value associated with this starting solution (as obtained in the following table) is

$$z = 15 \times 2 + 0 \times 11 + 15 \times 9 + 10 \times 20 + 5 \times 4 + 5 \times 18 = ₹\ 475.$$

Table 3.13: Starting solution by VAM

	Mill 1	Mill 2	Mill 3	Mill 4	Supply
Silo 1	10	2 15	20	11 0	15
Silo 2	12	7	9 15	20 10	25
Silo 3	4 5	14	16	18 5	10
Demand	5	15	15	15	

Note that, in this case, the solution gives the same objective value as that in case of the least-cost method and not a better starting solution.

In the next subsection, we will see the method to determine the optimal solution of a transportation model once the starting solution is determined by any of the above three methods.

3.2.6 Iterative computations of the Transportation algorithm

Once the starting solution of a transportation model is determined using any of the above discussed three methods, the optimum solution can be determined using the following algorithm:

Step 1. Use the simplex optimality condition to determine the entering variable. If the optimality condition is satisfied, stop. Otherwise go to step 2.

Step 2. Determine the leaving variable using the simplex feasibility condition to obtain the new basic solution. Go back to step 1.

Though the optimality and feasibility conditions are same as that of simplex method, the computations are not similar to simplex method. The special structure of the transportation model helps us to carry out simpler computations.

In the example below, we obtain optimum solution of the Sunray Transport model (Example 3.2.5) by using the starting solution provided by the Northwest-corner method.

Example 3.2.9. Solve the model of Example 3.2.5, starting with the northwest-corner solution.

Solution. The following table gives the starting solution provided by the northwest-corner method. The cells in which allocations are made indicate the basic variables. In our example there are 6 basic variables and they are $(x_{11}, x_{12}, x_{22}, x_{23}, x_{24}, x_{34})$. The non-allocated cells correspond to the non-basic variables.

The determination of the entering variable from among the non-basic variables is done by computing the z-row coefficients of the non-basic variables using the **method of multipliers**.

Table 3.14: Starting iteration

	Mill 1	Mill 2	Mill 3	Mill 4	Supply
Silo 1	10 5	2 10	20	11	15
Silo 2	12	7 5	9 15	20 5	25
Silo 3	4	14	16	18 10	10
Demand	5	15	15	15	

In the method of multipliers, the multipliers u_i and v_j are associated with the i^{th} and j^{th} column of the transportation table. For each basic variable x_{ij}, they satisfy the following equation:

$$u_i + v_j = c_{ij}.$$

We have 6 basic variables but 7 multiples, i.e. u_1, u_2, u_3 and v_1, v_2, v_3, v_4. This leaves us with 6 equations and 7 unknowns. To solve these equations, one of the multiplier is set to zero. We arbitrarily take $u_1 = 0$, then we solve for the remaining variables as shown in the following table.

Basic variable	(u,v)-equation	Solution
x_{11}	$u_1 + v_1 = 10$	Set $u_1 = 0 \Rightarrow v_1 = 10$
x_{12}	$u_1 + v_2 = 2$	$u_1 = 0 \Rightarrow v_2 = 2$
x_{22}	$u_2 + v_2 = 7$	$v_2 = 2 \Rightarrow u_2 = 5$
x_{23}	$u_2 + v_3 = 9$	$u_2 = 5 \Rightarrow v_3 = 4$
x_{24}	$u_2 + v_4 = 20$	$u_2 = 5 \Rightarrow v_4 = 15$
x_{34}	$u_3 + v_4 = 18$	$v_4 = 15 \Rightarrow v_3 = 3$

After having computed u_i's and v_j's, we use them to evaluate the z-coefficients of the non-basic variables x_{ij} by finding

$$u_i + v_j - c_{ij}.$$

The coefficient computations of non-basic variables are shown in the following table.

Nonbasic variable	$u_i + v_j - c_{ij}$
x_{13}	$u_1 + v_3 - c_{13} = 0 + 4 - 20 = -16$
x_{14}	$u_1 + v_4 - c_{14} = 0 + 15 - 11 = 4$
x_{21}	$u_2 + v_1 - c_{21} = 5 + 10 - 12 = 3$
x_{31}	$u_3 + v_1 - c_{31} = 3 + 10 - 4 = \mathbf{9}$
x_{32}	$u_3 + v_2 - c_{32} = 3 + 2 - 14 = -9$
x_{33}	$u_3 + v_3 - c_{33} = 3 + 4 - 16 = -9$

We know that, z-coefficient of all basic variables is zero. Here for all basic variables x_{ij}, $u_i + v_j - c_{ij} = 0$. Then the above two tables actually give the z-row of the simplex table represented as follows:

Basic	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	↓ x_{31}	x_{32}	x_{33}	x_{34}
z	0	0	-16	4	3	0	0	0	9	-9	-9	0

Since the objective of the transportation model is to minimize the cost, the entering variable is the variable with most positive coefficient in the z-row. Hence, here x_{31} is the entering variable.

The computations of the multipliers u_i, v_j can be directly done on the transportation table as shown in the following table, i.e. there is no need to write the (u, v)-equations separately.

Table 3.15: Iteration 1 Calculations

	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 = 0$	10 5	2 10	20 $\boxed{-16}$	11 $\boxed{4}$	15
$u_2 = 5$	12 $\boxed{3}$	7 5	9 15	20 5	25
$u_3 = 3$	4 $\boxed{9}$	14 $\boxed{-9}$	16 $\boxed{-9}$	18 10	10
Demand	5	15	15	15	

Once x_{31} is the entering variable, the leaving variable is determined from how much maximum we can assign to cell $(3, 1)$. If θ is the amount that can be shipped on route $(3, 1)$, then maximum value of θ depends on the following two conditions:

1. Supply and demand requirements remain satisfied.
2. Shipments through all the routes remain non-negative.

To determine the maximum value of θ and the leaving variable, we construct a closed loop (clockwise or counterclockwise) that starts and ends at the entering variable. The loop consists of connected horizontal and vertical segments only whose corner points are the basic variables as shown in the following table. We assign θ to the entering variable cell. Since the supply and demand limits remain satisfied, we subtract and add θ alternatively at the successive corner steps as shown in the table given below:

Table 3.16: Determination of closed loop for x_{31}

	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 = 0$	10 $5-\theta$	2 $10+\theta$	20 $\boxed{-16}$	11 $\boxed{4}$	15
$u_2 = 5$	12 $\boxed{3}$	7 $5-\theta$	9 15	20 $5+\theta$	25
$u_3 = 3$	4 θ $\boxed{9}$	14 $\boxed{-9}$	16 $\boxed{-9}$	18 $10-\theta$	10
Demand	5	15	15	15	

Since $\theta \geq 0$, the new variables remain non-negative if

$$x_{11} = 5 - \theta \geq 0$$
$$x_{22} = 5 - \theta \geq 0$$
$$x_{34} = 10 - \theta \geq 0$$

Thus, the maximum value of θ is 5 and this happens when both x_{11} and x_{22} reach at zero level (i.e. they both become zero). This means, either x_{11} or x_{22} can be leaving variable and we arbitrarily choose x_{11} as the leaving variable. The entering variable x_{13} is assigned value $\theta = 5$ and the values of the corner variables are adjusted accordingly. Since each unit shipped through route $(3,1)$ reduces the cost by ₹ 9 $(= u_3 + v_1 - c_{13})$, the total cost reduces by $c_{ij} \times x_{ij} = 9 \times 5 = 45$. Hence, the new cost is ₹ $520 - 45 = 475$. We get the new basic solution and a new table representing the same as follows:

Table 3.17: Iteration 2 Calculations

	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 = 0$	10 $\boxed{-9}$	2 $15-\theta$	20 $\boxed{-16}$	11 θ $\boxed{4}$	15
$u_2 = 5$	12 $\boxed{-6}$	7 $0+\theta$	9 15	20 $10-\theta$	25
$u_3 = 3$	4 5	14 $\boxed{-9}$	16 $\boxed{-9}$	18 5	10
Demand	5	15	15	15	

We repeat the process of computing the multipliers u and v by assuming $u_1 = 0$ arbitrarily. We find that x_{14} is the entering variable as $u_1 + v_4 - c_{14} = 4$ is the most positive. Constructing a closed loop as above, we get $x_{14} = 10$ and that the leaving variable is x_{24}. The new solution, shown in the table below, is hence $c_{14} \times x_{14} = 4 \times 10 =$ ₹ 40 less than the previous one. Hence the new cost is now, ₹ $475 - 40 =$₹ 435.

Table 3.18: Iteration 3 Calculations (Optimal)

	$v_1 = -3$	$v_2 = 2$	$v_3 = 4$	$v_4 = 11$	Supply
$u_1 = 0$	10 $\boxed{-13}$	2 5	20 $\boxed{-16}$	11 10	15
$u_2 = 5$	12 $\boxed{-10}$	7 10	9 15	20 $\boxed{-4}$	25
$u_3 = 7$	4 5	14 $\boxed{-5}$	16 $\boxed{-5}$	18 5	10
Demand	5	15	15	15	

The new values of $u_i + v_j - c_{ij}$ for all the nonbasic variables x_{ij} are all non-negative. Hence, the new solution obtained (represented in the above table) is optimal. The following table summarizes the optimum solution and the optimal objective value.

From silo	To mill	Number of truckloads
1	2	5
1	4	10
2	2	10
2	3	15
3	1	5
3	4	5
Optimal cost = ₹ 435		

□

3.3 The Assignment Model

The classical assignment problem is about matching the workers (with different skills) to the given jobs. It is assumed that varied skills of workers affects the cost of completing a job. The objective of the model is to determine the minimum-cost assignment of workers to jobs. The general assignment model with n workers and n jobs can be represented as an LP model follows: Let c_{ij} be the unit cost of assigning worker i to the job j, and define

$$x_{ij} = \begin{cases} 1, & \text{if worker } i \text{ is assigned to job } j; \\ 0, & \text{otherwise.} \end{cases}$$

The LP model of assignment problem can be given as

$$\text{Minimize } z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to

$$\sum_{i=1}^{n} x_{ij} = 1, \; i = 1,2,\ldots,n$$
$$\sum_{i=1}^{n} x_{ij} = 1, \; j = 1,2,\ldots,n$$
$$x_i = 0 \text{ or } 1.$$

We can assume, without the loss of generality, that the number of workers and the number of jobs are equal. If that is not the case, then we can add dummy workers or dummy jobs to satisfy this assumption.

The general assignment model with n workers and n jobs is represented in tabular form as shown below:

Table 3.19: Assignment Model

		Jobs				
		1	2	⋯	n	
Workers	1	c_{11}	c_{12}	⋯	c_{1n}	**1**
	2	c_{21}	c_{22}	⋯	c_{2n}	**1**
	⋮	⋮	⋮	⋮	⋮	⋮
	n	c_{n1}	c_{n2}	⋯	c_{nn}	**1**
		1	**1**	⋯	**1**	

The assignment model is a special case of the transportation model, where workers represent the sources and jobs represent the destinations. The supply and demand at each source and destination respectively is exactly equal to 1. The cost of "transporting" (assigning) worker i to job j is c_{ij}. Thus, the assignment model can be solved directly as a regular transportation model (or as a regular LP). However, the fact that all the supply and demand amounts are exactly equal to 1 leads to a simple solution algorithm for the assignment model called the **Hungarian method**. The solution method though appears to be different from the transportation model, the algorithm is rooted in the simplex method, just as in case of transportation problem.

3.3.1 The Hungarian Method

We demonstrate the application of Hungarian method by the means of the two examples considered in this subsection.

Example 3.3.1 (DOMSPU Model)**.** Department of Mathematics, SPU is renovating the Assembly Hall which involves three types of jobs, which includes flooring, painting and furniture work. There are three workers available, Worker A, Worker B and Worker C who can do these jobs and their cost (charges) for these jobs (in thousands of ₹) are summarized in the table below. The assignment model seeks the best assignment of workers to jobs which minimizes the cost of renovation.

Table 3.20: DOMSPU Problem

	Flooring	Painting	Furniture
Worker A	₹ 15	₹ 10	₹ 9
Worker B	₹ 9	₹ 15	₹ 10
Worker C	₹ 10	₹ 12	₹ 8

Solution. We solve the problem by Hungarian method. The steps of the method are described below.

Step 1. Determine p_i, the minimum cost element of row i in the original cost matrix, and subtract it from all the elements of row i, for all $i = 1,2,3$.

Step 2. For the matrix obtained from step 1, determine q_j, the minimum cost element of column j and subtract it from all the elements of column j, for all $j = 1,2,3$.

Step 3. From the matrix obtained in step 2, attempt to find a *feasible* assignment among all the resulting zero entries.

- **3 a.** If such an assignment can be determined, it is optimal.
- **3 b.** If such assignment does not exists, then additional calculations are required (as explained in Example 3.3.2 below).

The tables associated with the computations in the above steps are shown below:

Table 3.21: Step 1

	Flooring	Painting	Furniture	Row minimum
Worker A	15	10	9	$p_1 = 9$
Worker B	9	15	10	$p_2 = 9$
Worker C	10	12	8	$p_3 = 8$

Table 3.22: Step 2

	Flooring	Painting	Furniture
Worker A	6	1	0
Worker B	0	6	1
Worker C	2	4	0
Column minimum	$q_1 = 0$	$q_2 = 1$	$q_3 = 0$

Table 3.23: Step 3

	Flooring	Painting	Furniture
Worker A	6	**0**	0
Worker B	**0**	5	1
Worker C	2	3	**0**

The last (above) table shows the feasible solution of this assignment model. In the third column of the above table has two zeros. This indicates that Furniture job can be assigned to Worker A or Worker C. Likewise the two zeros in the first row suggests that Worker A can be given Painting job or Furniture job. However, if Worker A is assigned Furniture job, then first row is crossed out and there is no worker to execute the Painting job. Hence, the feasible solution is given by

- Worker A is assigned Paint job.
- Worker B gets to do the Flooring.
- Worker C gets the Furniture job.

Comparing this assignment with the cost Table 3.20 (first table), the total cost incurred to the Department will be $9+10+8 =$ ₹ 27 (thousand).

Note that this amount will be always equal to

$$(p_1+p_2+p_3)+(q_1+q_2+q_3) = (9+9+8)+(0+1+0) = 27.$$

□

In the above problem, the zero entries of the final matrix gives a feasible solution. However, as discussed in the step 3 of the Hungarian method above, the zeros created by step 1 and step 2 may not give a feasible solution directly. In that case, further steps are needed to find the optimal assignment. This is explained in the following example.

Example 3.3.2. Suppose that the situation in Example 3.3.1 is extended to four workers and four jobs. Table given below summarizes the cost elements of the problem.

Table 3.24: Assignment problem

	Jobs			
	1	2	3	4
A	₹ 1	₹ 4	₹ 6	₹ 3
Workers B	₹ 9	₹ 7	₹ 10	₹ 9
C	₹ 4	₹ 5	₹ 11	₹ 7
D	₹ 8	₹ 7	₹ 8	₹ 5

Solution. We solve the problem by Hungarian method. Steps 1, 2 and 3 are shown below in tabular form to obtained the reduced assignment matrix.

Table 3.25: Step 1

	Jobs 1	2	3	4	Row minimum
Workers A	1	4	6	3	$p_1 = 1$
B	9	7	10	9	$p_2 = 7$
C	4	5	11	7	$p_3 = 4$
D	8	7	8	5	$p_4 = 5$

Table 3.26: Step 2

	Jobs 1	2	3	4
Workers A	0	3	5	2
B	2	0	3	2
C	0	1	7	3
D	3	2	3	0
Column minimum	$q_1 = 0$	$q_2 = 0$	$q_3 = 3$	$q_4 = 0$

Table 3.27: Reduced assignment matrix (step 3)

	Jobs 1	2	3	4
Workers A	**0**	3	2	2
B	2	**0**	**0**	2
C	**0**	1	4	3
D	3	2	**0**	**0**

The location of zeros in the above reduced matrix does not allow unique feasible assignment of jobs to workers. For example, if Worker A takes job 1, then first column is satisfied and crossed out. As a result, Worker C is not assigned any job. Likewise if Worker B takes job 3, then there is no one who is assigned job 2. To overcome this difficulty, the following additional steps are carried out.

Step 3 b. If no feasible zero-element assignments can be found, then

1. Draw the minimum number of horizontal and vertical lines in the last reduced matrix to cover all the zero entries.
2. Select the smallest uncovered entry, subtract it from every uncovered entry, and then add it to every entry at the intersection of two lines.

3. If still no feasible assignment can be found among the resulting zero entries, then repeat step **3 b.** Otherwise, go to step **3 a.** (described in the above example)

The tabular computations of step **3 b** are shown below.

Table 3.28: Application of Step 3 b

		Jobs			
		1	2	3	4
	A	0	3	2	2
Workers	B	**2**	0	0	2
	C	0	$\boxed{1}$	4	3
	D	**3**	2	0	0

The smallest uncovered entry is 1 which is added to the two intersection cells and subtracted from all other uncovered entries to give the following optimal table.

Table 3.29: Optimal assignment

		Jobs			
		1	2	3	4
	A	**0**	2	1	1
Workers	B	3	0	**0**	2
	C	0	**0**	3	2
	D	4	2	0	**0**

Thus, the optimal solution is given by

- Worker A is assigned Job 1
- Worker B is assigned Job 3
- Worker C is assigned Job 2
- Worker D is assigned Job 4

The associated optimal cost is $1+10+5+5=$₹ 21. The same cost is also determined by the sum

$$\begin{aligned} & (p_1+p_2+p_3+p_4)+(q_1+q_2+q_3+q_4)+(\text{least uncovered entry in 3b}) \\ = & (1+7+4+5)+(0+0+3+0)+(1)=21. \end{aligned}$$

□

Exercises

Exercise 3.1

Solve the following LP problems by dual simplex method:

(a) Minimize $z = 5x_1 + 6x_2$
subject to

$$\begin{aligned} x_1 + x_2 &\geq 20 \\ 4x_1 + x_2 &\geq 40 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

(b) Minimize $z = 4x_1 + 2x_2$
subject to

$$\begin{aligned} x_1 + x_2 &= 10 \\ 3x_1 - x_2 &\geq 20 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

(c) Minimize $z = 2x_1 + 3x_2$
subject to

$$\begin{aligned} 2x_1 + x_1 &\geq 30 \\ x_1 + x_2 &= 20 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

Exercise 3.2

Solve the following LP by dual simplex method.

$$\text{Minimize } \ z = 6x_1 + 7x_2 + 3x_3 + 5x_4$$

subject to

$$\begin{aligned} 5x_1 + 6x_2 - 3x_3 + 4x_4 &\geq 12 \\ x_2 - 5x_3 - 6x_4 &\geq 10 \\ 2x_1 + 5x_2 + x_3 + x_4 &\geq 8 \\ x_1,\ x_2,\ x_3,\ x_4 &\geq 0. \end{aligned}$$

Exercise 3.3

Determine and compare the starting solutions obtained by the northwest-corner, the least-cost method, and the Vogel approximation methods for each of the following models:

(a)

0	2	1	6
2	1	5	7
2	4	3	7
5	5	10	

(b)

1	2	6	12
0	4	2	7
3	1	5	11
10	10	10	

(c)

5	1	8	12
2	4	0	14
3	6	7	4
9	10	11	

Exercise 3.4

Consider the following transportation models:

(a) Use the northwest-corner method to find the starting solution.
(b) Develop the iterations that lead to the optimum solution.

(i)

₹ 0	₹ 2	₹ 1	6
₹ 2	₹ 1	₹ 5	9
₹ 2	₹ 4	₹ 3	5
5	5	10	

(ii)

₹ 10	₹ 4	₹ 2	8
₹ 2	₹ 3	₹ 4	5
₹ 1	₹ 2	₹ 0	6
7	6	6	

(iii)

–	₹ 3	₹ 5	4
₹ 7	₹ 4	₹ 9	7
₹ 1	₹ 8	₹ 6	19
5	6	19	

Exercise 3.5

Solve the following assignment models by Hungarian method.

(i)

₹ 9	₹ 8	₹ 2	₹ 10	₹ 3
₹ 6	₹ 5	₹ 2	₹ 7	₹ 5
₹ 6	₹ 3	₹ 2	₹ 7	₹ 5
₹ 8	₹ 4	₹ 12	₹ 3	₹ 5
₹ 7	₹ 8	₹ 6	₹ 7	₹ 7

(ii)

₹ 3	₹ 12	₹ 2	₹ 2	₹ 7
₹ 6	₹ 1	₹ 5	₹ 8	₹ 6
₹ 9	₹ 4	₹ 7	₹ 13	₹ 3
₹ 2	₹ 5	₹ 4	₹ 2	₹ 1
₹ 10	₹ 6	₹ 1	₹ 4	₹ 6

CHAPTER 4

Nonlinear Programming

In nonlinear programming, unlike linear programming, the objective function and the constraints (if present) need not be in linear form.

As seen in all the earlier models, there is an objective function that is to be optimized subject to some constraints. In this chapter, we shall see two types of algorithms, *unconstrained algorithms* (in which there are no constraints) and *constrained algorithms* (in which constraints are present but not necessarily linear).

4.1 Unconstrained Algorithms

We shall see two different methods to obtain optimum for unconstrained algorithms. They are the *direct method* and the *gradient method*.

4.1.1 Direct Search Method

Recall that, a function $f(x)$ is said to be a unimodal function if there exists m such that $f(x)$ is monotonically increasing for $x \leq m$ and monotonically decreasing for $x \geq m$. Thus, $f(x)$ has maximum value $f(m)$ and no other local maxima.

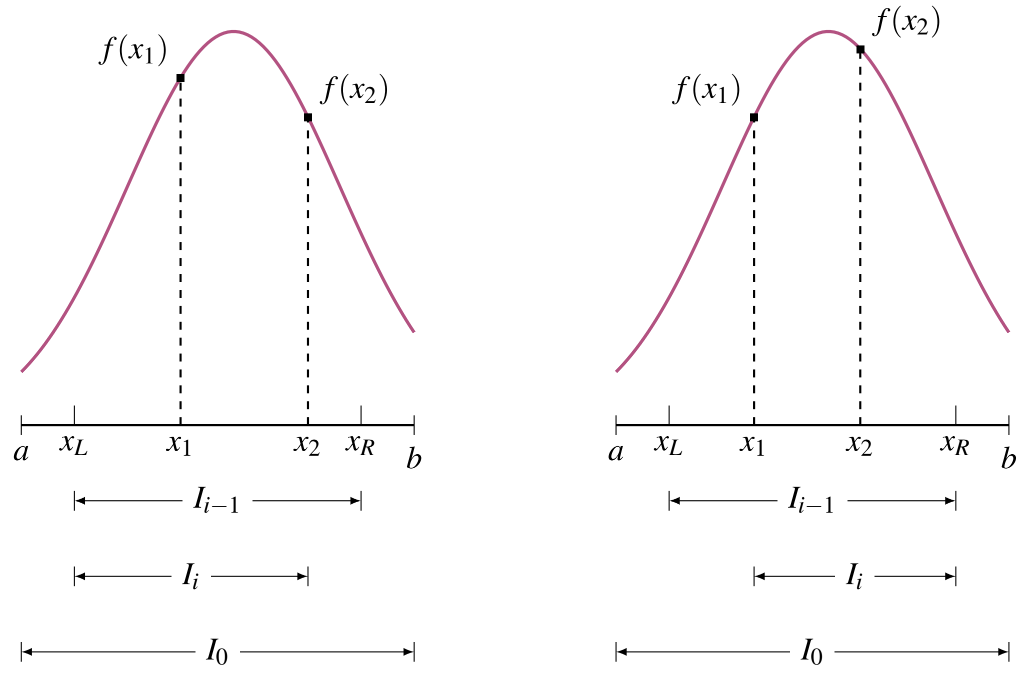
Direct method can be applied to single variable strictly unimodal functions. The procedure is to determine the interval of uncertainty, at each iteration, which contains optimum point. Iteratively, the length of the intervals of uncertainty decreases and can be brought down to a desired level of accuracy.

In this section, we shall see two closely related algorithms, **dichotomous** and **golden section**, that maximizes a unimodal objective function $f(x)$ over the interval $a \leq x \leq b$ which includes the optimum point x^*. Both the methods start by taking the initial interval of uncertainty as $I_0 = (a,b)$. The general iteration i is given as follows:

General iteration *i*. Suppose the interval in the current iteration (iteration $i-1$) is $I_{i-1} = (x_L, x_R)$ (at iteration 0, $I_0 = (a,b)$, i.e. $x_L = a$, $x_R = b$). The following table shows how x_1 and x_2 are determined in both the algorithms, i.e. in dichotomous and in golden section.

Dichotomous method	Golden section method
$x_1 = \frac{1}{2}(x_R + x_L - \Delta)$	$x_1 = x_R - \left(\frac{\sqrt{5}-1}{2}\right)(x_R - x_L)$
$x_2 = \frac{1}{2}(x_R + x_L + \Delta)$	$x_2 = x_L + \left(\frac{\sqrt{5}-1}{2}\right)(x_R - x_L)$

The selection of x_1 and x_2 defined in the above table is such that $x_L < x_1 < x_2 < x_R$.



(a) (b)

The next interval of uncertainty is determined by the following rule:

1. If $f(x_1) > f(x_2)$, then $x_L < x^* < x_2$. Let $x_R = x_2$ and set $I_i = (x_L, x_2)$.
2. If $f(x_1) < f(x_2)$, then $x_1 < x^* < x_R$. Let $x_R = x_2$ and set $I_i = (x_1, x_R)$.
3. If $f(x_1) = f(x_2)$, then $x_1 < x^* < x_2$. Let $x_L = x_1$ and $x_R = x_2$ and set $I_i = (x_1, x_2)$.

The way x_1 and x_2 are determine ensures that the length of the interval $I_i < I_{i-1}$. Let Δ be the desired level of accuracy. If the length of the interval $I_k \le \Delta$, then terminate the algorithm.

Dichotomous method

In the dichotomous method, the values x_1 and x_2 are exactly symmetrically on the opposite side of the midpoint of current interval of uncertainty, i.e. the length of the interval in iteration $i+1$ is

$$I_{i+1} = 0.5(I_i + \Delta) = \frac{I_i + \Delta}{2}.$$

Golden section method

Note that in the dichotomous method, at every iteration, we computed two values $f(x_1)$ and $f(x_2)$ and discarded the smaller one. In golden section method, to save computations, the discarded value is reused in the next iteration.

For $0 < \alpha < 1$, define

$$x_1 = x_R - \alpha(x_R - x_L) = \alpha x_L + (1-\alpha)x_R$$
$$x_2 = x_L + \alpha(x_R - x_L) = \alpha x_R + (1-\alpha)x_L.$$

Then as in case of previous method, the interval of uncertainty in the iteration i is $I_i = (x_L, x_2)$ or (x_1, x_R). Consider the case $I_i = (x_L, x_2)$, i.e. $x_L < x_1 < x_2$ which means that $x_1 \in I_i$. In the iteration $i+1$, we consider x_2 equal to x_1 of the iteration i. This gives the following relation:

$$x_2(\text{iteration } i+1) = x_1(\text{iteration } i).$$

Substituting these values in the above two equations, we get

$$x_L + \alpha[x_2(\text{iteration } i) - x_L] = x_R - \alpha(x_R - x_L)$$

or

$$x_L + \alpha[x_L + \alpha(x_R - x_L) - x_L] = x_R - \alpha(x_R - x_L).$$

On simplification, we get

$$\alpha^2 + \alpha - 1 = 0 \Rightarrow \alpha = \frac{-1 \pm \sqrt{5}}{2}.$$

Since, $0 < \alpha < 1$, the positive root $\alpha = \frac{-1+\sqrt{5}}{2} \approx 0.618$ is selected. This gives

$$I_{i+1} = \alpha I_i.$$

The golden section method converges to optimum faster than the dichotomous method, as the intervals of uncertainty are narrowing faster than in the case of dichotomous method. Besides, the golden section method requires only half of the computations because it uses the discarded values of $f(x_1)$ or $f(x_2)$ in the immediately preceding iteration.

Example 4.1.1.

$$\text{Maximize } f(x) = \begin{cases} 3x, & 0 \leq x \leq 2 \\ \frac{1}{3}(-x+20), & 2 \leq x \leq 3. \end{cases}$$

The maximum value of $f(x)$ occurs at $x = 2$. Desired level of accuracy $\Delta = 0.1$.

Solution. The following tables show two iterations of dichotomous method and golden section method with level of accuracy $\Delta = 0.1$.

Dichotomous method

Iteration 1

$I_0 = (0,3) = (x_L, x_R)$

$x_1 = 0 + 0.5(3 - 0 - 0.1) = 1.45,\ f(x_1) = 4.35$

$x_2 = 0 + 0.5(3 - 0 + 0.1) = 1.55,\ f(x_2) = 4.65$

$f(x_1) > f(x_2) \Rightarrow x_L = 1.45,\ I_1 = (1.45, 3)$

Iteration 2

$I_1 = (1.45, 3) = (x_L, x_R)$

$x_1 = 1.45 + 0.5(3 - 1.45 - 0.1) = 2.175,\ f(x_1) = 5.942$

$x_2 = \frac{3+1.45+0.1}{2} = 2.275,\ f(x_2) = 5.908$

$f(x_1) > f(x_2) \Rightarrow x_R = 2.275,\ I_2 = (1.45, 2.275)$

Golden section method

Iteration 1

$I_0 = (0,3) = (x_L, x_R)$

$x_1 = 3 - 0.618(3 - 0) = 1.146,\ f(x_1) = 3.438$

$x_2 = 0 + 0.618(3 - 0) = 1.854,\ f(x_2) = 5.562$

$f(x_2) > f(x_1) \Rightarrow x_L = 1.146,\ I_1 = (1.146, 3)$

Iteration 2

$I_1 = (1.146, 3) = (x_L, x_R)$

$x_1 = x_2$ in iteration $1 = 1.854,\ f(x_1) = 5.562$

$x_2 = 1.146 + 0.618(3 - 1.146) = 2.292,\ f(x_2) = 5.903$

$f(x_2) > f(x_1) \Rightarrow x_L = 1.854,\ I_2 = (1.854, 3)$

Continuing this way, the length of the intervals of uncertainty eventually reduces to the desired tolerance Δ. ☐

Remark 4.1.2. In both of the above (direct) methods, if the desired level of accuracy Δ is very small, then the number of iterations becomes considerably larger. It is not possible, then to compute all the iterations by hand and often the optimum is obtained using Excel or other programming software.

4.1.2 Gradient Method

In this section, we shall study a method for optimizing twice continuously differentiable functions, called the **steepest ascent** method. The idea is to generate points in the direction of gradient and the method terminates when the gradient vector becomes zero. This is only necessary condition for optimality. The process is as follows:

Suppose $f(\mathbf{X})$ is maximized. Let $\mathbf{X}_0$ be the initial point from which the procedure starts, and define $\nabla f(\mathbf{X_k})$ as the gradient of f at a point $\mathbf{X}_k$. The successive point $\mathbf{X}_{k+1}$ is obtained from the following relation:

$$\mathbf{X}_{k+1} = \mathbf{X}_k + r_k \nabla f(\mathbf{X}_k),$$

where r_k is the optimal step size at $\mathbf{X}_k$.

The step size r_k is determined such that the next point $\mathbf{X}_{k+1}$ gives the maximum improvement in f. This is equivalent to determining $r = r_k$ that maximizes the function

$$h(r) = f[\mathbf{X}_k + r\nabla f(\mathbf{X}_k)].$$

The process terminates when two successive points $\mathbf{X}_k$ and $\mathbf{X}_{k+1}$ are approximately equal. Equivalently, $r_k \nabla f(\mathbf{X}_k) \approx \mathbf{0}$, i.e. $f(\mathbf{X}_k) \approx \mathbf{0}$.

Example 4.1.3. Maximize $f(x_1,x_2) = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$.
The exact optimum occurs at $(x_1^*, x_2^*) = \left(\frac{1}{3}, \frac{4}{3}\right)$. Take the starting point $\mathbf{X}_0 = (1,1)$.

Solution. The gradient of f is

$$\nabla f(\mathbf{X}) = (4 - 4x_1 - 2x_2, 6 - 2x_1 - 4x_2). \tag{4.1}$$

The starting point $\mathbf{X}_0 = (1,1)$. The iterations obtained are as follows:

Iteration 1**:**

$$\nabla f(\mathbf{X}_0) = (-2,0).$$

The next point $\mathbf{X}_1$ is obtained by considering

$$\mathbf{X} = (1,1) + r(-2,0) = (1-2r,1).$$

Thus,

$$h(r) = f(1-2r,1) = -2(1-2r)^2 + 2(1-2r) + 4.$$

As described before, the value of r such that X_1 is the next largest improvement in f is equivalent to the value of r such that $h(r)$ is maximum. This r can be obtained by taking $\nabla h(r) = 0$ i.e. $h'(r) = 0$. Thus, $8(1-2r) - 4 = 0 \Rightarrow r = \frac{1}{4}$. Substituting this value of r in above expression of the point $\mathbf{X}$, we get $\mathbf{X}_1 = \left(\frac{1}{2}, 1\right)$.

Iteration 2**:** Substituting $\mathbf{X}_1 = \left(\frac{1}{2}, 1\right)$ obtained in the previous iteration in equation (4.1), we get

$$\begin{aligned}
\nabla f(\mathbf{X}_1) &= (0,1) \\
\mathbf{X} &= \left(\frac{1}{2}, 1\right) + r(0,1) = \left(\frac{1}{2}, 1+r\right) \\
h(r) &= -2(1+r)^2 + 5(1+r) + \frac{3}{2}
\end{aligned}$$

Thus, taking $h'(r) = 0$, we get $r_2 = \frac{1}{4}$ and hence $\mathbf{X}_2 = \left(\frac{1}{2}, \frac{5}{4}\right)$.

Iteration 3**:**

$$\begin{aligned}
\nabla f(\mathbf{X}_2) &= \left(-\frac{1}{2}, 0\right) \\
\mathbf{X} &= \left(\frac{1}{2}, \frac{5}{4}\right) + r\left(-\frac{1}{2}, 0\right) = \left(\frac{1-r}{2}, \frac{5}{4}\right)
\end{aligned}$$

$$h(r) = -\frac{1}{2}(1-r)^2 + \frac{3}{4}(1-r) + \frac{35}{8}$$

Hence, $r_3 = \frac{1}{4}$ and $\mathbf{X}_3 = \left(\frac{3}{8}, \frac{5}{4}\right)$.

Iteration 4:

$$\begin{aligned} \nabla f(\mathbf{X}_3) &= \left(0, \frac{1}{4}\right) \\ \mathbf{X} &= \left(\frac{3}{8}, \frac{5}{4}\right) + r\left(0, \frac{1}{4}\right) = \left(\frac{3}{8}, \frac{5+r}{4}\right) \\ h(r) &= -\frac{1}{8}(5+r)^2 + \frac{21}{16}(5+r) + \frac{39}{32} \end{aligned}$$

Hence, $r_4 = \frac{1}{4}$ and $\mathbf{X}_4 = \left(\frac{3}{8}, \frac{21}{16}\right)$.

Iteration 5:

$$\begin{aligned} \nabla f(\mathbf{X}_4) &= \left(-\frac{1}{8}, 0\right) \\ \mathbf{X} &= \left(\frac{3}{8}, \frac{21}{16}\right) + r\left(-\frac{1}{8}, 0\right) = \left(\frac{3-r}{8}, \frac{21}{16}\right) \\ h(r) &= -\frac{1}{32}(3-r)^2 + \frac{11}{64}(3-r) + \frac{567}{128} \end{aligned}$$

Thus, $r_5 = \frac{1}{4}$ and $\mathbf{X}_5 = \left(\frac{11}{32}, \frac{21}{16}\right)$.

Iteration 6:

$$\nabla f(\mathbf{X}_5) = \left(0, \frac{1}{16}\right)$$

The process can be terminated at this stage as $\nabla f(\mathbf{X}_5 \approx 0$. The approximate maximum obtained is $\mathbf{X}_5 = \left(\frac{11}{32}, \frac{21}{16}\right) = (0.3438, 1.3125)$. Note that the exact optimum was given to be $\mathbf{X}^* = \left(\frac{1}{3}, \frac{4}{3}\right) = (0.3333, 1.3333)$. □

4.2 Constrained Algorithms

The general constrained nonlinear programming problem is defined as

$$\text{Maximize (or Minimize) } z = f(\mathbf{X})$$

subject to

$$\mathbf{g}(\mathbf{X}) \leq \mathbf{0}.$$

The non-negativity conditions $\mathbf{X} \geq \mathbf{0}$ are part of the constraints. Since it is a nonlinear problem, at least one of the functions $f(\mathbf{X})$ and $\mathbf{g}(\mathbf{X})$ is nonlinear. In addition, all the functions are continuously differentiable.

4.2.1 Separable Programming

Before describing the method to solve a non-linear (separable) problem, we first define a **separable** function.

Definition 4.2.1 (Separable function)**.** A function $f(x_1,x_2,\ldots,x_n)$ is *separable* if it can be expressed as the sum of n single variable functions $f_1(x_1)$, $f_2(x_2),\ldots,f_n(x_n)$, i.e.

$$f(x_1,x_2,\ldots,x_n) = f_1(x_1)+f_2(x_2)+\cdots+f_n(x_n).$$

For example, any linear function is clearly separable. On the other hand, a function of the form

$$h(x_1,x_2,x_3) = x_1^2 + x_1\sin(x_2+x_3)+x_2e^{x_3}$$

is not separable as it cannot be written as sum of single variable functions.

Some nonlinear functions can be made separable using appropriate substitutions. For example, consider the case of maximizing the function $z = x_1x_2$. In this case, we take $y = x_1x_2$ and so taking logarithm on both sides, we get $\ln y = \ln x_1 + \ln x_2$. Consequently, the separable problem can be framed as

$$\text{Maximize } z = y$$

subject to

$$\ln y = \ln x_1 + \ln x_2.$$

The substitution assumes that x_1 and x_2 are *positive* variables as logarithm cannot be defined for non-positive values. We can consider the case where x_1 and x_2 can assume zero values by taking the approximations

$$w_1 = x_1+\delta_1 > 0$$
$$w_2 = x_2+\delta_2 > 0.$$

The constants δ_1 and δ_2 are arbitrarily small positive values.

In this section, we shall see how an approximate solution for any separable problem can be obtained using linear approximation and simple method of LP. The single-variable function $f(x)$ can be approximated by a piece-wise linear function using a method called *mixed integer programming*.

Now, suppose the function $f(x)$ is approximate over the interval $[a,b]$. Define a_k, $k = 1,2,\ldots,K$ as the kth break point on the x-axis such that $a_1 < a_2 < \cdots < a_K$, where $a_1 = a$ and $a_K = b$. Thus, $f(x)$ is approximated as

$$f(x) \approx \sum_{k=1}^{K} f(a_k)w_k$$
$$x = \sum_{k=1}^{K} a_k w_k.$$

The non-negative weights w_k must satisfy the condition

$$\sum_{k=1}^{K} w_k = 1,\ w_k \geq 0,\ k = 1,2,\ldots,K.$$

A valid approximate solution can be obtained by mixed integer programming by imposing the following two additional conditions:

1. At most two w_k are positive.
2. If w_k is positive, then only an adjacent w_{k+1} or w_{k-1} can assume a positive value (*adjacency requirement*).

Consider a separable problem

$$\text{Maximize (or minimize) } z = \sum_{j=1}^{n} f_j(x_j)$$

subject to

$$\sum_{j=1}^{n} g_{ij}(x_j) \leq b_i,\ i = 1,2,\ldots,m.$$

Let

$$\left.\begin{array}{rl} a_{jk} = & \text{breakpoint } k \text{ for variable } x_j \\ w_{jk} = & \text{weight with breakpoint } k \text{ of variable } x_j \end{array}\right\} k = 1,2,\ldots,K_j,\ j = 1,2,\ldots,n.$$

This approximation model can also be solved by another method which is the regular simplex method with **restricted basis**. The restricted basis modifies the optimality condition of the simplex method by selecting the entering variable w_j with the best z-coefficient (most negative for maximization problems and most positive for minimization problems) that does not violate the adjacency requirement of the w-variables with positive values.

The process is repeated until the optimality condition is satisfied or until it is impossible to satisfy the restricted basis condition, whichever happens first.

Example 4.2.2. Consider the separable nonlinear problem

$$\text{Maximize } z = x_1 + x_2^4$$

subject to

$$\begin{aligned} 3x_1 + 2{x_2}^2 &\leq 9 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

The exact optimum solution (solved by computer program) is given to be $x_1 = 0$, $x_2 = 2.12132$ and $z^* = 20.25$. Find the approximate solution.

Solution. Consider the separable functions

$$\begin{aligned} f_1(x_1) &= x_1 \\ f_2(x_2) &= x_2^4 \\ g_1(x_1) &= 3x_1 \\ g_2(x_2) &= 2x_2^2 \end{aligned}$$

The variable x_1 is not approximated as the functions $f_1(x_1)$ and $g_1(x_1)$ of variable x_1 are already linear. The functions $f_2(x_2)$ and $g_2(x_2)$ are nonlinear in the variable x_2 and hence we approximate x_2. Let $a_{21} = 0, a_{22} = 1, a_{23} = 2, a_{24} = 3$ be four breakpoints of x_2. Given $x_2 \leq 3$, we have

k	a_{2k}	$f_2(a_{2k}) = a_{2k}^4$	$g_2(a_{2k}) = a_{2k}^2$
1	0	0	0
2	1	1	2
3	2	16	8
4	3	81	18

Thus, f_2 is approximated as

$$\begin{aligned} f_2(x_2) &\approx w_{21}f_2(a_{21}) + w_{22}f_2(a_{22}) + w_{23}f_2(a_{23}) + w_{24}f_2(a_{24}) \\ &\approx 0w_{21} + 1w_{22} + 16w_{23} + 81w_{24} = w_{22} + 16w_{23} + 81w_{24}. \end{aligned}$$

Similarly, we have

$$g_2(x_2) \approx 2w_{22} + 8w_{23} + 18w_{24}.$$

The approximation problem thus becomes,

$$\text{Maximize } z = x_1 + w_{22} + 16w_{23} + 81w_{24}$$

subject to

$$\begin{aligned} 3x_1 + 2w_{22} + 8w_{23} + 18w_{24} &\leq 9 \\ w_{21} + w_{22} + w_{23} + w_{24} &= 1 \end{aligned}$$

$$x_1 \geq 0,\ w_{2k} \geq 0,\ k = 1,2,3,4,$$

where the values w_{2k} $(k = 1,2,3,4)$ must satisfy the restricted basis condition. A slack variable s_1 is added to the first constraint and since w_{21} is not present in the objective function, it is treated as a slack variable in the second constraint to give the starting solution. The starting simplex table is as follows:

Basic	x_1	w_{22}	w_{23}	w_{24}	s_1	w_{21}	Solution
z	-1	-1	-16	-81	0	0	0
s_1	3	2	8	18	1	0	9
w_{21}	0	1	1	1	0	1	1

Clearly w_{24} is the entering variable. The ratios indicate that s_1 is leaving variable. But this is not possible as w_{21} is already basic with positive value. Both w_{21} and w_{24} cannot simultaneously assume positive value because it will violate adjacency condition.

The next best choice of entering variable is w_{23} which makes w_{21} as the leaving variable. This is possible and the next simplex table thus becomes

Basic	x_1	w_{22}	w_{23}	w_{24}	s_1	w_{21}	Solution
z	-1	15	0	-65	0	16	16
s_1	3	-6	0	10	1	-8	1
w_{23}	0	1	1	1	0	1	1

Next, w_{24} is the entering variable which is allowed as w_{23} is positive. The feasibility condition shows that s_1 is the leaving variable. The resultant table is given as follows:

Basic	x_1	w_{22}	w_{23}	w_{24}	s_1	w_{21}	Solution
z	$\frac{37}{2}$	-24	0	0	$\frac{13}{2}$	-36	$\frac{45}{2}$
w_{24}	$\frac{3}{10}$	$-\frac{6}{10}$	0	1	$\frac{1}{10}$	$-\frac{8}{10}$	$\frac{1}{10}$
w_{23}	$-\frac{3}{10}$	$\frac{16}{10}$	1	0	$-\frac{1}{10}$	$\frac{18}{10}$	$\frac{9}{10}$

The above table shows that w_{21} is the entering variable which is not possible as it is neither adjacent to w_{23} or w_{24}. Next possibility for entering variable is w_{22}. This gives w_{23} is the leaving variable which is not possible as w_{22} and w_{24} cannot simultaneously assume positive values.

Thus, the above table is the best restricted-basis solution for the given problem. The optimum solution is

$$\begin{aligned}
x_1 &= 0\\
x_2 &\approx a_{21}w_{21} + a_{22}w_{22} + a_{23}w_{23} + a_{24}w_{24}\\
x_2 &\approx 2w_{23} + 3w_{24} = 2\left(\frac{9}{10}\right) + 3\left(\frac{1}{10}\right) = 2.1\\
z &= 0 + (2.1)^4 = 19.45
\end{aligned}$$

□

4.2.2 Quadratic Programming

A quadratic programming model is defined as

$$\text{Maximize } z = \mathbf{CX} + \mathbf{X}^T\mathbf{DX}$$

subject to

$$\mathbf{AX} \leq \mathbf{b},\ \mathbf{X} \geq \mathbf{0},$$

where

$$\begin{aligned}
\mathbf{X} &= (x_1, x_2, \ldots, x_n)^T\\
\mathbf{C} &= (c_1, c_2, \ldots, c_n)\\
\mathbf{b} &= (b_1, b_2, \ldots, b_n)^T\\
\mathbf{A} &= \begin{pmatrix} a_{11} & \cdots & a_{1n}\\ \vdots & \vdots & \vdots\\ a_{m1} & \cdots & a_{mn}\end{pmatrix}\\
\mathbf{D} &= \begin{pmatrix} d_{11} & \cdots & d_{1n}\\ \vdots & \vdots & \vdots\\ d_{n1} & \cdots & d_{nn}\end{pmatrix}
\end{aligned}$$

The function $\mathbf{X}^T\mathbf{D}\mathbf{X}$ defined a quadratic form. The matrix $\mathbf{D}$ is assumed to be **symmetric** and negative definite.

The quadratic programming problem is considered for the maximization case. Conversion of minimization problem to maximization problem is straightforward.

Example 4.2.3. Express the following problem in matrix form:

$$\text{Maximize } z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 2 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

Solution. Here,

$$\begin{aligned} \mathbf{X} &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \mathbf{C} &= (c_1, c_2) = (4, 6) \\ \mathbf{D} &= \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \\ \mathbf{A} &= (1, 2) \end{aligned}$$

The problem can be put in the matrix form as follows:

$$\text{Maximize } z = (4,6)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1, x_2)\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

subject to

$$\begin{aligned} (1,2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\leq 2 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

□

Example 4.2.4. Express the following quadratic programming problem in the matrix form:

$$\text{Maximize } z = 6x_1 + 3x_2 - 4x_1x_2 - 2x_1^2 - 3x_2^2$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 1 \\ 2x_1 + 3x_2 &\leq 4 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

Solution. The problem can be put into matrix form as follows:

$$\text{Maximize } z = (6,3)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1, x_2)\begin{pmatrix} -2 & -2 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

subject to

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$
$$x_1,\, x_2 \geq 0.$$

□

4.2.3 Linear Combinations Method

The linear combinations method is useful in solving the nonlinear problems in which all the constraints are linear. Consider such a problem of the form:

$$\text{Maximize } z = f(\mathbf{X})$$

subject to

$$\mathbf{AX} \leq \mathbf{b},\ \mathbf{X} \geq \mathbf{0}.$$

The procedure is based on the steepest-ascent method (i.e. the gradient method) as seen in the unconstrained case. The method is modified for the constrained case as follows:

Let $\mathbf{X}_k$ be the feasible point at iteration k. The objective function $f(\mathbf{X})$ can be expanded in the neighborhood of $\mathbf{X}_k$ using Taylor's series as

$$f(\mathbf{X}) \approx f(\mathbf{X}_k) + \nabla f(\mathbf{X}_k)(\mathbf{X} - \mathbf{X}_k) = (f(\mathbf{X}_k) - \nabla f(\mathbf{X}_k)\mathbf{X}_k) + \nabla f(\mathbf{X}_k)\mathbf{X}.$$

The process determines a feasible point $\mathbf{X} = \mathbf{X}^*$ such that $f(\mathbf{X})$ is maximized subject to the given linear constraints. Note that, the term $f(\mathbf{X}_k) - \nabla f(\mathbf{X}_k)\mathbf{X}_k$ is constant. Hence, the problem for determining the point $\mathbf{X}^*$ reduces to finding solution of the following linear programming problem:

$$\text{Maximize } w_k(\mathbf{X}) = \nabla f(\mathbf{X}_k)\mathbf{X}$$

subject to

$$\mathbf{AX} \leq \mathbf{b},\ \mathbf{X} \geq \mathbf{0}.$$

Given that $w_k(\mathbf{X}^*) > w_k(\mathbf{X}_k)$, there must exists a point $\mathbf{X}_{k+1}$ on the line segment $(\mathbf{X}_k, \mathbf{X}^*)$ such that $f(\mathbf{X}_{k+1}) > f(\mathbf{X}_k)$. The objective is to determine $\mathbf{X}_{k+1}$. Define

$$\mathbf{X}_{k+1} = r\mathbf{X}^* + (1-r)\mathbf{X}_k = \mathbf{X}_k + r(\mathbf{X}^* - \mathbf{X}_k), \quad 0 < r \leq 1.$$

This means that $\mathbf{X}_{k+1}$ is a linear combination of $\mathbf{X}_k$ and $\mathbf{X}^*$ and it is determined such that $f(\mathbf{X})$ is maximized. Because $\mathbf{X}_{k+1}$ is a function of r only, it is determined by maximizing

$$h(r) = f(\mathbf{X}_k + r(\mathbf{X}^* - \mathbf{X}_k)).$$

The process is repeated until, at the kth iteration, we have $w_k(\mathbf{X}^*) \leq w_k(\mathbf{X}_k)$ and at this stage $\mathbf{X}_k$ is the best solution point.

Example 4.2.5. Consider the quadratic programming as seen in the above example

$$\text{Maximize } f(\mathbf{X}) = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

subject to

$$x_1 + 2x_2 \leq 2$$
$$x_1,\, x_2 \geq 0.$$

Let the initial trial point be $\mathbf{X}_0 = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Solution. Here,

$$\nabla f(\mathbf{X}) = (4-4x_1-2x_2, 6-2x_1-4x_2)$$

Iteration 1. $$\nabla f(\mathbf{X}_0) = (1,3).$$

The associated linear program is

$$w_1 = x_1 + 3x_2$$

subject to the constraints of the original problem. The optimal solution (obtained by graphical method) of this problem is $\mathbf{X}^* = (0,1)$. Now, since

$$w_1(\mathbf{X}_0) = w_1\left(\tfrac{1}{2},\tfrac{1}{2}\right) = 2 < w_1(\mathbf{X}^*) = w_1(0,1) = 3$$

the next point $\mathbf{X}_1$ is to be determined. We have

$$\mathbf{X}_1 = r\mathbf{X}^* + (1-r)\mathbf{X} + 0 = r(0,1) + (1-r)\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \left(\tfrac{1-r}{2},\tfrac{1+r}{2}\right).$$

Therefore,

$$\begin{aligned} h(r) &= f\left(\tfrac{1-r}{2},\tfrac{1+r}{2}\right) \\ &= 4\left(\tfrac{1-r}{2}\right) + 6\left(\tfrac{1+r}{2}\right) - 2\left(\tfrac{1-r}{2}\right)^2 - 2\left(\tfrac{1-r}{2}\right)\left(\tfrac{1+r}{2}\right) - 2\left(\tfrac{1+r}{2}\right)^2 \\ &= -\tfrac{r^2}{2} + r + \tfrac{7}{2}. \end{aligned}$$

To determine the step size r, we take $h'(r) = 0 \Rightarrow -r+1 = 0 \Rightarrow r = 1$. Substituting this value of r in $\mathbf{X}_1$, we get $\mathbf{X}_1 = (0,1)$.

Iteration 2. $$\nabla f(\mathbf{X}_1) = (2,2)$$

The objective function of the new linear programming problem is $w_2 = 2x_1 + 2x_2$ and the optimum solution subject to the given constraints is $\mathbf{X}^* = (2,0)$. Now, since

$$w_1(\mathbf{X}_1) = 2 < 4 = w_1(\mathbf{X}^*)$$

new trial point $\mathbf{X}_2$ must be determined. Thus,

$$\mathbf{X}_2 = r(2,0) + (1-r)(0,1) = (2r, 1-r).$$

The maximization of the function

$$h(r) = f(2r, 1-r)$$

as above gives $r = \frac{1}{6}$. This gives $\mathbf{X}_2 = \left(\frac{1}{3},\frac{5}{6}\right)$ and $f(\mathbf{X}_2) \approx 4.16$.

Iteration 3. $$\nabla f(\mathbf{X}_2) = (1,2)$$

The corresponding objective function of the linear problem is $w_3 = x_1 + 2x_2$. This problem has alternative optima, $\mathbf{X}^* = (0,1)$ and $\mathbf{X}^* = (2,0)$. The value of w_3 at both these points is 2 which is same as the value of w_3 at $\mathbf{X}_2$. Thus,

$$w_3(\mathbf{X}^*) \leq w_3(\mathbf{X}_2).$$

Hence, no further improvement is possible and the point $\mathbf{X}_2 = \left(\frac{1}{3},\frac{5}{6}\right)$ is optimum with optimum value $f(\mathbf{X}_2) \approx 4.16$. □

Exercises

Exercise 4.1

Carry out five iterations for each of the following problems using the method of steepest descent (ascent). Assume that the initial point $\mathbf{X}_0 = 0$ in each case.

(a) min $f(\mathbf{X}) = (x_2 - x_1^2)^2 + (1 - x_1)$

(b) max $f(\mathbf{X}) = \mathbf{cX} + \mathbf{X}^T\mathbf{AX}$, where

$$\mathbf{c} = (1, 3, 5)$$
$$\mathbf{A} = \begin{pmatrix} -5 & -3 & -\frac{1}{2} \\ -3 & -2 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

(c) min $f(\mathbf{X}) = x_1 - x_2 + x_1^2 - x_1x_2$

Exercise 4.2

Approximate the following problem using the restricted basis method.

$$\text{Maximize } z = e^{-x_1} + x_1 + (x_2 + 1)^2$$

subject to

$$x_1^2 + x_2 \leq 3$$
$$x_1,\ x_2 \geq 0$$

Exercise 4.3

Show how the following problem can be made separable.

$$\text{Maximize } z = x_1x_2 + x_3 + x_1x_3$$

subject to

$$x_1x_2 + x_2 + x_1x_3 \leq 10$$
$$x_1,\ x_2x_3 \geq 0.$$

Exercise 4.4

Show how the following problem can be made separable.

$$\text{Maximize } z = e^{2x_1 + x_2^2} + (x_3 - 2)^2$$

subject to

$$x_1 + x_2 + x_3 \leq 6$$
$$x_1,\ x_2x_3 \geq 0.$$

Exercise 4.5

Show how the following problem can be made separable.

$$\text{Maximize } z = e^{x_1x_2} + x_2^2x_3 + x_4$$

subject to

$$\begin{aligned} x_1 + x_2x_3 + x_3 &\le 10 \\ x_1,\ x_2,\ x_3 &\ge 0 \\ x_4 &\ \text{unrestricted.} \end{aligned}$$

Exercise 4.6

Put the following problem in the matrix form

$$\text{Minimize } z = 2x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_2x_3 + x_1 - 3x_2 - 5x_3$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\ge 1 \\ 3x_1 + 2x_2 + x_3 &\le 6 \\ x_1,\ x_2,\ x_3 &\ge 0. \end{aligned}$$

Index

Symbols

(u,v)-method . . . 99

A

Alternative Optima . . . 61
Artificial starting solution . . . 42
    Big M-method . . . 43
    Two-Phase method . . . 48
Assignment model . . . 103
assignment table . . . 104

B

balancing transportation model . . . 92
basic feasible solution . . . 30
basic solution . . . 30
basic variables . . . 30
Big M-method . . . 43

C

Canonical form of an LPP . . . 27

D

Degeneracy . . . 59
degenerate basic solution . . . 33
Dichotomous method . . . 112
dual . . . 65
dual feasibility condition . . . 85
dual optimality condition . . . 85
Dual problem . . . 65
    Examples . . . 66
    Method for construction . . . 66
    rules for constructing . . . 66
dual simplex method . . . 85

E

Economic Interpretation
    Dual constraints . . . 77
    Dual variables . . . 78
    Duality . . . 76
Equation (Standard) form of an LPP . . . 28
Exercises
    Chapter 1 . . . 53
    Chapter 2 . . . 79
    Chapter 3 . . . 109
    Chapter 4 . . . 124
extremum points . . . 17

F

feasible region . . . 16

G

Gauss-Jordan row operations . . . 38
General linear programming problem . . . 26
Golden section method . . . 112
Graphical method for LPP . . . 15
    disadvantages . . . 22
Graphical sensitivity analsysis . . . 22

H

Hungarian method . . . 104

I

Infeasible Solution . . . 64
ISO-profit method . . . 21

L

Least-cost method . . . 95
Linear Programming Problem . . . 12

  - Canonical form 27
  - Equation (Standard) form 28
    - Matrix form, 29

LP model
  - Reddy Mikks model 12

LPP 12

M

matrix form of standard LPP 29
method of multipliers 99

N

non-basic variables 30
non-degenerate basic solution 33
Non-linear programming 111
  - Constrained algorithms 116
  - Direct search method 111
  - Gradient method 114
  - Linear combination method 122
  - Quadratic programming 120
  - Separable programming 117
  - Unconstrained algorithms 111

Northwest-corner method 94

O

Optimal dual solution 69

P

primal 65

R

Reddy Mikks model 13

S

Sensitivity analysis 22
Separable function 117
Simple tableau layout 68
Simplex method
  - Alternative optima 61
  - Big M-method 43
  - Degeneracy 59
  - Feasiblility condition 38
  - Gauss-Jordan row operations 38
  - Infeasible solution 64
  - Optimality condition 38
  - Two-Phase method 48
  - Unbounded solution 62

Slack variable 29
Surplus variable 29

T

Terminology of OR 11
Transportation algorithm 93
Transportation model 90
  - determining starting solution 94
  - iterative computations 99

Two-Phase method 48

U

Unbounded Solution 62

V

Vogel approximation method 96